Stochastic Iterative Learning Control for Lumped- and Distributed-Parameter Systems: A Wiener-Filtering Approach

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Stochastic Iterative Learning Control for Lumped- and Distributed-Parameter Systems: A Wiener-Filtering Approach

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***Abstract*—This paper presents a stochastically optimal iterative learning control (ILC) approach by designing a general integral learning operator which minimizes the expected mean-squares output error. The proposed learning law generalizes existing optimal PD-type learning laws and the resulting optimal learning operator turns out to be the solution of the non-causal Wiener-Hopf equation. The proposed solution can be interpreted as a systematic dual to traditional norm-optimal ILC schemes with superior asymptotic properties under stochastic perturbations. While the fully optimal solution is inherently iteration-varying, a simpler sub-optimal learning operator with less computational effort is introduced. Moreover, a numerically very efficient strategy based on the fast Fourier transform (FFT) is presented to obtain numerical solutions of the learning operator. By avoiding the need of spectral factorizations or solutions to Riccati equations, this approach is directly applicable to a certain class of distributed-parameter systems. Finally, the Wiener-filter-based ILC algorithm is demonstrated on finite- and infinite-dimensional example problems.**

***Index Terms*—Iterative learning control, distributed-parameter systems, non-causal Wiener-filter.**

I. Introduction

All types of ILC algorithms try to perform some kind of approximate inversion of the input-output behaviour of the system. Since this inversion has to be performed online based on measurements, it is inherently constrained by the presence of stochastic quantities such as process disturbances and measurement noise. The systematic treatment of iterative learning within a stochastic framework has received some attention from the scientific community, with most contributions being reviewed in [1]. Within such a stochastic framework, the usual approach is to find optimal solutions in a minimum mean-square-error (MMSE) sense. The majority of these contributions utilize some kind of P-type or D-type learning scheme with stochastically optimal learning gains most notably due to Saab (e.g., [2], [3], [4], [5]). All these contributions focus on a minimum level of model knowledge and are particularly well-suited for poorly known systems. In contrast, the desire to improve the convergence speed has led to the development of model-based (or inversion-based) ILC methods such as [6] which systematically exploit knowledge of the system in the form of a model. In particular, different versions of norm-optimal ILC schemes have been thoroughly studied in recent years by using the framework of so-called lifted systems (e.g., [7], [8]) in a discrete-time setting, in a general Hilbert space setting [9], [10], or by using frequency-domain methods [11]. While offering superior convergence properties [8], the effects of measurement noise or process disturbances are usually not explicitly treated or only seen as a side issue in these contributions, with the exception of [12] using pre-filtered variants of inversion-based ILC for SISO plants.

Recently, the control of distributed-parameter systems (DPSs) has become increasingly relevant not only in control theory [13], [14], [15] but also in applications such as large-scale manipulators (e.g., [16]), flexible structures and adaptive optics [17], chemical reactors [18] or pulse shaping in laser applications [19], [20], just to name a few. The existing literature on ILC for DPSs relies either on model-free ILC versions like D-type and PD-type ILC (e.g., [21], [22], [19]), or on non-constructive concepts as shown in [23]. Applications of model-based ILC schemes for DPSs are rather scarce and typically employ the classical early-lumping approach, where the full system dynamics is approximated by a finite-dimensional model first and the ILC design is then performed for the resulting system of ordinary differential equations (ODEs), e.g., as in [17]. This has the advantage that the well-established control theory of finite-dimensional systems can be applied but usually leads to high-dimensional system representations to keep the approximation accuracy at a sufficient level, particularly for weakly damped or purely dispersive dynamics [20]. Considering the full system structure during the design process by employing a late-lumping approach is clearly beneficial, as long as the computational effort of the resulting learning law is kept at a reasonable level.

Drawing from these directions, we systematically investigate MMSE-optimal solutions to the iterative learning problem in a stochastic setting applicable to lumped- and distributed-parameter systems. Following this idea ultimately leads to non-causal Wiener-filters introduced in [24] for the SISO case that are commonly used in image processing and image restoration, see, e.g., [25], or deconvolution applications as in [26]. The resulting learning law is a generalization of existing stochastically optimal P-type and D-type laws [4], [2], [27] and can be interpreted as a systematic dual to traditional norm-optimal ILC designs that shares their convergence properties but achieves superior asymptotic behaviour under stochastic perturbations. Although Wiener-filtering methods are applicable to general linear systems, we restrict ourselves to time-invariant systems for simplicity. Since the resulting non-causal

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Wiener-Hopf equation can be easily solved regardless of the distributed nature of the system in this case, the proposed learning law directly extends to DPSs in a late-lumping sense.

The structure of this paper is as follows: after the problem setting is stated in Section II, a stochastical MMSE optimal learning operator is derived in Section III. The resulting Wiener-filter-based learning operator requires knowledge of the stochastic properties of the input error, which can either be calculated to yield the fully optimal result or roughly estimated by a-priori knowledge to achieve a sub-optimal result. Both approaches are analyzed in Section IV and a simple and efficient numerical solution procedure is presented in Section V. The resulting algorithms are finally evaluated in simulation scenarios for finite- and infinite-dimensional systems in Section VI.

Nomenclature

In the following, we will make use of the bilateral Laplace transform

$$f(s) = \mathcal{B}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\exp(-st)\mathrm{d}t \tag{1}$$

for all s inside the region of convergence (ROC). If the imaginary axis is included in the ROC, the bilateral Laplace transform is linked to the Fourier transform by $f(s)|_{s=j\omega} = \mathcal{F}\{f(t)\}$. For two vector-valued stochastic signals $\mathbf{a}^k(t)$ and $\mathbf{b}^l(t)$ with the iteration indices k and l, their cross-correlation function is given by $\mathbf{R}_{a^k b^l}(\tau) = \mathrm{E}\{\mathbf{a}^k(t+\tau)(\mathbf{b}^l(t))^{\mathrm{T}}\}$ where $\mathrm{E}\{\cdot\}$ denotes the expectation operator. The corresponding power spectral density (PSD) reads as $\mathbf{S}_{a^k b^l}(s) = \mathcal{B}\{\mathbf{R}_{a^k b^l}(\tau)\}$. In case $k = l$, the common index is written as superscript, i.e., $\mathbf{R}^k_{ab}(\tau) = \mathbf{R}_{a^k b^k}(\tau)$ and $\mathbf{S}^k_{ab}(s) = \mathbf{S}_{a^k b^k}(s)$.

II. Problem Statement

Within an ILC framework, we consider a class of L^2-stable continuous-time linear systems, whose k-th iteration can be formally described by the input-output relation

$$\mathbf{y}^k(t) = \boldsymbol{\mathcal{G}}\mathbf{u}^k(t) + \mathbf{w}^k(t) \tag{2}$$

with the system operator $\boldsymbol{\mathcal{G}} : L^2(\mathbb{R};\mathbb{R}^l) \to L^2(\mathbb{R};\mathbb{R}^m)$, the input $\mathbf{u}^k(t) \in \mathbb{R}^l$, and the output $\mathbf{y}k(t) \in \mathbb{R}^m$ whereby $t \in \mathbb{R}$. The system description (2) in particular includes infinite-dimensional dynamics exhibited by distributed-parameter systems (DPSs) with finite-dimensional input and output quantities, which is the case in most practically relevant systems. The ideal behavior of the plant $\boldsymbol{\mathcal{G}}$ is perturbed by the exogenous disturbance $\mathbf{w}^k(t) \in \mathbb{R}^m$ which is assumed to be a zero-mean wide-sense stationary (WSS) stochastic process. In the time-invariant case, the system operator $\boldsymbol{\mathcal{G}} : L^2(\mathbb{R};\mathbb{R}^l) \to L^2(\mathbb{R};\mathbb{R}^m)$ can be written as

$$\boldsymbol{\mathcal{G}}\mathbf{u}(t) = \int_{-\infty}^{\infty} \mathbf{G}(t-\tau)\mathbf{u}(\tau)\mathrm{d}\tau \tag{3}$$

with the corresponding impulse response matrix $\mathbf{G}(t)$.

Remark 1. *Frequency-domain methods at least tacitly assume an infinite-time horizon which can be truncated suitably, see [6], [28], while retaining stability assertions.*

For a given desired output trajectory $\mathbf{y}^d(t) \in L_2(\mathbb{R};\mathbb{R}^m)$, which is the solution of the unperturbed system $\mathbf{y}^d(t) = \boldsymbol{\mathcal{G}}_u\mathbf{u}^d(t)$ with the unknown desired input $\mathbf{u}^d(t)$, we want to iteratively track this desired output trajectory by means of a general linear learning law of the form

$$\mathbf{u}^{k+1}(t) = \mathbf{u}^k(t) + \boldsymbol{\mathcal{L}}^k\boldsymbol{\eta}^k(t) \tag{4a}$$

with the output error $\boldsymbol{\eta}^k(t) = \mathbf{y}^d(t) - \mathbf{y}^k(t)$ for the iteration k. The corresponding linear learning operator $\boldsymbol{\mathcal{L}}^k : L^2(\mathbb{R};\mathbb{R}^m) \to L^2(\mathbb{R};\mathbb{R}^l)$ is given by

$$\boldsymbol{\mathcal{L}}^k\boldsymbol{\eta}^k(t) = \int_{-\infty}^{\infty} \mathbf{L}^k(t-\tau)\boldsymbol{\eta}^k(\tau)\mathrm{d}\tau \tag{4b}$$

with the learning kernel $\mathbf{L}^k(t)$. Thus, our goal is to find an optimal learning operator such that the system output asymptotically converges to the desired output, i.e., $\mathbf{y}^k(t) \to \mathbf{y}^d(t)$. Since the system output is a stochastic quantity, we aim at doing this in a stochastically optimal way, i.e. $\mathbf{u}^{k+1}(t)$ should be determined such that it minimizes the expected value of the mean-square output error $\mathrm{E}\Big\{\big(\boldsymbol{\eta}^{k+1}(t)\big)^{\mathrm{T}}\boldsymbol{\eta}^{k+1}(t)\Big\}$.

III. A Wiener-Filtering Approach

In view of (2) and the unperturbed system, the output error of the iteration k is given by

$$\boldsymbol{\eta}^k(t) = \boldsymbol{\mathcal{G}}\boldsymbol{\nu}^k(t) - \mathbf{w}^k(t) \tag{5}$$

using the input error $\boldsymbol{\nu}^k(t) = \mathbf{u}^d(t) - \mathbf{u}^k(t)$. Due to the learning law (4), the evolution of the input error is described by

$$\boldsymbol{\nu}^{k+1}(t) = \boldsymbol{\nu}^k(t) - \boldsymbol{\mathcal{L}}^k\boldsymbol{\eta}^k(t). \tag{6}$$

Thus, the output error of the iteration $k+1$ yields

$$\boldsymbol{\eta}^{k+1}(t) = \Big(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}^k\Big)\boldsymbol{\eta}^k(t) + \mathbf{w}^k(t) - \mathbf{w}^{k+1}(t) \tag{7}$$

using the identity operator $\boldsymbol{\mathcal{I}}$. The problem of learning in a stochastically optimal sense can be written as the optimization problem

$$\min_{\boldsymbol{\mathcal{L}}^k} \mathrm{E}\Big\{\big(\boldsymbol{\eta}^{k+1}(t)\big)^{\mathrm{T}}\boldsymbol{\eta}^{k+1}(t)\Big\}. \tag{8}$$

Due to the error dynamics (5), the stochastic quantities $\boldsymbol{\eta}^k$ and $\mathbf{w}^k$ appearing on the right-hand side of (7) are correlated. Plugging (5) into (7), one obtains

$$\boldsymbol{\eta}^{k+1}(t) = \Big(\boldsymbol{\mathcal{G}} - \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}^k\boldsymbol{\mathcal{G}}\Big)\boldsymbol{\nu}^k(t) + \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}^k\mathbf{w}^k(t) - \mathbf{w}^{k+1}(t). \tag{9}$$

Before deriving the optimal solution, we assume that

A1 the input error $\boldsymbol{\nu}^k$ is uncorrelated with the exogenous disturbance of the current and the following iteration, i.e. $\mathrm{E}\{\boldsymbol{\nu}^k(t+\tau)(\mathbf{w}^i(t))^{\mathrm{T}}\} = \mathbf{0}$ for $i \in \{k, k+1\}$.

Remark 2. *Assumption A1 essentially truncates the depth of stochastic reasoning. This means that any implicit correlation of the current input error $\boldsymbol{\nu}^k(t)$ and the exogenous disturbances $\mathbf{w}^i(t)$ with $i \in \{k, k+1\}$ due to possible correlations between different instances of the exogenous disturbance in the past, i.e. $\mathrm{E}\{\mathbf{w}^i(t+\tau)(\mathbf{w}^j(t))^{\mathrm{T}}\} \neq \mathbf{0}$ with $i, j < k$ and*

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$i \neq j$ are neglected. If one assumes that the disturbances are uncorrelated as we will do later, Assumption A1 is equivalent to $\mathrm{E}\{\boldsymbol{\nu}^0(t+\tau)(\mathbf{w}^i(t))^{\mathrm{T}}\} = \mathbf{0}$*, which is common in the literature on stochastic ILC, see, [4], [3].*

Since the system dynamics (2) is linear and assumed to be time-invariant for simplicity and the stochastic quantities $\mathbf{w}^k$ are WSS, the optimization problem (8) can be treated in the Laplace domain (cf. [29]) by applying the bilateral Laplace transform, which yields

$$\begin{aligned}\min_{\boldsymbol{\mathcal{L}}^k} \mathrm{E}\Big\{\left(\boldsymbol{\eta}^{k+1}(t)\right)^{\mathrm{T}} \boldsymbol{\eta}^{k+1}(t)\Big\} &= \min_{\boldsymbol{\mathcal{L}}^k} \mathrm{Tr}\left\{\mathbf{R}^{k+1}_{\eta\eta}(0)\right\} \\ &= \min_{\mathbf{L}^k(s)} \frac{1}{j2\pi} \mathrm{Tr}\left\{\int_{-j\infty}^{j\infty} \mathbf{S}^{k+1}_{\eta\eta}(s)\mathrm{d}s\right\}, \end{aligned} \tag{10}$$

where $\mathrm{Tr}\{\cdot\}$ denotes the trace operator. For the following derivation, the Laplace variable s is omitted for brevity. Due to (9) and the assumptions A1, the PSD of the output error in iteration $(k+1)$ is given by

$$\begin{aligned}\mathbf{S}^{k+1}_{\eta\eta} = &\left(\mathbf{G} - \mathbf{G}\mathbf{L}^k\mathbf{G}\right) \mathbf{S}^k_{\nu\nu} \left(\mathbf{G} - \mathbf{G}\mathbf{L}^k\mathbf{G}\right)^{\mathrm{H}} \\ &+ \mathbf{G}\mathbf{L}^k \mathbf{S}^k_{ww} (\mathbf{L}^k)^{\mathrm{H}}\mathbf{G}^{\mathrm{H}} + \mathbf{S}^{k+1}_{ww} \\ &- \mathbf{G}\mathbf{L}^k \mathbf{S}_{w^k w^{k+1}} - \mathbf{S}_{w^{k+1}w^k} (\mathbf{L}^k)^{\mathrm{H}}\mathbf{G}^{\mathrm{H}} \end{aligned} \tag{11}$$

with the Hermitian conjugate $(\cdot)^{\mathrm{H}}$. Employing the calculus of variations, the optimization problem (10) can be solved analytically by considering the first-order optimality condition, i.e. for the optimal learning kernel $\mathbf{L}^k(s)$ the Gâteaux derivative has to vanish

$$\delta \mathrm{E}\Big\{\left(\boldsymbol{\eta}^{k+1}(t)\right)^{\mathrm{T}} \boldsymbol{\eta}^{k+1}(t)\Big\} = 0. \tag{12}$$

Following the typical variational approach, we substitute $\mathbf{L}^k(s) \rightarrow \mathbf{L}^k(s) + \epsilon\boldsymbol{\Gamma}(s)$ into (11), whereby $\epsilon\boldsymbol{\Gamma}(s)$ is a small variation of the optimal learning operator $\mathbf{L}^k(s)$. The optimality criterion (12) finally yields

$$\begin{aligned} 0 = \mathrm{Tr}\Bigg\{ &\int_{-j\infty}^{j\infty} \mathbf{G}\boldsymbol{\Gamma}\Big[- \mathbf{G}\mathbf{S}^k_{\nu\nu} \left(\mathbf{G} - \mathbf{G}\mathbf{L}^k\mathbf{G}\right)^{\mathrm{H}} \\ &+ \mathbf{S}^k_{ww}(\mathbf{L}^k)^{\mathrm{H}}\mathbf{G}^{\mathrm{H}} - \mathbf{S}_{w^k w^{k+1}}\Big]\mathrm{d}s \\ &\int_{-j\infty}^{j\infty} \Big[-\left(\mathbf{G} - \mathbf{G}\mathbf{L}^k\mathbf{G}\right)\mathbf{S}^k_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \\ &+ \mathbf{G}\mathbf{L}^k\mathbf{S}^k_{ww} - \mathbf{S}_{w^{k+1}w^k}\Big]\boldsymbol{\Gamma}^{\mathrm{H}}\mathbf{G}^{\mathrm{H}}\mathrm{d}s \Bigg\}. \end{aligned}$$

Since we allow a potentially non-causal solution, $\boldsymbol{\Gamma}(s)$ can vary freely. Using that $\mathrm{Tr}\left(\mathbf{M} + \mathbf{M}^{\mathrm{H}}\right) = 2\,\mathrm{Tr}\left(\mathrm{Re}\{\mathbf{M}\}\right)$ for any square matrix $\mathbf{M}$ and invoking the fundamental lemma of calculus of variations, the optimal learning operator has to fulfill

$$\mathbf{G}\mathbf{L}^k\left[\mathbf{G}\mathbf{S}^k_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \mathbf{S}^k_{ww}\right] = \mathbf{G}\mathbf{S}^k_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \mathbf{S}_{w^{k+1}w^k}. \tag{13}$$

If the system is square, i.e., $m = l$, and if $\mathbf{G}(s)$ is invertible, (13) can be solved for $\mathbf{L}^k(s)$ explicitly which yields

$$\mathbf{L}^k = \left(\mathbf{S}^k_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \mathbf{G}^{-1}\mathbf{S}_{w^{k+1}w^k}\right)\left[\mathbf{G}\mathbf{S}^k_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \mathbf{S}^k_{ww}\right]^{-1}. \tag{14}$$

This type of learning operator takes preemptive actions according to known correlations of the external disturbances given by $\mathbf{S}_{w^{k+1}w^k}$. In practice, a detailed description of the correlation properties of the disturbance $\mathbf{w}$ is hardly known and the constraint that $\mathbf{G}(s)$ is invertible drastically limits the scope of the approach. However, there is a more severe implication: The stability of a learning law can be analysed in the frequency domain using the well known (deterministic) stability criterion [30], [31]

$$\sup_{\omega} \bar{\rho}\left\{\mathbf{I} - \mathbf{G}(j\omega)\mathbf{L}(j\omega)\right\} < 1 \tag{15}$$

where $\bar{\rho}$ denotes the eigenvalue with the largest absolute value, i.e. the spectral radius. Plugging the general learning operator (14) into (15), one can easily see that its spectral radius is not necessarily less than one for arbitrary $\mathbf{S}_{w^{k+1}w^k}$. The fact that the learning operator (14) does not *necessarily* fulfill the stability condition (15) does not imply that the output error PSD may not converge to some finite limit $\mathbf{S}^{\infty}_{\eta\eta}(s) = \lim_{k\to\infty} \mathbf{S}^k_{\eta\eta}(s)$ in the nominal case. However, imperfect knowledge of the stochastic properties can destabilize the learning law (14) which is problematic for robustness reasons. Thus, we neglect correlations of the stochastic disturbances in the following by assuming that

A2 different instances of the disturbance are uncorrelated, i.e. $\mathrm{E}\{\mathbf{w}^i(t+\tau)(\mathbf{w}^j(t))^{\mathrm{T}}\} = \mathbf{0}$ for $i \neq j$

to obtain the iteration-varying learning operator

$$\mathbf{L}^k(s) = \mathbf{S}^k_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s)\left[\mathbf{G}(s)\mathbf{S}^k_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s) + \mathbf{S}^k_{ww}(s)\right]^{-1}. \tag{16}$$

The type of solution (16) for uncorrelated disturbances is in fact well known in the context of linear filtering problems [26]. By transforming the optimal filter $\mathbf{L}^k(s)$ back into the time domain, one obtains the non-causal Wiener-Hopf equation

$$\int_{-\infty}^{\infty} \mathbf{L}^k(\xi)\mathbf{R}^k_{\eta\eta}(\tau - \xi)\mathrm{d}\xi = \mathbf{R}^k_{\nu\eta}(\tau) \tag{17}$$

with the auto-correlation function $\mathbf{R}^k_{\eta\eta}(\tau) = \mathcal{B}^{-1}\{\mathbf{G}(s)\mathbf{S}^k_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s) + \mathbf{S}^k_{ww}(s)\}$ of the output error and the cross-correlation function $\mathbf{R}^k_{\nu\eta}(\tau) = \mathcal{B}^{-1}\{\mathbf{S}^k_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s)\}$ between input error and output error. Thus, the proposed approach can be understood as the systematic dual to traditional norm-optimal ILC methods, as one is effectively using a non-causal Wiener-filter to *estimate* the input deviation that optimally *explains* the measured output error.

IV. Choosing the input error PSD $\mathbf{S}^k_{\nu\nu}(s)$

A common problem of Wiener-filter-based approaches is that the optimal solution (16) requires knowledge of the input error PSD $\mathbf{S}^k_{\nu\nu}(s)$, which is typically handled using a-priori knowledge of the problem. In the case of ILC, however, the iterative structure of the learning problem can be exploited to obtain an estimate of the current PSD. In the following, we will present two approaches: first, by assuming a fixed input error PSD, a stochastically sub-optimal but iteration-invariant learning operator is obtained. Second, a forward prediction of the current input error PSD yields an optimal but iteration-varying learning operator.

For the following analysis, we assume that the stochastic properties of $\mathbf{w}^k$ are independent of the iteration index k, i.e., $\mathbf{S}^k_{ww}(s) = \mathbf{S}_{ww}(s)$ for all k.

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A. A sub-optimal iteration-invariant learning law

As highlighted above, the stochastically optimal learning operator (16) depends on the expected input error, i.e., the input error PSD $\mathbf{S}_{\nu\nu}^k(s)$, which is generally unknown. Since iteration-varying learning laws usually increase the effort of implementation, one may prefer to accept the sub-optimal performance of an iteration-invariant learning law by using a fixed input error PSD $\mathbf{S}_{\nu\nu}^k(s) = \mathbf{S}_{\nu\nu}(s)$ according to a-priori knowledge. This yields the iteration-invariant learning operator

$$\mathbf{L}(s) = \mathbf{S}_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s)\left[\mathbf{G}(s)\mathbf{S}_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s) + \mathbf{S}_{ww}(s)\right]^{-1}. \tag{18}$$

The following theorem ensures under mild conditions that the resulting iteration is stable.

Theorem 1. *If the given PSD of the disturbance $\mathbf{S}_{ww}(s)$ and the chosen input error PSD $\mathbf{S}_{\nu\nu}(s)$ are positive definite and the system transfer matrix $\mathbf{G}(s)$ does not exhibit transmission zeros on the imaginary axis $s = j\omega$, i.e. if $\mathbf{G}(j\omega)\mathbf{u} \neq \mathbf{0}$ for any $\mathbf{u} \neq \mathbf{0}$ and all frequencies ω, the learning operator (18) yields a stable learning iteration that converges to a positive definite asymptotic ouput error PSD $\mathbf{S}_{\eta\eta}^{\infty}(s)$ given by the solution of*

$$\begin{aligned}(\mathbf{I}-\mathbf{G}(s)\mathbf{L}(s))\,\mathbf{S}_{\eta\eta}^{\infty}(s)\,(\mathbf{I}-\mathbf{G}(s)\mathbf{L}(s))^{\mathrm{H}} - \mathbf{S}_{\eta\eta}^{\infty}(s)\\ +\mathbf{G}\mathbf{L}(s)\mathbf{S}_{ww}(s) + \mathbf{S}_{ww}(s)\mathbf{L}^{\mathrm{H}}(s)\mathbf{G}^{\mathrm{H}}(s) = \mathbf{0}.\end{aligned} \tag{19}$$

Proof. Omitting the Laplace variable s for simplicity, the definition of the sub-optimal learning operator (18) yields

$$\begin{aligned}\mathbf{I} - \mathbf{G}\mathbf{L} &= \mathbf{I} - \mathbf{G}\mathbf{S}_{\nu\nu}\mathbf{G}^{\mathrm{H}}\left[\mathbf{G}\mathbf{S}_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \mathbf{S}_{ww}\right]^{-1}\\ &= \mathbf{S}_{ww}\left[\mathbf{G}\mathbf{S}_{\nu\nu}\mathbf{G}^{\mathrm{H}} + \mathbf{S}_{ww}\right]^{-1}.\end{aligned}$$

Since $\mathbf{S}_{ww}$ is positive definite by assumption, there exists a square root $\mathbf{S}_{ww}^{1/2}$ that is positive definite and therefore

$$\mathbf{I} - \mathbf{G}\mathbf{L} = \mathbf{S}_{ww}^{1/2}\left[\mathbf{\Psi} + \mathbf{I}\right]^{-1}\mathbf{S}_{ww}^{-1/2},$$

with $\mathbf{\Psi} = \mathbf{S}_{ww}^{-1/2}\mathbf{G}\mathbf{S}_{\nu\nu}\mathbf{G}^{\mathrm{H}}\mathbf{S}_{ww}^{-1/2}$. As $\mathbf{G}$ does not exhibit transmission zeros, $\mathbf{\Psi}$ is positive definite and it follows that

$$\begin{aligned}\sup_{\omega}\bar{\rho}\left(\mathbf{I} - \mathbf{G}\mathbf{L}\right) &= \sup_{\omega}\bar{\rho}\left(\left(\mathbf{\Psi} + \mathbf{I}\right)^{-1}\right)\\ &= \sup_{\omega}\left(\underline{\rho}\left(\mathbf{\Psi} + \mathbf{I}\right)\right)^{-1} < 1\end{aligned}$$

where $\underline{\rho}$ denotes the smallest eigenvalue. Thus, the sub-optimal learning law satisfies the deterministic stability condition (15). Additionally, the iteration of the output error PSD (cf. (7))

$$\mathbf{S}_{\eta\eta}^{k+1} = (\mathbf{I}-\mathbf{G}\mathbf{L})\,\mathbf{S}_{\eta\eta}^{k}\,(\mathbf{I}-\mathbf{G}\mathbf{L})^{\mathrm{H}} + \mathbf{G}\mathbf{L}\mathbf{S}_{ww} + \mathbf{S}_{ww}\mathbf{L}^{\mathrm{H}}\mathbf{G}^{\mathrm{H}}$$

converges to a unique asymptotic output error PSD $\mathbf{S}_{\eta\eta}^{\infty}$ given by the discrete Lyapunov-type equation (19). □

Remark 3. *The limitation of transmission zeros is always violated at $\omega \to \infty$ for strictly proper systems. This is a necessary consequence of the optimality principle of the Wiener-filter, since for systems with low-pass characteristics it should be avoided to learn high frequencies. This is hardly a problem in practice as one is typically interested in a limited frequency range as considered in [23]. Alternatively, the system operator can be modified such that it includes a small feedthrough term as suggested in [6].*

The learning operator (18) is structurally similar to feedforward norm-optimal ILC. Contrary to the presented approach, norm-optimal schemes are typically set up in a finite-horizon framework. Applying a limit argument as used in [32], [11] for comparison, one obtains a frequency-domain representation of feedforward norm-optimal learning laws as

$$\mathbf{L}(s) = \left[\mathbf{R} + \mathbf{G}^{\mathrm{H}}(s)\mathbf{Q}\mathbf{G}(s)\right]^{-1}\mathbf{G}^{\mathrm{H}}(s)\mathbf{Q}, \tag{20}$$

where $\mathbf{Q}$ is the weighting matrix of the output error and $\mathbf{R}$ is the weighting matrix of the input difference $\mathbf{u}^{k+1}(t) - \mathbf{u}^{k}(t)$ (see, e.g., [9], [8]). For the special case $\mathbf{S}_{\nu\nu}(s) = \sigma_\nu\mathbf{I}$ and $\mathbf{S}_{ww}(s) = \sigma_w\mathbf{I}$, one can easily show that (18) is equivalent to (20) for $\mathbf{R} = \sigma_R\mathbf{I}$ and $\mathbf{Q} = \sigma_Q\mathbf{I}$ with $\sigma_\nu\sigma_R = \sigma_w\sigma_Q$[1]. In other words: if all components of the stochastic perturbations are uncorrelated, white, and of equal noise power and if no information about the input signal is available, Wiener-filter-based learning laws and norm-optimal learning laws are equivalent. Both implement a kind of (Tikhonov-) regularized inversion of the system operator $\mathbf{G}$. This idea is directly addressed in [6] by using a pseudo-inverse learning law that can be written as

$$\mathbf{L}(s) = \left[\alpha\mathbf{I} + \mathbf{G}^{\mathrm{H}}(s)\mathbf{G}(s)\right]^{-1}\mathbf{G}^{\mathrm{H}}(s). \tag{21}$$

As one can see from (21) and (20), norm-optimal schemes introduce regularization by penalizing variations of the input and thus reduce the effective speed of learning - which is somewhat artificial. In contrast, the Wiener-filter-based learning operator (18) is regularizing according to the expected PSD of the perturbations $\mathbf{S}_{ww}(s)$. This is exactly what one would expect: learning is avoided when the stochastic perturbations are strong compared to the expected signal components $\mathbf{G}(s)\mathbf{S}_{\nu\nu}(s)\mathbf{G}^{\mathrm{H}}(s)$.

From a practical point of view, the Wiener-filtering-based approach has two significant benefits: first, one is not bothered with tuning unphysical weighting matrices but rather guided by experimentally accessible quantities. Second, the spectral shaping due to the choice of $\mathbf{S}_{\nu\nu}(s)$ is often appreciated to easily restrict learning to certain frequency ranges or to systematically account for a varying fidelity of the process model $\mathbf{G}(s)$ along the whole frequency range.

B. An optimal iteration-varying learning law

Due to the learning process, the measured output error $\boldsymbol{\eta}^k(t)$ is increasingly dominated by stochastic disturbances. Therefore, a stochastically optimal learning law will reduce its learning behaviour with increasing iterations (see [4]). Such a behaviour is intrinsic to the general solution (16) due to the decreasing input error PSD $\mathbf{S}_{\nu\nu}^k(s)$. Since

$$\boldsymbol{\nu}^{k+1}(t) = \left(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{L}}^k\boldsymbol{\mathcal{G}}\right)\boldsymbol{\nu}^k(t) - \boldsymbol{\mathcal{L}}^k\mathbf{w}^k(t), \tag{22}$$

[1]Notice the interchanged dimensions of the matrices, i.e. $\mathbf{R}$, $\mathbf{S}_{\nu\nu}(s) \in \mathbb{R}^{l\times l}$ and $\mathbf{Q}$, $\mathbf{\Phi}(s) \in \mathbb{R}^{m\times m}$.

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cf. (5) and (6), one can calculate the evolution of the expected input error PSD in the form

$$\mathbf{S}_{\nu\nu}^{k+1}(s) = \left(\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)\right)\mathbf{S}_{\nu\nu}^{k}(s)\left(\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)\right)^{\mathrm{H}} + \mathbf{L}^k(s)\mathbf{S}_{ww}(s)(\mathbf{L}^k(s))^{\mathrm{H}}. \tag{23}$$

Thus, only an initial PSD $\mathbf{S}_{\nu\nu}^0(s)$ is required and the following PSDs can be updated according to the equation above. Using the proposed learning operator (16) under the assumption of iteration-independent stochastic properties of $\mathbf{w}^k$, i.e.

$$\mathbf{L}^k(s) = \mathbf{S}_{\nu\nu}^k(s)\mathbf{G}^{\mathrm{H}}(s)\left[\mathbf{G}(s)\mathbf{S}_{\nu\nu}^k(s)\mathbf{G}^{\mathrm{H}}(s) + \mathbf{S}_{ww}(s)\right]^{-1}, \tag{24a}$$

and plugging it into the forward iteration (23) results in

$$\mathbf{S}_{\nu\nu}^{k+1}(s) = \left(\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)\right)\mathbf{S}_{\nu\nu}^{k}(s). \tag{24b}$$

Due to the dependence of the learning operator on $\mathbf{S}_{\nu\nu}^k$, (24b) is a nonlinear iteration. In the following theorem, we show that the learning operator (24a) with (24b) is stable and additionally that $\mathbf{u}^k(t) \to \mathbf{u}^d(t)$ in a stochastical sense, i.e. that $\lim_{k\to\infty} \mathbf{S}_{\nu\nu}^k = \mathbf{0}$.

Theorem 2. *If $\mathbf{S}_{ww}(s)$ and the initial input error PSD $\mathbf{S}_{\nu\nu}^0(s)$ are positive definite and the system's transfer matrix $\mathbf{G}(s)$ does not exhibit transmission zeros on the imaginary axis (see Theorem 1), the learning operator* (24a) *together with* (24b) *yields a stable learning law that ensures convergence to the optimal error PSDs*

$$\mathbf{S}_{\nu\nu}^{\infty}(s) = \mathbf{0}, \quad \mathbf{S}_{\eta\eta}^{\infty}(s) = \mathbf{S}_{ww}(s). \tag{25}$$

Proof. The resulting nonlinear iteration (24) is structurally very similar to [4] for a stochastic discrete-time D-type law and the following proof proceeds along its lines although the involved quantities have considerably different meaning. First, one can show by induction that $\mathbf{S}_{\nu\nu}^k(s)$ is positive definite and that

$$\sup_{\omega} \bar{\rho}\left(\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)\right) < 1 \tag{26}$$

for all $k \in \mathbb{N}$. Assume that $\mathbf{S}_{\nu\nu}^k(s)$ is positive definite. Due to the structure of (23), $\mathbf{S}_{\nu\nu}^{k+1}(s)$ is necessarily positive semi-definite. To analyze the eigenvalues of $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$, we use the identity $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s) = \left[\mathbf{I} + \mathbf{S}_{\nu\nu}^k(s)\mathbf{G}^{\mathrm{H}}(s)\left(\mathbf{S}_{ww}(s)\right)^{-1}\mathbf{G}(s)\right]^{-1}$. The matrix $\mathbf{G}^{\mathrm{H}}(s)\left(\mathbf{S}_{ww}(s)\right)^{-1}\mathbf{G}(s)$ is positive definite due to the assumptions made. Although the product of positive definite matrices is not necessarily positive definite again, it is still true that the eigenvalues of the product are greater than zero, i.e. $\underline{\rho}\left(\mathbf{S}_{\nu\nu}^k(s)\mathbf{G}^{\mathrm{H}}(s)(\mathbf{S}_{ww}(s))^{-1}\mathbf{G}(s)\right) > 0$. Thus $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$ has full rank and (26) hold true by the same line of reasoning as in Theorem 1. Moreover, one can show that the eigenvectors of $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$ remain invariant for all k when using the forward iteration of the input error PSD according to (24b), i.e., each eigenvector of $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$ is also an eigenvector of $\mathbf{I} - \mathbf{L}^m(s)\mathbf{G}(s)$ for $m \neq k$. The full-rank property of $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$ implies that $\mathbf{S}_{\nu\nu}^{k+1}(s)$ is indeed positive definite again which - since $\mathbf{S}_{\nu\nu}^0(s)$ is positive definite - finishes the induction.

While all eigenvalues of $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$ are necessarily less than one, they approach one as $\mathbf{L}^k(s)$ tends to zero for $k \to \infty$. Thus, the limit of $\mathbf{S}_{\nu\nu}^k(s)$ has to be analyzed more carefully. Since all eigenvalues are inside the unit circle and their corresponding eigenvectors remain unchanged, there exists a norm irrespective of k such that $\left\|\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)\right\| < 1$ and we thus know that $\left\|\mathbf{S}_{\nu\nu}^{k+1}(s)\right\| < \left\|\mathbf{S}_{\nu\nu}^{k}(s)\right\|$ for all k using the *same* norm.

Together with the lower bound $\|\mathbf{S}_{\nu\nu}^k(s)\| > 0$ it is clear that there exists a limit $\|\mathbf{S}_{\nu\nu}^{\infty}(s)\|$, and the following shows that this limit equals zero by contradiction. Assuming that $\mathbf{S}_{\nu\nu}^{\infty}(s) \neq 0$ implies that all eigenvalues of $\mathbf{I} - \mathbf{L}^k(s)\mathbf{G}(s)$ are inside the unit circle in the limit. In this case

$$\lim_{k\to\infty} \sup_{\omega} \frac{\|\mathbf{S}_{\nu\nu}^{k+1}(s)\|}{\|\mathbf{S}_{\nu\nu}^{k}(s)\|} < 1 \tag{27}$$

and thus $\|\mathbf{S}_{\nu\nu}^{\infty}(s)\| = 0$, which contradicts the assumption. Thus, it follows that $\mathbf{S}_{\nu\nu}^k(s)$ converges indeed to zero for $k \to \infty$ and that the input error converges to zero in mean. As a consequence, the learning kernel $\mathbf{L}^k(s)$ also converges to zero. In view of (11), the output error PSD then converges to the minimal asymptotic error given in (25). □

V. Numerical Implementation of the Learning Law

Solutions of the non-causal Wiener-Hopf equation (17) can be obtained with standard methods for integral equations like straightforward discretization or using a finite number of basis functions. In the LTI case, however, both the learning operator and the plant have comparatively simple descriptions in the Laplace domain compared to the time domain. In the following, we want to use this fact to find numerical solutions of the learning kernel $\mathbf{L}(t)$.

Based on the assumption that the system (2) is L_2-stable, there exists some $\varepsilon > 0$ such that the region of convergence (ROC) of $\mathbf{G}(s)$ includes all s with $\mathrm{Re}\{s\} > -\varepsilon$ (cf. [33]). Conversely, for the adjoint system $\mathbf{G}^{\mathrm{H}}(s)$ it holds that the stable ROC (cf. backward integration) includes all s with $\mathrm{Re}\{s\} < \varepsilon$. Hence, the imaginary axis $j\omega$ is included in the ROC associated with the learning operator $\mathbf{L}(s)$. As a result, the learning kernel $\mathbf{L}(t)$ can be obtained by the inverse Fourier transform

$$\mathbf{L}(t) = \mathcal{F}^{-1}\left\{\mathbf{L}(j\omega)\right\}, \tag{28}$$

which is equivalent to forward integration of the stable part and backward integration of the unstable part (see [34]).

The continuous Fourier transform is often approximately computed using the discrete Fourier transform (DFT) on a discrete grid. Since a discretization of t or ω determines a discretization of the other quantity, a spectrally broad operator $\mathbf{L}(j\omega)$ usually requires a high number of sampling points although $\mathbf{L}(t)$ is (almost) zero at most sampling points.

A simple solution to this problem building on a so-called "fractional Fourier transform" is given in [35]. Assuming that $\mathbf{L}(j\omega)$ is zero outside the finite interval $[-\omega_S/2, \omega_S/2]$, the relation (28) can be approximately solved for $t \in [-t_S/2, t_S/2]$ on the discrete grid $t_n = (n - N/2)\beta$ and $\omega_m = (m - N/2)\gamma$ with $0 \leq m, n < N$ by

$$\mathbf{L}(t_n) = \frac{1}{2\pi}\int_{-\omega_S/2}^{\omega_S/2} \mathbf{L}(j\omega)\mathrm{e}^{j\omega t_n}\mathrm{d}\omega \tag{29}$$

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$$\approx \frac{\gamma}{2\pi} \mathrm{e}^{-j\pi(n-N/2)N\delta} \sum_{m=0}^{N-1} \mathbf{L}(j\omega_m) \mathrm{e}^{-j2\pi\delta m(n-N/2)}$$
$$= \frac{\gamma}{2\pi} \mathrm{e}^{-j\pi(n-N/2)N\delta} G_n \left\{ \mathbf{L}(j\omega_m) \mathrm{e}^{-j\pi m N\delta}, -\delta \right\}$$

where $\beta = t_S/N$, $\gamma = \omega_S/N$ and $\delta = \beta\gamma/(2\pi)$. The operator G_n denotes the n-th element of the "fractional Fourier transform" evaluated by means of the DFT as given in [35, eq. (14)-(18)][2]. Using fast Fourier transform (FFT) algorithms, this procedure can be computed very efficiently.

This approach is particularly interesting for distributed-parameter systems, where inversion- or model-based ILC schemes have been discarded in many contributions (e.g., [22], [19]) due to their potential complexity. Using (29), one can compute the desired learning kernel $\mathbf{L}(t)$ with arbitrary precision regardless of the distributed nature. This is due to the chosen input-output description of the system (2) and (3) that is independent of the dimension of the state-space, whether it is finite- or infinite-dimensional. Thus, one avoids the computational tasks that typically arise in established state-space approaches [36], [10] such as the solution of (operator) Riccati equations and state-prediction equations that become quite delicate and computationally very expensive for infinite-dimensional systems. A variety of frequency-domain-based ILC approaches requires the implementation of non-causal learning operators (as (16) is). To implement such laws, the causal and anti-causal part of the learning operator are usually separated using spectral factorization techniques. Both parts are then solved by forward- and backward integration of the corresponding state-space representation due to non-causal stable inversion methods [37], [12], [6], which is quite demanding for DPSs (cf. [38]). Combining the spectral equivalence to stable inversion [34] with the numerical solution (29), one is effectively avoiding the spectral factorization by directly computing the learning kernel $\mathbf{L}(t)$ of the learning operator (4b) rather than finding a representation of the learning operator in form of two systems of (partial) differential equations.

VI. Simulation Examples

A. A sub-optimal ILC for the damped wave equation

First, we want to illustrate the proposed sub-optimal method on a boundary-controlled DPS defined on a one-dimensional spatial domain $z \in [0, 1]$ given by the hyperbolic PDE

$$\frac{\partial^2 x}{\partial t^2}(z,t) + \gamma \frac{\partial x}{\partial t}(z,t) = \frac{\partial^2}{\partial z^2} x(z,t), \tag{30}$$

with the initial conditions $x(z,0) = \frac{\partial x}{\partial t}(z,0) = 0$, the boundary conditions $x(0,t) = u(t)$, $\frac{\partial x}{\partial z}(1,t) = 0$, and the measured output $y(t) = x(1,t) + n(t)$ corrupted by band-limited measurement noise

$$S_{nn}(s) = \begin{cases} \sigma_n^2 \frac{\pi}{\omega_n} & |\mathrm{Im}\{s\}| \leq \omega_n \\ 0 & \text{else,} \end{cases} \tag{31}$$

with noise power $\sigma_n^2 = 1 \times 10^{-3}$ and bandwidth $\omega_n = 2.15 \times 10^3$. Applying the bilateral Laplace transform to (30)

[2]The term "fractional Fourier transform" in [35] is denoting a special case of the Z-transform and has fallen out of use since.

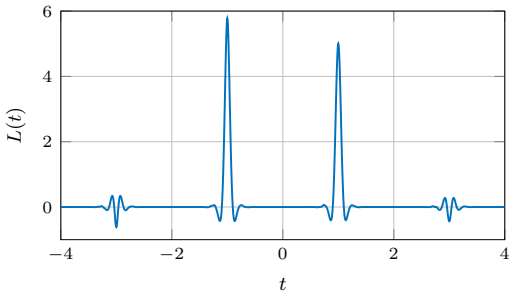

Fig. 1. The (non-causal) learning kernel $L(t)$ for the damped wave equation (32) calculated using (29) with $N = 512$, $t_S = 8$, and $\omega_S = 100$.

and solving the resulting spatial ODE using the boundary conditions above yields the well-known transfer function

$$G(s) = \cosh\left(\sqrt{s(s+\gamma)}\right)^{-1}, \tag{32}$$

i.e. the input-output behaviour is governed by an infinite number of poles. Following the sub-optimal approach presented in Section IV-A, we have to choose the input error PSD $S_{\nu\nu}(s)$ according to a priori knowledge. Assuming that the input corrections are mainly below some frequency ω_{co}, one can shape the learning behavior spectrally by using $S_{\nu\nu}(s) = |1 + s/\omega_{co}|^{-6}$. Thus, a numerical solution of the learning kernel $L(t)$ considering the full infinite-dimensional dynamics of (30), i.e., using a late-lumping approach, is given by (29) with (18) and (32) as shown in Fig. 1 for $\gamma = 0.1$ and $\omega_{co} = 10$. Using the calculated learning kernel with (4b) to track a desired output trajectory $y^d(t) = \exp(-t^2)\sin(7t)$ starting from $u^0(t) = 0$, the output error converges close to the lower bound given by the measurement noise after the first iteration as shown in Fig. 2. Following a conventional early-lumping approach, one could approximate the infinite-dimensional dynamics of (30) by some finite-dimensional dynamics of order N_{FD} that can be treated with standard norm-optimal or stable-inversion-based learning laws. Using a pseudo-inverse-based learning law according to (21) with $N_{FD} = 30$ and $\alpha = \sigma_n^2 \pi/\omega_n$, the learning iteration diverges quickly due to model errors at higher frequencies. The standard approach to increase the robustness to model errors utilizes additional Q-filters (see, e.g., [8]) which yields stable learning iterations at the cost of significantly increased stationary errors as shown in Fig. 2 for a forth-order zero-phase low-pass filter with cutoff frequency $\omega_Q = 50$. Applying the Wiener-filtering-based ILC approach using the approximated finite-dimensional dynamics for comparison nicely illustrates its spectrally selective learning property. However, the presence of significant model errors due to the early-lumping approach requires quite drastic choices of $S_{\nu\nu}(s)$ to achieve stable iterations without Q-filtering.

Remark 4. *Using such a regularized inversion methods might be an interesting option also for feedforward applications in case of linear distributed-parameter systems, since flatness-based techniques are significantly more involved using so-called Weierstraß canonical products, may ultimately require resummation techniques dealing with diverging series, and*

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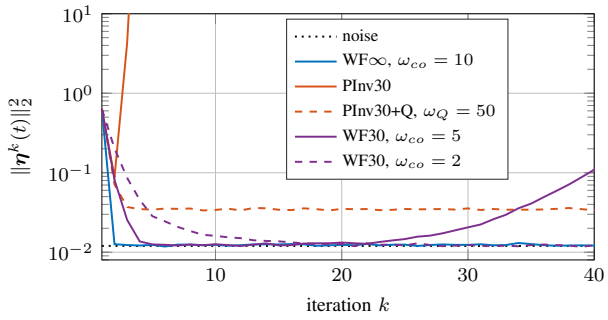

Fig. 2. Comparison of the convergence behavior of Wiener-filtering (WF) and pseude-inverse (PInv) learning laws based on finite ($N_{FD} = 30$) and infinite-dimensional description during the design process. The cut-off frequencies of the chosen input error PSD ω_{co} or the additional Q-filter ω_Q is given for reference.

have known limitations for systems with clustering poles such as Euler-Bernoulli beams with Kelvin-Voight damping [14].

B. Comparing the sub-optimal with the optimal ILC scheme

To illustrate the differences between the sub-optimal (18) and optimal learning (24) approaches, we consider a finite-dimensional MIMO system of the form (2) with zeros in the right half-plane and disturbances $\mathbf{S}_{ww} = \mathbf{G}_v(s)\mathrm{S}_{vv}(s)\mathbf{G}_v^{\mathrm{H}}(s) + \mathbf{S}_{nn}$ using

$$\mathbf{G}(s) = \begin{bmatrix} \frac{s}{s^2+s+1} & \frac{1}{s^2+s+1} \\ \frac{s+10}{s^2+2s+4} & \frac{1}{s+1} \end{bmatrix}, \mathbf{G}_v(s) = \begin{bmatrix} \frac{s}{s^2+s+1} \\ \frac{3s^2+15/2s+6}{s^3+3s^2+6s+4} \end{bmatrix}.$$

The measurement noise n is again assumed to be band-limited white noise according to (31) with $\sigma_n^2 = 1/100$ and $\omega_n = 321.7$ is chosen to be at the Nyquist frequency of the simulation. Similarly, the process disturbance v is given by

$$\mathbf{S}_{vv}(s) = \begin{cases} 1/10 & |\mathrm{Im}\{s\}| \in [\pi, 2\pi] \\ 0 & \text{else.} \end{cases}$$

To specify the learning operators, we have to choose the initial input error PSD $\mathbf{S}_{\nu\nu}^0(s)$ for the optimal case (24) and the fixed input error PSD $\mathbf{S}_{\nu\nu}(s)$ for the sub-optimal case (18), respectively. Because these PSDs are usually unknown, they are effectively tuning parameters that have to be chosen as part of the design process. Since we expect the input to consist of predominantly lower frequencies, we choose both to be

$$\mathbf{S}_{\nu\nu}^0(s) = \mathbf{S}_{\nu\nu}(s) = \frac{0.01}{|1+s|^2}\mathbf{I}.$$

Finally, the two Wiener-filter-based approaches are compared to existing norm-optimal approaches. By choosing $\mathbf{R} = \gamma\mathbf{I}$ and $\mathbf{Q} = \beta\mathbf{I}$, one can see from the analysis in Section IV-A that the resulting norm-optimal learning law is equivalent to the pseudo-inverse solution [6] given by the frequency-domain representation (21) with a single remaining tuning parameter $\alpha = \gamma/\beta$.

In Fig. 3, the evolution of the L^2-norms of the input and output errors are shown for all three learning operators. As expected, the optimal learning operator (24) settles around the expected steady-state level given by the optimal asymptotic

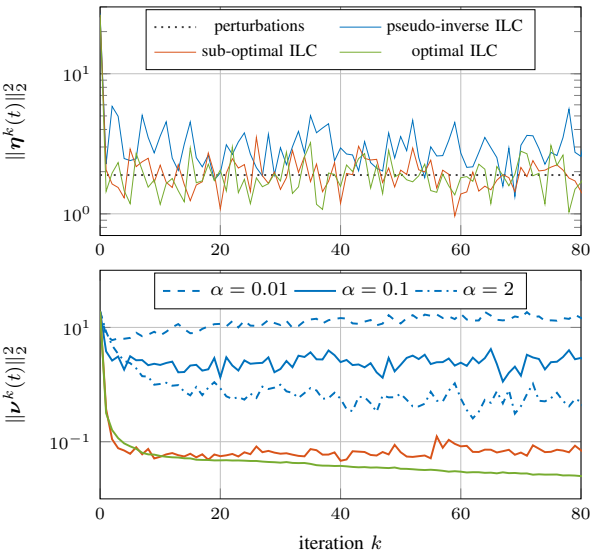

Fig. 3. Evolution of the input and output error's L^2-norms for three different learning operators: pseudo-inverse (blue), sub-optimal (red) and optimal (green). For clarity, the evolution of the output error is only depicted for the case $\alpha = 1/10$.

output error PSD $\mathbf{S}_{\eta\eta}^{\infty}$ in (25) indicated by the dashed line in Fig. 3. Since the asymptotic error PSD of the sub-optimal ILC (19) is quite close to the optimal value, both Wiener-filter-based learning operators perform roughly equal while the pseudo-inverse operator (using $\alpha = 1/10$) is not able to achieve a similarly low output error on average.

The reason for this can be seen in the bottom plot of Fig. 3: lacking stochastic information, the pseudo-inverse operator is hardly able to separate stochastic perturbations from signal information and thus settles at a significantly higher steady-state level of the input error compared to the Wiener-filter-based operators that avoid learning stochastic perturbations (cf. Fig. 4). Lowering the regularization parameter (i.e. increasing $\mathbf{Q}$ w.r.t. $\mathbf{R}$) to $\alpha = 1/100$ deteriorates the performance, while increasing it to $\alpha = 2$ reduces the asymptotic input error at the cost of slower convergence. Comparing the sub-optimal learning operator with the optimal learning operator shows that the input error of the sub-optimal approach reaches a steady-state level, while the optimal learning operator asymptotically approaches zero as predicted by the theory. However, the convergence rate is usually quite slow. Depending on the exact application scenario, one may choose to avoid the extra computational effort of implementing an iteration-varying learning law as its impact on the output error is quite limited, at least in this example.

VII. Conclusions and Outlook

In this paper, a general linear learning law for continous-time linear systems was developed that is stochastically optimal in a MMSE sense. The proposed learning operator is a generalization of existing MMSE-optimal approaches and represents a systematic dual to existing feedforward norm-optimal approaches by using a kind of Wiener-filter to optimally estimate the input correction of each iteration based on

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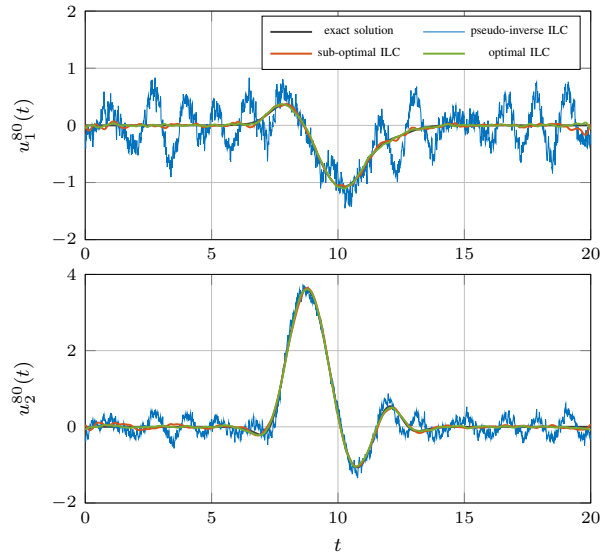

Fig. 4. The learned input signals after 80 iterations $\mathbf{u}^{80}(t)$ using a pseudo-inverse (blue), a sub-optimal (red) or an optimal (green) learning operator compared to the exact inverse solution (black) for the desired output $\mathbf{y}^d(t)$.

the observed measurement. Thus, the proposed learning law combines the superior convergence speed of inversion-based or norm-optimal approaches with the stochastic optimality of various stochastic ILC schemes. Using the numerically efficient FFT-based algorithm presented in Section V to determine the learning operator given by the non-causal Wiener-Hopf equation, one is able to avoid both spectral factorization techniques and solutions to Riccati equations. Therefore, the general approach is applicable to DPSs in a late-lumping sense which have not been treated directly with model-based ILC methods so far.

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