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@Article{Prinz15,
  Title = {Mathematical modelling of a diesel common-rail system},
  Author = {Prinz, Katharina and Kemmetm\"uller, Wolfgang and Kugi, Andreas},
  Journal = {Mathematical and Computer Modelling of Dynamical Systems},
  Pages = {311--335},
  Volume = {21},
  Year = {2015},
  Number = {4},
  Doi = {10.1080/13873954.2014.968590}
}
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To appear in Mathematical and Computer Modelling of Dynamical Systems
Vol. 21, No. 4, June 2015, 311–335

Mathematical Modeling of a Diesel Common-Rail System

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(*Received 16 June 2014; accepted 19 September 2014*)

In diesel common-rail systems, the exact knowledge of the injection pressure is important to accurately control the injected diesel mass and thus the combustion process. This paper focuses on the mathematical modeling of the hydraulic and mechanical components of a common-rail system in order to describe the dynamics of the diesel rail pressure. Based on this model, an average model is derived to reduce the model complexity and to allow for a fast calculation of the mass flow into the rail for different crank shaft revolution speeds and openings of the fuel metering unit. The main purpose of this average model is to serve as a basis for a model based (nonlinear) controller design. The stationary accuracy of the models is validated by means of measurement data.

Keywords: diesel engine; common-rail injection; control oriented modeling; physics-based modeling

AMS Subect Classification: 37N10; 37N35; 93C10; 93C95

1. Introduction

Common-rail injection technology is typically used in modern diesel engines. The major advantage compared to prior injection systems as e.g. in-line injection pumps is that the process of injection and high pressure generation are decoupled. This enables injections with arbitrary timing and quantity, even so-called multiple injections, which can significantly increase the efficiency and reduce exhaust emissions of the engine [1].

Fig. 1 depicts a schematic diagram of the considered common-rail injection system. The central component of the system is a rail volume filled with highly pressurized diesel, which delivers diesel to the injectors. By opening and closing the injectors, the desired amount of diesel can be injected into the combustion chambers of the engine. The high pressure in the rail is generated by a radial piston pump, which is actuated by one or more eccentrics at the camshaft of the engine. The amount of diesel flow into the radial piston pump can be controlled by an electromagnetically actuated variable displacement valve (fuel metering unit, FMU). The fuel metering unit is supplied by a constant pressure, which is generated by a gear pump and an overflow valve. Due to leakages of the FMU it is necessary to install a zero delivery throttle, which enables zero volume flows of the radial piston pump. In many configurations, a pressure control valve (PCV) is installed in the rail to transport diesel from the rail to the tank and thus to actively reduce the

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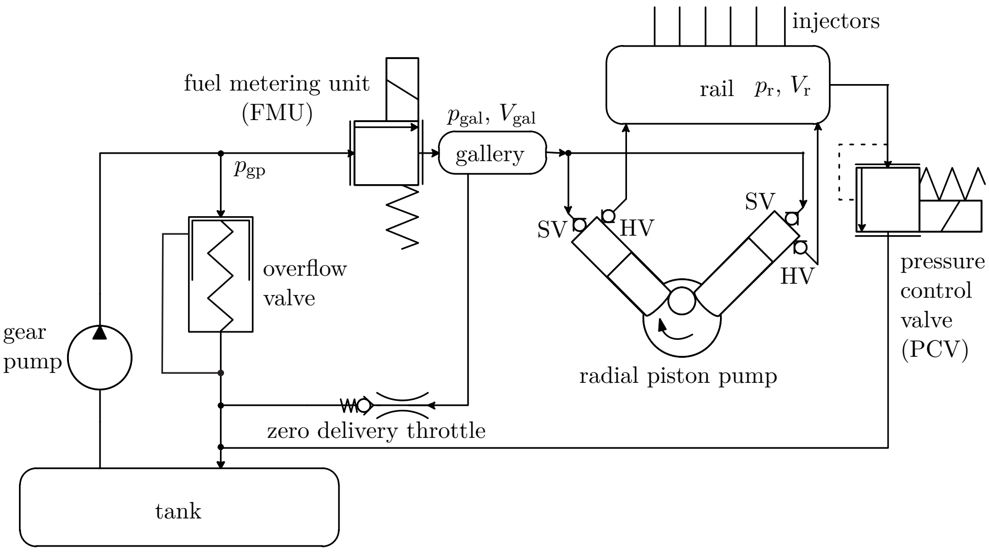

Figure 1. Schematic diagram of a common-rail system.

rail pressure.

Accurately controlling the rail pressure is an important prerequisite to achieve the desired quantity of fuel injected. Mathematical models provide the basis for the analysis of the dynamical behavior and the design of control strategies. Stationary characteristic maps are frequently used to describe the system behavior. These models must be calibrated by measurements and do not account for the dynamical behavior. These models are thus not directly applicable for the controller design. First-principle models are more suitable for this purpose.

The authors of [2] developed a model of a diesel injection system, where the focus was set on the injection process itself. The models developed in [3] concentrate on a detailed description of the rail and the injection process as well. In [4, 5], models of common-rail systems are presented which were primarily intended for the control design. These models are characterized by a low dimension and are based on a number of simplifications, which neglect some important effects, e.g. of the dynamics of the radial piston pump. Moreover, [2, 4, 5] do not consider a fuel metering unit such that the models are not suitable for the considered common-rail system. The authors of [6, 7] include the FMU in their model but do not consider a physics based model of the radial piston pump. A more detailed model is given in [8] where a hybrid model for the common-rail system is developed. Although an approach for the description of cavitation in the radial piston pump is presented, a measured curve is utilized to describe the flow rate of the radial piston pump as a function of the current of the FMU.

Therefore, the aim of this paper is to develop a mathematical model of the common-rail system which (i) allows for a fast and yet accurate simulation of the dynamical behavior and (ii) which can be easily parameterized by means of construction parameters of the system. Section 2 summarizes the mathematical equations of this model and gives some simulation results. A comparison with measurements shows that the model is capable of accurately reproducing the real system behavior. Due to its complexity this model is, however, not directly applicable to design a controller. Thus, a reduced model of the system is developed in Section 3, which is based on averaging the system variables over one cycle of the piston motion of the radial piston pump. Simulation results show that the reduced model still exhibits a high accuracy and captures the essential dynamical behavior

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of the system.

2. Detailed Model

In this section, a detailed mathematical model of the common-rail system is derived. This model is based on the following assumptions and simplifications:

(i) The pressure at the high pressure side of the FMU is kept almost constant by means of the overflow valve. Thus, models of the gear pump and the overflow valve are not required. Instead, a constant pressure p_{gp} is presumed.
(ii) The effect of zero diesel flow for closed FMU, which is achieved by the zero delivery throttle in the real system, is approximated by an idealized FMU without leakage flow.
(iii) In this paper, a common-rail system without pressure control valve (PCV) is considered. This is a common configuration in cost-sensitive applications. The control of the rail pressure is thus only achieved by means of the FMU.
(iv) The exact time evolution of the injected diesel flow into the cylinders of the internal combustion engine strongly depends on the actual load of the engine. The model presented in this paper is primary developed for the design of a controller for the rail pressure. As will be seen later, it is not possible to influence the fast pressure fluctuations in the rail caused by the injection process. Therefore, a simplified model will be given, which, however, captures the main effects of the injection process.

2.1 *Isentropic Fluid Model*

Diesel pressure in a common-rail system varies from lower than atmospheric pressure (e. g. in the cylinders of the radial piston pump) to very high pressures in the rail. At pressures lower than vapor pressure the fluid starts to vaporize. Moreover, the compressibility of diesel considerably decreases for high pressures. As these effects cause significant changes in the fluid properties, a model which describes the fluid for both cavitation and high pressure cases is strived for, see also [9, 10].

The subsequent model is based on an isentropic fluid, where the mass density ϱ and the bulk modulus β (both depending on the pressure p) meet the relation

$$\frac{\partial \varrho(p)}{\partial p} = \frac{\varrho(p)}{\beta(p)} \,. \tag{1}$$

To characterize diesel depending on its pressure p, three cases are distinguished:

1) Diesel at pressures higher than the upper vapor pressure $p_{\mathrm{vap,u}}$, $p \geq p_{\mathrm{vap,u}}$, is liquid.
2) In the pressure range $p_{\mathrm{vap,l}} < p < p_{\mathrm{vap,u}}$ diesel starts to vaporize, i. e. there is some liquid and some vapor part.
3) At pressures lower than the lower vapor pressure $p_{\mathrm{vap,l}}$, $p \leq p_{\mathrm{vap,l}}$, all diesel mass is vaporized.

The next sections deal with these three cases in detail and provide equations for $\beta(p)$ and $\varrho(p)$.

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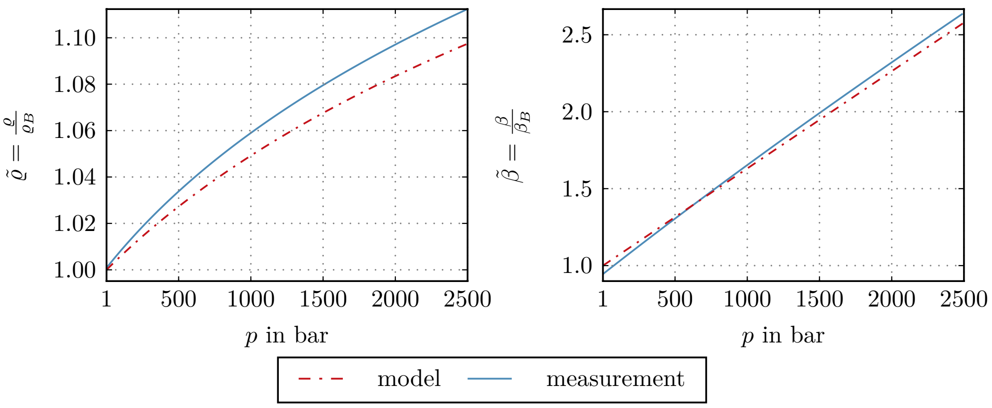

Figure 2. Mass density ϱ and bulk modulus β.

2.1.1 *Case 1: $p \geq p_{\mathrm{vap,u}}$*

When describing fluids at moderate pressures, a constant bulk modulus β_0 can be used, see, e.g., [11]. However, for diesel pressures in a common-rail system, which can increase beyond 2000 bar, an increasing bulk modulus is observed in measurements, see Fig. 2.[1] The bulk modulus β is described as

$$\beta(p) = \beta_0 + b_1 (p - p_0) \ , \tag{2}$$

with the bulk modulus β_0 at reference pressure p_0 and the gradient b_1. With (1) the corresponding mass density $\varrho(p)$ is given by

$$\varrho(p) = \varrho_{\mathrm{f0}} \left(\frac{\beta(p)}{\beta_0} \right)^{\frac{1}{b_1}} \ , \tag{3}$$

where ϱ_{f0} is the mass density at reference pressure $p = p_0$.

2.1.2 *Case 2: $p_{\mathrm{vap,l}} < p < p_{\mathrm{vap,u}}$*

In this case the fluid is considered to be partly liquid and partly vaporized. Given the overall mass m, let us consider that a mass fraction $m_{\mathrm{vap}} = \Phi_{\mathrm{vap}} m$ is vaporized, while the mass fraction $m_{\mathrm{f}} = (1 - \Phi_{\mathrm{vap}})m$ is still liquid. It is assumed that all fluid is liquid at $p = p_{\mathrm{vap,u}}$, i. e. $\Phi_{\mathrm{vap}} = 0$, and all fluid is vaporized at $p = p_{\mathrm{vap,l}}$, i.e. $\Phi_{\mathrm{vap}} = 1$. In general, the vaporized fraction Φ_{vap} is a nonlinear function of the pressure p. In this work, it is supposed that Φ_{vap} increases linearly with decreasing pressure, see, e.g., [9, 11]

$$\Phi_{\mathrm{vap}} = 1 - \frac{p - p_{\mathrm{vap,l}}}{p_{\mathrm{vap,u}} - p_{\mathrm{vap,l}}} \ . \tag{4}$$

[1]Quantities labeled with tilde refer to normalized quantities. Here, the mass density ϱ and the bulk modulus β are normalized to their values ϱ_B and β_B at ambient pressure $p = 1$ bar. All other quantities $(\cdot)$ are normalized to their respective maximum values $(\cdot)_B$.

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The bulk modulus β_f and mass density ϱ_f of the liquid fraction are given by (2) and (3)

$$\beta_\mathrm{f} = \beta_0 + b_1 (p - p_0) \tag{5a}$$

$$\varrho_\mathrm{f} = \varrho_\mathrm{f0} \left(\frac{\beta_\mathrm{f}}{\beta_0}\right)^{\frac{1}{b_1}} , \tag{5b}$$

such that the volume V_f of the liquid fraction is given by

$$V_\mathrm{f} = \frac{m_\mathrm{f}}{\varrho_\mathrm{f}} = \frac{(1-\Phi_\mathrm{vap})m}{\varrho_\mathrm{f}} = \frac{(1-\Phi_\mathrm{vap})m}{\varrho_\mathrm{f0}} \left(\frac{\beta_\mathrm{f}}{\beta_0}\right)^{-\frac{1}{b_1}} . \tag{6}$$

The behavior of the vaporized diesel is described by an isentropic process of ideal gas with the isentropic exponent $\kappa > 1$

$$p \left(\frac{V_\mathrm{vap}}{\Phi_\mathrm{vap} m}\right)^{\kappa} = p_\mathrm{vap,u} \left(\frac{V_\mathrm{vap,u}}{\Phi_\mathrm{vap,u} m}\right)^{\kappa} = p_\mathrm{vap,u} \left(\frac{1}{\varrho_\mathrm{vap,u}}\right)^{\kappa} , \tag{7}$$

where $\varrho_\mathrm{vap,u}$ is the vapor density and $V_\mathrm{vap,u}$ is the volume at $p = p_\mathrm{vap,u}$. Hence, the volume filled with vaporized diesel V_vap is given in the form

$$V_\mathrm{vap} = \left(\frac{p_\mathrm{vap,u}}{p}\right)^{\frac{1}{\kappa}} \frac{\Phi_\mathrm{vap} m}{\varrho_\mathrm{vap,u}} . \tag{8}$$

The density ϱ of the liquid-vapor-mixture in this case reads as

$$\varrho(p) = \frac{m}{V_\mathrm{f} + V_\mathrm{vap}} = \varrho_\mathrm{f0} \varrho_\mathrm{vap,u} \frac{p_\mathrm{vap,u} - p_\mathrm{vap,l}}{\varrho_\mathrm{vap,u} \left(\frac{\beta_\mathrm{f}}{\beta_0}\right)^{-\frac{1}{b_1}} (p - p_\mathrm{vap,l}) + \varrho_\mathrm{f0} \left(\frac{p_\mathrm{vap,u}}{p}\right)^{\frac{1}{\kappa}} (p_\mathrm{vap,u} - p)} . \tag{9}$$

The bulk modulus β is derived using (1)

$$\beta(p) = \frac{V_\mathrm{vap} + V_\mathrm{f}}{V_\mathrm{vap}\left(\frac{1}{p\kappa} + \frac{1}{p_\mathrm{vap,u} - p}\right) + V_\mathrm{f}\left(\frac{1}{\beta_\mathrm{f}} - \frac{1}{p - p_\mathrm{vap,l}}\right)} . \tag{10}$$

2.1.3 Case 3: $p \leq p_\mathrm{vap,l}$

For very low pressures below $p_\mathrm{vap,l}$, the fluid is considered to be completely vaporized. This case is obtained by means of (7) and (3) resulting in the following simplified equations for the mass density ϱ and the bulk modulus β

$$\varrho(p) = \varrho_\mathrm{vap,u} \left(\frac{p}{p_\mathrm{vap,u}}\right)^{\frac{1}{\kappa}} \tag{11a}$$

$$\beta(p) = \kappa p . \tag{11b}$$

Figure 3 shows the mass density ϱ and the bulk modulus β for low pressures. Note that the logarithmic scaling of the β-axis has been used to depict the rapid rise at $p = p_\mathrm{vap,u}$. Moreover, the discontinuity of β at $p = p_\mathrm{vap,l}$ is due to the linear approximation of Φ_vap in (4). Using a continuously differentiable approximation

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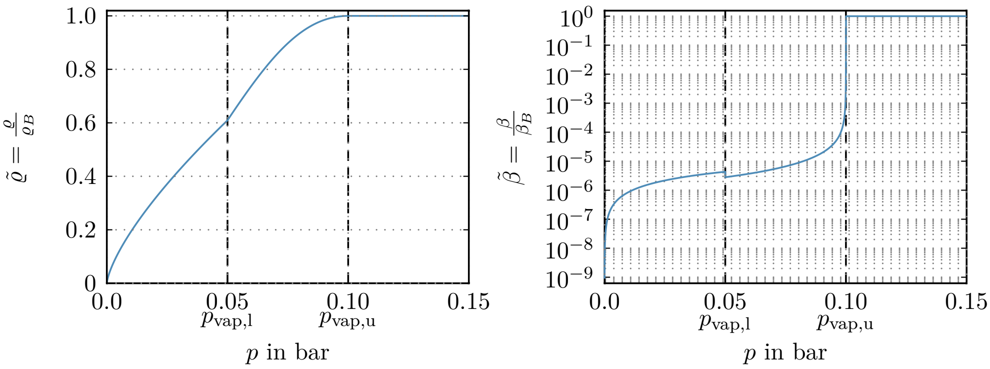

Figure 3. Mass density ϱ and bulk modulus β at low pressures p.

would eliminate this discontinuity. However, since continuity of β is not necessary in the following derivations, the linear approximation (4) is feasible.

2.2 *Fuel Metering Unit*

The fuel metering unit (FMU) is used to control the diesel flow into the radial piston pump. Figure 4 shows the hydraulic part of the FMU, given by an inlet orifice, the variable cross section and an outlet orifice connected in series. The turbulent flow through the orifices can be described by

$$\dot{m}_{\text{in}} = \alpha_{\text{in}} A_{\text{in}} \sqrt{2\varrho_{\text{in}}} \sqrt{\Delta p_{\text{in}}} \tag{12a}$$

$$\dot{m}_{\text{v}} = \alpha_{\text{v}} A_{\text{v}}(x_{\text{fmu}}) \sqrt{2\varrho_{\text{v}}} \sqrt{\Delta p_{\text{v}}} \tag{12b}$$

$$\dot{m}_{\text{out}} = \alpha_{\text{out}} A_{\text{out}} \sqrt{2\varrho_{\text{out}}} \sqrt{\Delta p_{\text{out}}} \ , \tag{12c}$$

where α_k, $k \in \{\text{in}, \text{v}, \text{out}\}$, are the discharge coefficients, A_k the corresponding opening areas, ϱ_k the mass densities of the diesel and Δp_k the pressure drops over the orifices.

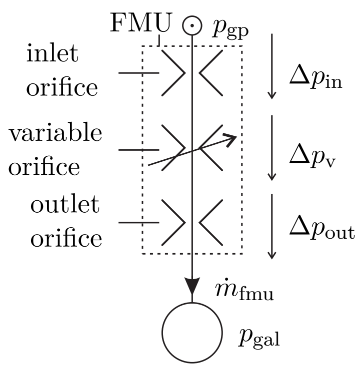

Figure 4. Hydraulic setup of the FMU.

Balance of mass gives $\dot{m}_{\text{in}} = \dot{m}_{\text{v}} = \dot{m}_{\text{out}} = \dot{m}_{\text{fmu}}$. It is further assumed that $\varrho_{\text{in}} = \varrho_{\text{v}} = \varrho_{\text{out}} = \varrho_{\text{fmu}}$. This is reasonable due to the small changes in pressure. Then the mass flow through the FMU reads as

$$\dot{m}_{\text{fmu}} = \alpha_{\text{fmu}} A_{\text{fmu}}(x_{\text{fmu}}) \sqrt{2\varrho_{\text{fmu}}} \sqrt{\Delta p_{\text{fmu}}} \ , \tag{13a}$$

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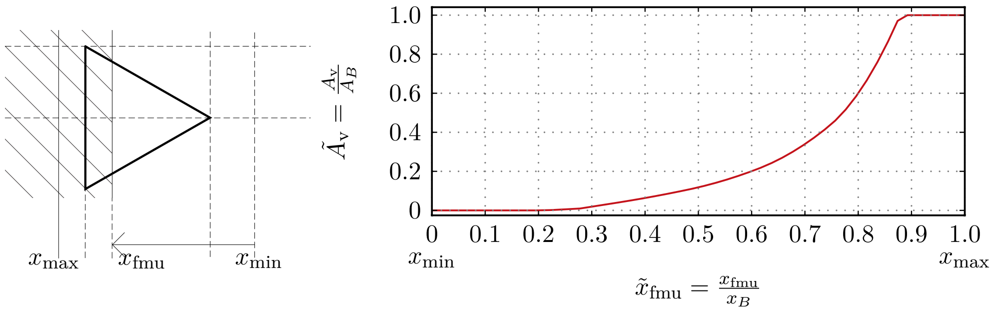

Figure 5. Shape of the variable cross section of the FMU.

with

$$A_{fmu}(x_{fmu}) = \frac{A_v(x_{fmu})}{\alpha_{in}^2 A_{in}^2 \alpha_v^2 A_v^2(x_{fmu}) + \alpha_{in}^2 A_{in}^2 \alpha_{out}^2 A_{out}^2 + \alpha_v^2 A_v^2(x_{fmu}) \alpha_{out}^2 A_{out}^2} \tag{13b}$$

$$\alpha_{fmu} = \alpha_v \alpha_{in} A_{in} \alpha_{out} A_{out} \tag{13c}$$

and $\Delta p_{fmu} = p_{gp} - p_{gal}$, with the gear pump pressure p_{gp} and the gallery pressure p_{gal}. The mass density ϱ_{fmu} is approximated by the density at the mean pressure according to Section 2.1 in the form

$$\varrho_{fmu} = \varrho\left(\frac{p_{gp} + p_{gal}}{2}\right) . \tag{14}$$

The variable area A_v is given as a function of the position x_{fmu} of the valve spool. Since the opening area exhibits a triangular shape, A_v increases quadratically with x_{fmu}, see Fig. 5. The movement of the valve spool is controlled by a solenoid, generating the (positive) magnetic force f_m. The restoring force is due to a preloaded spring with stiffness c_v. The mechanical system of the FMU can be described by

$$\frac{\mathrm{d}}{\mathrm{d}t} x_{fmu} = v_{fmu} \tag{15a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{fmu} = \frac{1}{m_v}\left(f_m - c_v\left(x_{fmu} - x_o\right) - d_v v_{fmu}\right) , \tag{15b}$$

with the position x_{fmu}, the velocity v_{fmu}, and the mass m_v of the spool (including the mass of all moving parts of the solenoid), and the damping coefficient d_v. The preload of the spring x_o is chosen as the position which corresponds to the maximum mass flow. The position x_{fmu} is limited within the mechanical stops $x_{min} \leq x_{fmu} \leq x_{max}$. The contact with the end stops is modeled by a perfectly inelastic collision, dissipating the whole kinetic energy. At the end stops, the equations of motion read as

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x_{fmu} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} v_{fmu} &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{if } (x_{fmu} = x_{min})\ \&\ (f_m - c_v(x_{min} - x_o) \leq 0) \\ &\text{or } (x_{fmu} = x_{max})\ \&\ (f_m - c_v(x_{max} - x_o) \geq 0) , \end{aligned} \tag{16}$$

and outside these stops, the motion is governed by (15).

The magnetic force f_m is generated by a solenoid, where the force is a function of both the position x_{fmu} and the current applied to the electromagnet of the

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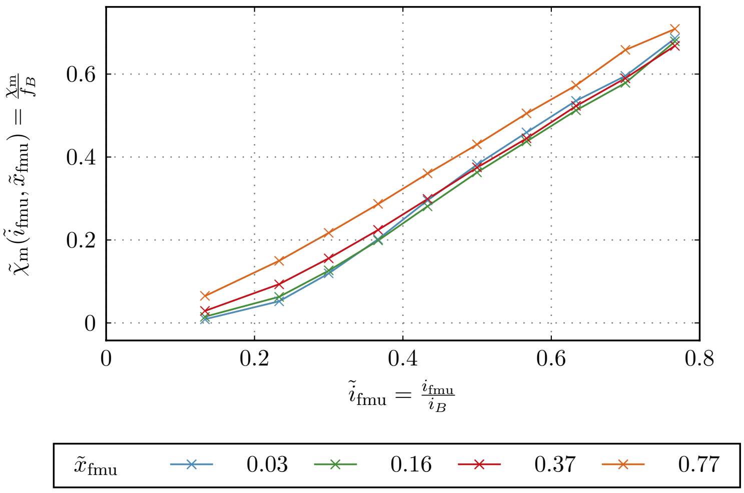

Figure 6. Magnetic force $\chi_{\mathrm{m}}(i_{\mathrm{fmu}}, x_{\mathrm{fmu}})$ as a function of the current set point i_{fmu} and the position x_{fmu}.

FMU. The current is controlled by a subordinate current controller such that the setpoint of the current i_{fmu} serves as the control input to the system. An analytic model of the overall dynamics of the solenoid, including the current controller, is not meaningful due to the resulting complexity. Instead, the dynamics of the current controlled solenoid is approximated by a first order lag element with the time constant $T_{\mathrm{m}} > 0$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\mathrm{m}} = \frac{1}{T_{\mathrm{m}}} \left(-f_{\mathrm{m}} + \chi_{\mathrm{m}}(i_{\mathrm{fmu}}, x_{\mathrm{fmu}}) \right) \ . \tag{17}$$

Here, χ_{m} is the stationary characteristics of the magnetic force f_{m} as a function of the position x_{fmu} and the current set point i_{fmu} of the current controller. Fig. 6 depicts the measured characteristics χ_{m}.

2.3 *Gallery Volume*

The mass flow of the FMU (13) is delivered into the small gallery volume V_{gal} between FMU and radial piston pump. The diesel mass flow out of V_{gal} is the sum of all flows into the N_{c} cylinders of the radial piston pump. Therefore, the pressure p_{gal} is given by the balance of mass

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\mathrm{gal}} = \frac{\beta_{\mathrm{gal}}}{V_{\mathrm{gal}}} \frac{\dot{m}_{\mathrm{fmu}} - \sum_{i=1}^{N_{\mathrm{c}}} \dot{m}_{\mathrm{sv},i}}{\varrho_{\mathrm{gal}}} \ , \tag{18}$$

with the bulk modulus β_{gal} and the density ϱ_{gal} of diesel in this volume according to Section 2.1, $\beta_{\mathrm{gal}} = \beta(p_{\mathrm{gal}})$, $\varrho_{\mathrm{gal}} = \varrho(p_{\mathrm{gal}})$. Moreover, $\dot{m}_{\mathrm{sv},i}$ is the mass flow through the inlet valve of the i-th cylinder.

2.4 *Radial Piston Pump*

The radial piston pump delivers diesel into the rail by a periodic motion of the pistons. Starting at the top dead center, the cylinder volume is increased due to the downward motion of the piston and the diesel in the cylinder is expanded until

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the inlet (suction) valve, SV in Fig. 1, opens. Then, diesel flows from the gallery volume into the cylinder until the inlet valve closes again, which is the case when the cylinder pressure is higher than the gallery pressure. With the upward motion the diesel is compressed until the outlet (high pressure) valve, HV in Fig. 1, into the rail opens and diesel flows into the rail.

The radial piston pump is driven by the combustion engine such that its shaft rotates with the angular velocity $\dot{\varphi}_{\mathrm{cr}} = \omega_{\mathrm{cr}}$ of the crank shaft. During one revolution of the shaft, the cylinders are moving N_{e} times up and down due to the eccentricities of the shaft. Moreover, the cylinders are mounted with a relative angle $\Delta\varphi_{\mathrm{c}}$ to each other, see Fig. 1, and the motion of the cylinders is sinusoidal. Thus, the volume $V_{\mathrm{c},i}$ of the i-th cylinder reads as

$$V_{\mathrm{c},i}(t) = V_{\mathrm{c}0} + \frac{V_{\mathrm{ch}}}{2}\left(1 - \cos(N_{\mathrm{e}}\varphi_{\mathrm{cr}}(t) - N_{\mathrm{e}}(i-1)\Delta\varphi_{\mathrm{c}})\right) \qquad i = 1, \ldots, N_{\mathrm{c}} \tag{19}$$

with the number of cylinders N_{c}. The volumes $V_{\mathrm{c}0}$ and V_{ch} are the dead volume of a cylinder and its displaced volume, respectively. In this paper, a radial piston pump with $N_{\mathrm{c}} = 2$ cylinders, $N_{\mathrm{e}} = 3$ cylinder strokes per revolution and $\Delta\varphi_{\mathrm{c}} = 60°$ is used. The resulting volumes $V_{\mathrm{c},1}$ and $V_{\mathrm{c},2}$ of the cylinders are given in Fig. 7.

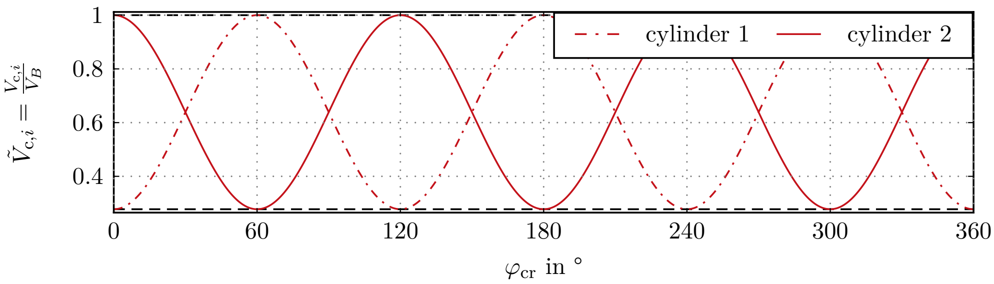

Figure 7. Cylinder volumes $V_{\mathrm{c},1}$ and $V_{\mathrm{c},2}$ during one revolution of the crank shaft.

Using the balance of mass, the pressure $p_{\mathrm{c},i}$ in the cylinders reads as

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\mathrm{c},i} = \frac{\beta_{\mathrm{c},i}}{V_{\mathrm{c},i}(t)}\left(\frac{\dot{m}_{\mathrm{sv},i} - \dot{m}_{\mathrm{hv},i} - \dot{m}_{\mathrm{l},i}}{\varrho_{\mathrm{c},i}} - \dot{V}_{\mathrm{c},i}(t)\right), \qquad i = 1, \ldots, N_{\mathrm{c}}\ , \tag{20}$$

with the diesel mass flow $\dot{m}_{\mathrm{sv},i}$ through the inlet (suction) valve into the cylinder, $\dot{m}_{\mathrm{hv},i}$ through the outlet (high pressure) valve into the rail and the leakage mass flow $\dot{m}_{\mathrm{l},i}$.

The areas $A_{\mathrm{sv},i}$ of the spring loaded inlet valves depend on the difference pressure $p_{\mathrm{gal}} - p_{\mathrm{c},i}$. If the pressure difference is lower than $p_{\mathrm{sv,c}}$, the valve is closed, i. e. $A_{\mathrm{sv},i} = 0$. For pressures higher than $p_{\mathrm{sv,o}}$ the valve is completely opened, i. e. $A_{\mathrm{sv},i} = A_{\mathrm{sv},0}$, with the maximum area $A_{\mathrm{sv},0}$ of the suction valve. For pressures between these two levels the area is assumed to increase linearly with the pressure drop. This results in the mathematical model of the form

$$A_{\mathrm{sv},i} = \begin{cases} 0 & \text{if } p_{\mathrm{gal}} - p_{\mathrm{c},i} \leq p_{\mathrm{sv,c}} \\ \dfrac{p_{\mathrm{gal}} - p_{\mathrm{c},i} - p_{\mathrm{sv,c}}}{p_{\mathrm{sv,o}} - p_{\mathrm{sv,c}}} A_{\mathrm{sv},0} & \text{if } p_{\mathrm{sv,c}} < p_{\mathrm{gal}} - p_{\mathrm{c},i} < p_{\mathrm{sv,o}} \\ A_{\mathrm{sv},0} & \text{if } p_{\mathrm{gal}} - p_{\mathrm{c},i} \geq p_{\mathrm{sv,o}}\ . \end{cases} \tag{21}$$

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The mass flow through the inlet valve is then given by

$$\dot{m}_{\mathrm{sv},i} = \alpha_{\mathrm{sv}} A_{\mathrm{sv},i} \sqrt{2\varrho_{\mathrm{sv},i}} \sqrt{p_{\mathrm{gal}} - p_{\mathrm{c},i}}, \qquad i = 1, \ldots, N_{\mathrm{c}} \ , \tag{22a}$$

with the discharge coefficient α_{sv} and the corresponding mass density

$$\varrho_{\mathrm{sv},i} = \varrho \left(\frac{p_{\mathrm{c},i} + p_{\mathrm{gal}}}{2} \right) \ . \tag{22b}$$

The high pressure valves are modeled analogously using the pressure $p_{\mathrm{c},i} - p_{\mathrm{r}}$, with the rail pressure p_{r}

$$A_{\mathrm{hv},i} = \begin{cases} 0 & \text{if } p_{\mathrm{c},i} - p_{\mathrm{r}} \leq p_{\mathrm{hv,c}} \\ \dfrac{p_{\mathrm{c},i} - p_{\mathrm{r}} - p_{\mathrm{hv,c}}}{p_{\mathrm{hv,o}} - p_{\mathrm{hv,c}}} A_{\mathrm{hv},0} & \text{if } p_{\mathrm{hv,c}} < p_{\mathrm{c},i} - p_{\mathrm{r}} < p_{\mathrm{hv,o}} \\ A_{\mathrm{hv},0} & \text{if } p_{\mathrm{c},i} - p_{\mathrm{r}} \geq p_{\mathrm{hv,o}} \end{cases} \tag{23a}$$

$$\dot{m}_{\mathrm{hv},i} = \alpha_{\mathrm{hv}} A_{\mathrm{hv},i} \sqrt{2\varrho_{\mathrm{hv},i}} \sqrt{p_{\mathrm{c},i} - p_{\mathrm{r}}} \tag{23b}$$

$$\varrho_{\mathrm{hv},i} = \varrho \left(\frac{p_{\mathrm{c},i} + p_{\mathrm{r}}}{2} \right), \qquad i = 1, \ldots, N_{\mathrm{c}} \ . \tag{23c}$$

Here, the high pressure valve is completely opened, i. e. $A_{\mathrm{hv},i} = A_{\mathrm{hv},0}$, if the cylinder pressure $p_{\mathrm{c},i}$ is higher than the rail pressure p_{r} plus the opening pressure of this valve $p_{\mathrm{hv,o}}$. For cylinder pressures lower than $p_{\mathrm{r}} + p_{\mathrm{hv,c}}$ the valve is closed. Again, a linearly increasing area is assumed if the cylinder pressure is between $p_{\mathrm{r}} + p_{\mathrm{hv,c}}$ and $p_{\mathrm{r}} + p_{\mathrm{hv,o}}$.

At the gaps between the cylinders and the pistons small leakage flows occur which flow into the housing of the pump and can be modeled by a laminar flow (leakage coefficient $k_{\mathrm{l},i}$) in the form

$$\dot{m}_{\mathrm{l},i} = k_{\mathrm{l},i} \left(p_{\mathrm{c},i} - p_{\mathrm{gp}} \right) \ . \tag{24}$$

Note that the housing of the pump is connected to p_{gp} in order to lubricate the pump.

2.5 *Rail*

The rail is used as a storage for pressurized diesel. It is connected to the radial piston pump, which supplies the mass flow $\sum_{i=1}^{N_{\mathrm{c}}} \dot{m}_{\mathrm{hv},i}$, and the injectors, which inject the mass flow $\sum_{i=1}^{N_{\mathrm{i}}} \dot{m}_{\mathrm{inj},i}$ into the combustion chambers of the engine. The diesel pressure in the (constant) rail volume V_{r} is formulated as

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\mathrm{r}} = \frac{\beta_{\mathrm{r}}}{V_{\mathrm{r}}} \frac{\sum\limits_{i=1}^{N_{\mathrm{c}}} \dot{m}_{\mathrm{hv},i} - \sum\limits_{i=1}^{N_{\mathrm{i}}} \dot{m}_{\mathrm{inj},i} - \dot{m}_{\mathrm{l,r}}}{\varrho_{\mathrm{r}}} \ , \tag{25}$$

with the bulk modulus $\beta_{\mathrm{r}} = \beta(p_{\mathrm{r}})$ and the density $\varrho_{\mathrm{r}} = \varrho(p_{\mathrm{r}})$ of diesel in the rail according to Section 2.1. The overall injector leakage mass flow $\dot{m}_{\mathrm{l,r}}$ is modeled in the form of a laminar flow into the tank (pressure p_0), i. e.

$$\dot{m}_{\mathrm{l,r}} = k_{\mathrm{l,r}} \left(p_{\mathrm{r}} - p_0 \right) \ , \tag{26}$$

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with the leakage coefficient $k_{\mathrm{l,r}}$.

As already mentioned at the beginning of this section, the injection process is rather complex and therefore an accurate description of the injector mass flows $\dot{m}_{\mathrm{inj},i}$, $i = 1, \ldots, N_{\mathrm{i}}$, with the number of injectors N_{i} (equal to the number of cylinders of the combustion engine), is beyond the scope of this paper. A simplified description is meaningful, since the exact time evolution of the injector mass flow is irrelevant for the design of a control strategy for the rail pressure. Instead, a model is proposed, which covers the following essential characteristics:

(i) The periodic injection of diesel yields fluctuations in the rail pressure with the frequency $\frac{N_{\mathrm{i}}}{2}\omega_{\mathrm{cr}} = \frac{N_{\mathrm{i}}}{2}\dot{\varphi}_{\mathrm{cr}}$. These variations of the rail pressure should be included approximately in the mathematical model.

(ii) Of course, the average mass flow $\bar{\dot{m}}_{\mathrm{inj}}$ of the injectors of one injection cycle must be accurately described by the model.

Based on these considerations, a saw tooth characteristic of the form

$$\dot{m}_{\mathrm{inj}} = \begin{cases} \dfrac{\frac{N_{\mathrm{i}}}{2}\varphi_{\mathrm{cr}} \bmod 2\pi}{\pi} 2\bar{\dot{m}}_{\mathrm{inj}} & \text{if } \dfrac{N_{\mathrm{i}}}{2}\varphi_{\mathrm{cr}} \bmod 2\pi \leq \pi \\ \left(1 - \dfrac{\frac{N_{\mathrm{i}}}{2}\varphi_{\mathrm{cr}} \bmod 2\pi}{\pi}\right) 2\bar{\dot{m}}_{\mathrm{inj}} & \text{if } \dfrac{N_{\mathrm{i}}}{2}\varphi_{\mathrm{cr}} \bmod 2\pi > \pi \end{cases} \tag{27}$$

is used.

2.6 *Simulation results*

In this section, simulation results are shown to analyze the behavior of the modeled system and the stationary mass flow is compared with measurements to validate the model.

Figure 8 depicts simulation results of the cylinder pressures $p_{\mathrm{c},i}$ and the gallery pressure p_{gal}, as well as the mass flows $\dot{m}_{\mathrm{sv},i}$ and $\dot{m}_{\mathrm{hv},i}$ for a constant FMU position $\tilde{x}_{\mathrm{fmu}} = 0.85$, a crank revolution speed $n_{\mathrm{cr}} = 1000\,\mathrm{min}^{-1}$ and a constant rail pressure $p_{\mathrm{r}} = 1300\,\mathrm{bar}$. Taking a closer look at cylinder 1 first, the pressure $p_{\mathrm{c},1}$ decreases during the downward stroke of the piston until the cylinder pressure $p_{\mathrm{c},1}$ is lower than the gallery pressure p_{gal} and the inlet valve opens. Since the FMU is rather widely opened, the increasing cylinder volume $V_{\mathrm{c},1}$ can be completely filled by the diesel flowing through the inlet valve, i. e. $\dot{m}_{\mathrm{sv},i} = \varrho_{\mathrm{c},i}\dot{V}_{\mathrm{c},i}$. After some time, however, the cylinder volume increases faster than it can be filled with diesel such that the diesel in the cylinder starts to cavitate. Then, the cylinder pressure is almost constant and thus also the mass flow into the cylinder remains almost constant. The cylinder pressure begins to rise again when the cylinder is completely filled with fluid. If this happens before the bottom dead center (BDC) (as it is the case in Fig. 8), the cylinder is completely filled at the bottom dead center and the mass flow of the radial piston pump is solely determined by the geometrical displacement and the angular velocity of the pump. Moreover, in this case the two cylinders do not influence each other since the inlet valve of cylinder 1 closes before the inlet valve of cylinder 2 opens.

In contrast to this, the FMU opening is smaller ($\tilde{x}_{\mathrm{fmu}} = 0.67$) in the simulation scenario depicted in Fig. 9. This brings along that the cavitation in the cylinder

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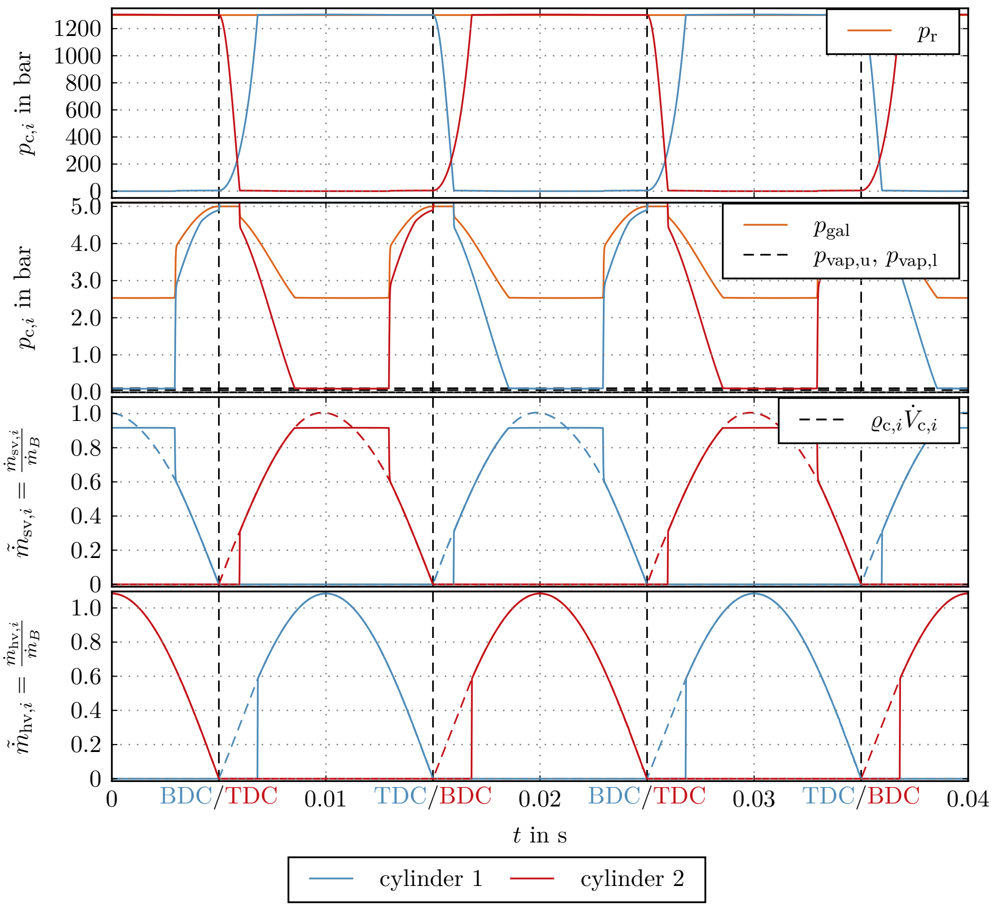

Figure 8. Pressure and mass flow for constant FMU position $\tilde{x}_{fmu} = 0.85$, crank revolution speed $n_{cr} = 1000\,\mathrm{min}^{-1}$ and rail pressure $p_r = 1300\,\mathrm{bar}$.

immediately starts after the inlet valve opens and the cylinder is not completely filled when passing the bottom dead center. Thus, in this case the mass flow of the radial piston pump is also influenced by the opening of the FMU. Time intervals do occur where the inlet valves of both cylinders are opened and thus both cylinders draw diesel from the gallery.

Measurements of the cylinder pressures and mass flows are not available for a validation of the model. Thus, the average mass flow of the radial piston pump for various FMU positions and crank shaft revolution speeds are used to validate the stationary accuracy of the model. The results in Fig. 10 show a comparison of the measurement data (cross symbols) with simulation results (solid lines). It can be seen that for small openings of the FMU, the mass flow is basically determined by the FMU opening and is almost independent of the angular velocity of the crank shaft. For higher openings, the mass flow is, however, determined by the geometrical displacement of the pump. A comparison of the measurement data with simulation results of the model shows that the maximum errors are well below 10 %, which is a very good result since the model has been parameterized by the nominal geometrical parameters of the system only.

Finally, in Fig. 11 the dynamics of the modeled system is discussed. Here, the current $\tilde{i}_{fmu}$ is switched from 0 to 1.00 at $t = 100\,\mathrm{ms}$ and back to $\tilde{i}_{fmu} = 0$ at $t = 200\,\mathrm{ms}$. Due to the pre-load of the spring, this implies that the FMU is entirely opened for $\tilde{i}_{fmu} = 0$ at the beginning and the end of the simulation and closed

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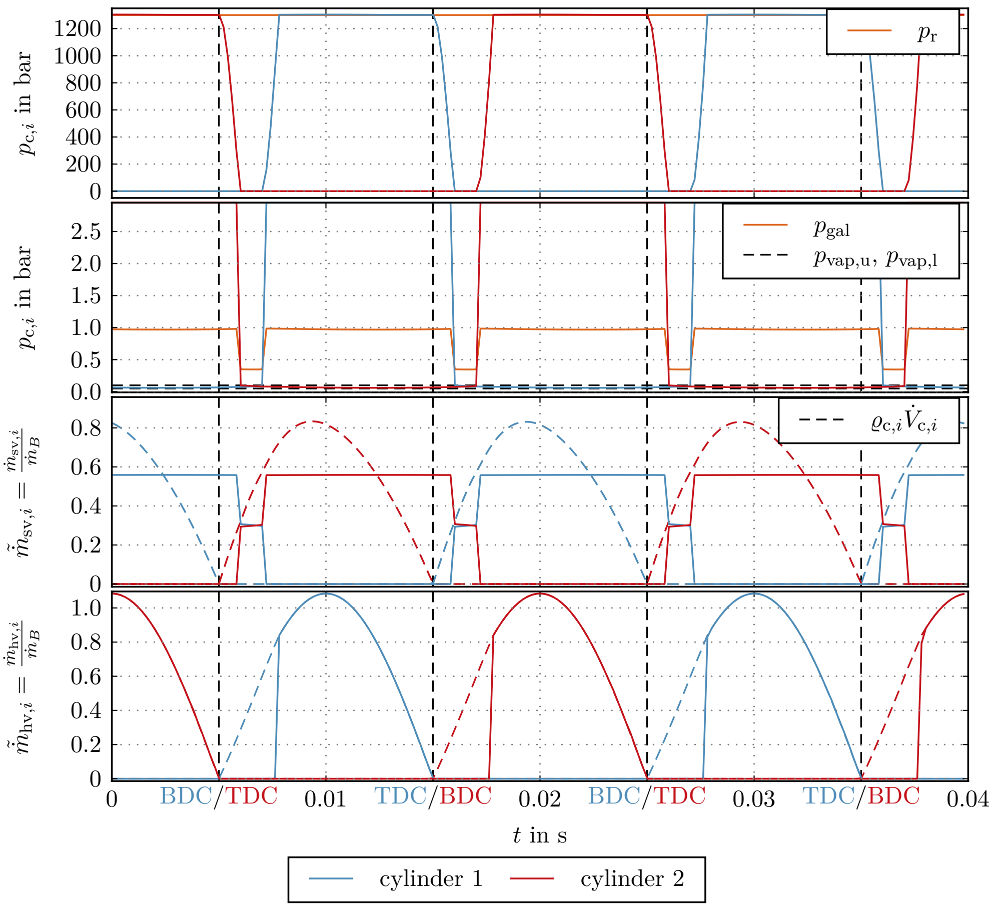

Figure 9. Pressure and mass flow for constant FMU position $\tilde{x}_{fmu} = 0.67$, crank revolution speed $n_{cr} = 1000\,\mathrm{min}^{-1}$ and rail pressure $p_r = 1300\,\mathrm{bar}$.

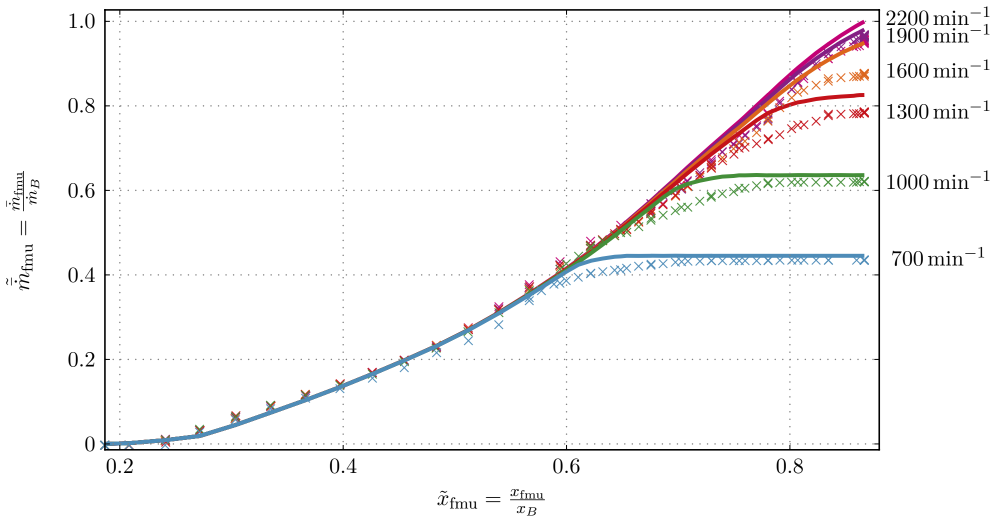

Figure 10. Mass flow of the detailed model (solid) in comparison with measurements (x).

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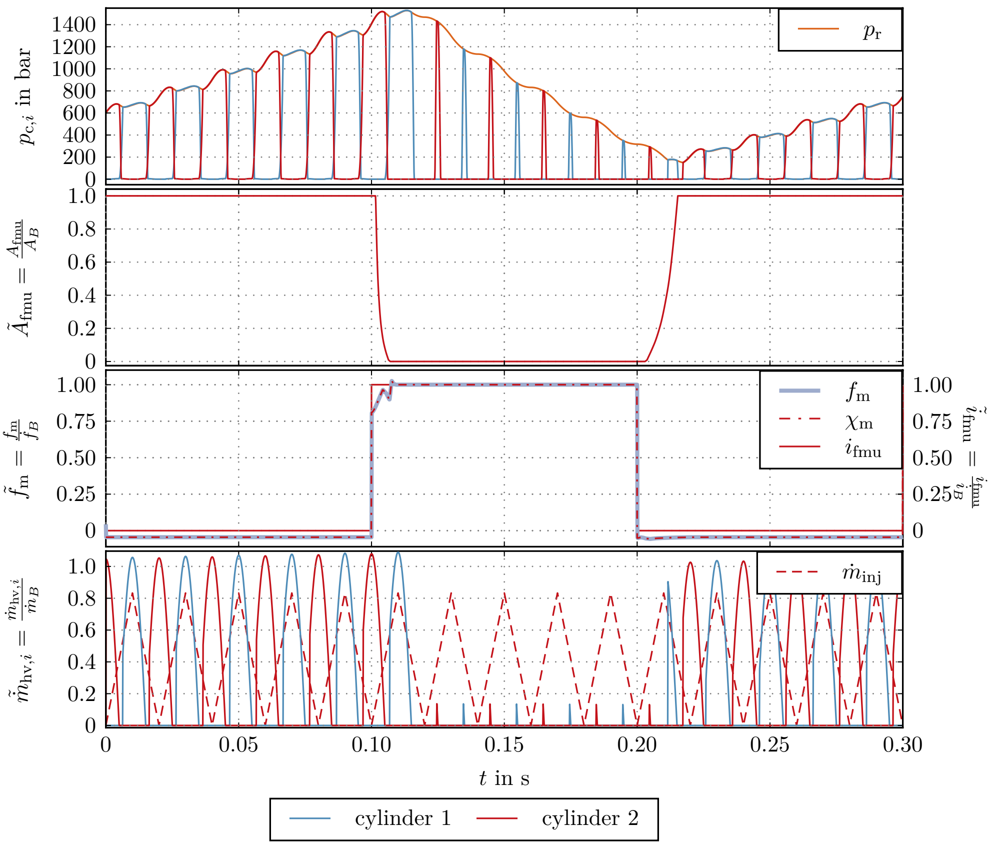

Figure 11. Pressure and mass flow for constant crank revolution speed $n_{cr} = 1000\,\mathrm{min}^{-1}$ and average injection mass flow $\tilde{\dot{m}}_{inj} = 0.42$, steps in the current set point i_{fmu}.

in between. The average injected diesel mass flow is set to $\tilde{\dot{m}}_{inj} = 0.42$. Since the injected mass flow is lower than the mass flow $\dot{m}_{hv}$ delivered by the radial piston pump from the beginning until $t = 100\,\mathrm{ms}$ and after $t = 200\,\mathrm{ms}$, the rail pressure is increasing. During $100\,\mathrm{ms} < t < 200\,\mathrm{ms}$ the FMU is closed and no diesel is delivered to the rail. Thus, the rail pressure decreases in this time interval. When the current in the solenoid is switched at $t = 100\,\mathrm{ms}$ the magnetic force f_m leads the valve spool to move and close the FMU, see the time evolution of A_{fmu}. Note that the rail pressure is still increasing a short time after the FMU is closed. This is due to the fact that one cylinder is still filled with diesel which is delivered into the rail. The same effect occurs when the FMU is opening. The cylinders have to be filled and the diesel compressed, before diesel is delivered to the rail. This results in some kind of dead time in the system, which is a function of the angular velocity of the crank shaft, see also [8].

The model developed in this section is not directly useful for a controller design due to its high complexity. As the periodic cycles of the pump mass flow are not of interest for the control strategy of the average rail pressure, a reduced model for the average values of the variables is derived in the next section.

3. Average Model

As the ripples in the rail pressure due to the piston strokes of the pump and the injection process cannot be affected by the FMU, only the average value of the rail

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pressure during one stroke is relevant for the control strategy.[1] Typically, control strategies for the rail pressure are based on a stationary (measured) characteristic map of the mass flow of the pump as a function of the FMU position x_{fmu} and the rotational speed of the pump n_{cr}. This approach, however, has two basic drawbacks:

(i) In the controller design, handling of characteristic maps might be difficult, since higher derivatives of these maps are frequently required, as e. g. in a flatness-based controller. The numerical calculation of these derivatives might become cumbersome.

(ii) It is difficult to analyze the influence of parameter variations or different installation sizes by means of maps obtained from measurements.

Thus, an average model based on the detailed model of the previous section is strived for. In order to derive an average model, only a single cylinder of the pump is considered in the first step. The influence of the second piston is taken into account in a second step.

3.1 *Single cylinder*

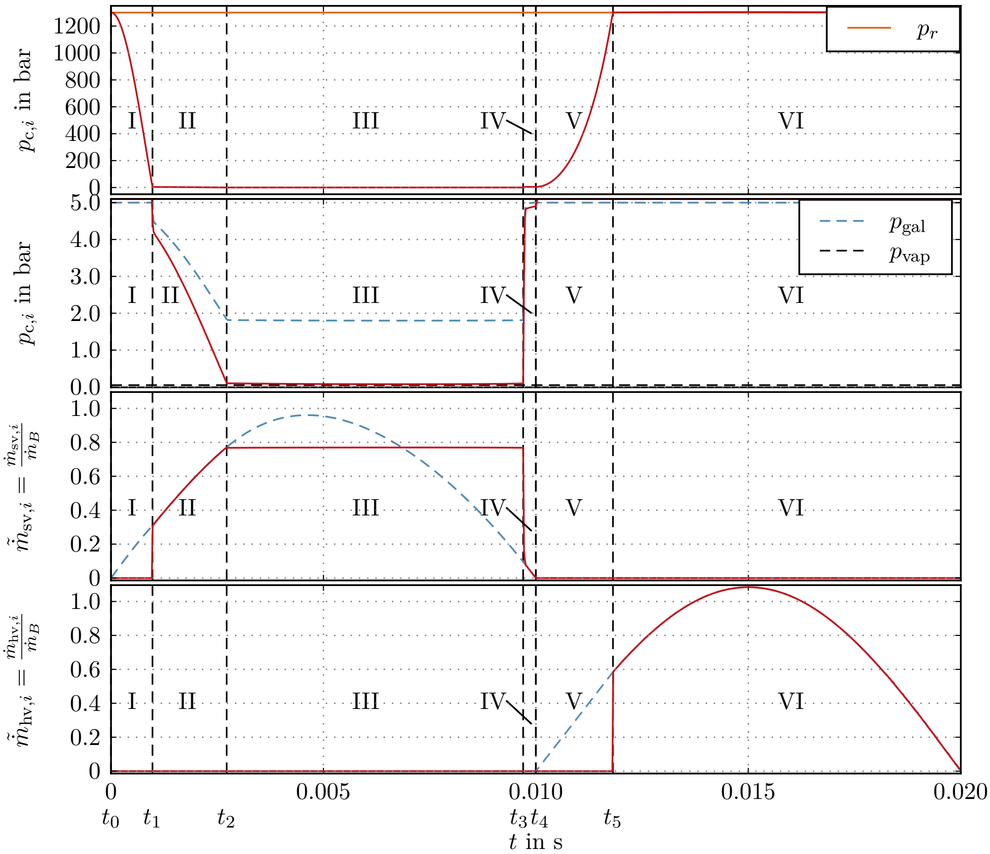

Figure 12. Pressure and mass flow of one cylinder for constant FMU position $\tilde{x}_{\text{fmu}} = 0.77$, crank revolution speed $n_{\text{cr}} = 1000\,\text{min}^{-1}$ and rail pressure $p_{\text{r}} = 1300\,\text{bar}$.

[1]Instead of controlling the average pressure it is often desired to control the peak pressure over one cycle of the cylinder stroke. Since the average and the peak pressure are directly coupled, controlling the peak pressure is equivalent to controlling the average pressure. Thus, subsequently only this case is considered.

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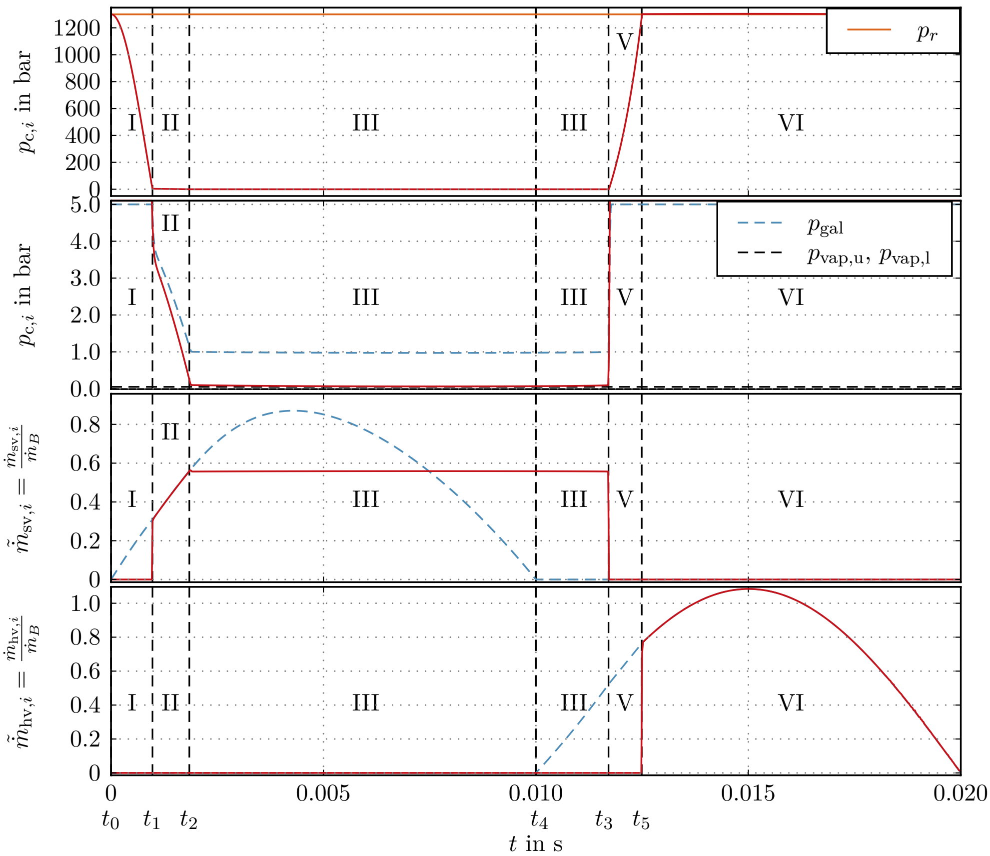

Figure 13. Pressure and mass flow of one cylinder for constant FMU position $\tilde{x}_{\mathrm{fmu}} = 0.67$, crank revolution speed $n_{\mathrm{cr}} = 1000\,\mathrm{min}^{-1}$ and rail pressure $p_{\mathrm{r}} = 1300\,\mathrm{bar}$.

Figures 12 and 13 show the pressure p_{c} and the mass flows $\dot m_{\mathrm{sv}}$ and $\dot m_{\mathrm{hv}}$ of one cylinder for one piston stroke, a constant FMU position (approx. 50 % opening in Fig. 12 and approx. 30 % in Fig. 13), and a constant rail pressure. The piston stroke can be divided into the subsequent phases:

(I) The high pressure valve is closed at the top dead center (TDC) and the pressure p_{c} in the cylinder is equal to the rail pressure at $t = t_0$. During phase I both valves are closed such that the diesel mass in the cylinder is constant and the pressure decreases as the volume increases.

(II) The inlet valve is opened at $t = t_1$ when the cylinder pressure is equal to the gallery pressure p_{gal}. In this phase, the mass flow through the inlet valve is high enough to completely fill the increasing volume of the cylinder.

(III) If the pressure in the cylinder further drops beyond the vapor pressure $p_{\mathrm{vap,u}}$, the diesel starts to vaporize. However, below the lower vapor pressure $p_{\mathrm{vap,l}}$, all diesel is vaporized and the cylinder pressure is basically constant. In order to simplify the treatment of cavitation, $p_{\mathrm{vap}} = p_{\mathrm{vap,l}} = p_{\mathrm{vap,u}}$ is set in the average model. This is reasonable, since $p_{\mathrm{vap,u}} - p_{\mathrm{vap,l}}$ is small and thus errors connected to this simplification are negligible.

(IV) In this phase, the volume of the cylinder can again be filled completely before the piston reaches its bottom dead center (BDC). Note that this phase is not present in Fig. 13, where the FMU is opened only approx. 30 % of its maximum value. Figure 12 showing the results for a wider opened FMU of 50 % of its maximum value contains such a phase.

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(V) When the pressure p_c rises above p_gal at $t = t_4$ the inlet valve is closed and the diesel is compressed until the pressure reaches $p_\mathrm{c} = p_\mathrm{r}$ at $t = t_5$.

(VI) In this phase, the outlet valve is open and diesel is delivered into the rail.

It can be seen that the mass of diesel delivered into the rail during one cycle of the piston stroke is given by the difference of the maximum diesel mass in the cylinder before the inlet valve closes and the mass at $t = t_1$, i. e. the time when the inlet valve opens, $\Delta m = \varrho V_\mathrm{c}\big(\max(t_3, t_4)\big) - \varrho V_\mathrm{c}(t_1)$. This directly gives the average mass flow in the form

$$\begin{aligned}\bar{\dot{m}}_\mathrm{sv} = \frac{1}{T_\mathrm{c}}\int_0^{T_\mathrm{c}} \dot{m}_\mathrm{sv}(t)\mathrm{d}t &= N_\mathrm{e} n_\mathrm{cr}\Big(\varrho V_\mathrm{c}\big(\max(t_3, t_4)\big) - \varrho V_\mathrm{c}(t_1)\Big) \\ &= N_\mathrm{e} n_\mathrm{cr}\varrho\frac{V_\mathrm{ch}}{2}\Big(\cos(N_\mathrm{e}\omega_\mathrm{cr} t_1) - \cos\big(N_\mathrm{e}\omega_\mathrm{cr}\max(t_3, t_4)\big)\Big)\ .\end{aligned} \tag{28}$$

Here, $T_\mathrm{c} = \frac{2\pi}{N_\mathrm{e}\omega_\mathrm{cr}}$ is the time period of a complete stroke of the cylinder. In the subsequent section, the different phases will be described in detail and the corresponding times t_i, $i = 0, \ldots, 5$ are derived.

3.1.1 Phase I: $t_0 < t < t_1$ Expansion

At the top dead center $t = t_0$ the high pressure valve is closed and the pressure in the cylinder is $p_\mathrm{c}(t_0) = p_\mathrm{r} + p_\mathrm{hv,c}$. With the assumptions $t_0 = 0$, $\varphi_\mathrm{cr}(t_0) = 0$, $\dot{m}_\mathrm{l,i} = 0$ and using $\beta_{c,i}$ due to (2), (20) with (19) can be solved analytically in the interval $t_0 \leq t \leq t_1$ to

$$p_\mathrm{c}(t) = \left(\frac{V_\mathrm{c0}}{V_\mathrm{c0} + \frac{V_\mathrm{ch}}{2}\left(1 - \cos(N_\mathrm{e}\omega_\mathrm{cr} t)\right)}\right)^{b_1}\left(p_\mathrm{r} + p_\mathrm{hv,c} - p_0 + \frac{\beta_0}{b_1}\right) - \frac{\beta_0}{b_1} + p_0\ . \tag{29}$$

The inlet valve is opened when the pressure is lower than $p_\mathrm{gal} - p_\mathrm{sv,c}$. Since the gallery pressure p_gal also varies with time, it is difficult to calculate $p_\mathrm{gal} - p_\mathrm{sv,c}$ at time t_1. Instead, the expansion process is calculated until the pressure in the cylinder is equal to p_vap, i. e. $p_\mathrm{c} = p_\mathrm{vap}$, which results in a small but negligible error for t_1. Then, t_1, i. e. the time when the inlet valve opens, is approx. given by

$$t_1 = \frac{1}{N_\mathrm{e}\omega_\mathrm{cr}}\arccos\left(\left(1 - \left(\frac{\beta_0 + b_1\left(p_\mathrm{r} + p_\mathrm{hv,c} - p_0\right)}{\beta_0 + b_1(p_\mathrm{vap} - p_0)}\right)^{\frac{1}{b_1}}\right)\frac{2V_\mathrm{c0}}{V_\mathrm{ch}} + 1\right)\ . \tag{30}$$

3.1.2 Phase II: $t_1 < t < t_2$ Cylinder completely filled

During this phase the inlet valve is open and the diesel mass flow through the inlet valve is determined by the change in cylinder volume, i. e.

$$\dot{m}_\mathrm{sv}(t) = \varrho_\mathrm{sv}\dot{V}_\mathrm{c}(t) = \varrho_\mathrm{sv} N_\mathrm{e}\omega_\mathrm{cr}\frac{V_\mathrm{ch}}{2}\sin(N_\mathrm{e}\omega_\mathrm{cr} t)\ . \tag{31}$$

To derive an analytical expression for the time t_2, when the cavitation in the cylinder starts ($p_\mathrm{c} = p_\mathrm{vap}$), a number of additional simplifications are made.

(i) The gallery volume V_gal is very small and is therefore neglected. Then, balance of mass directly results in $\dot{m}_\mathrm{fmu} = \dot{m}_\mathrm{sv}$, see (18).

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(ii) The diesel pressure outside the cylinder (i. e. in the FMU and the gallery) is well above vapor pressure p_{vap}, such that setting $\varrho_{\mathrm{fmu}} = \varrho_{\mathrm{sv}} = \bar{\varrho} = \mathrm{const.}$ is reasonable.

(iii) It is assumed that the inlet valve is completely opened in the phases II-IV, i. e. $A_{\mathrm{sv}} = A_{\mathrm{sv},0}$. This is equivalent to setting $p_{\mathrm{sv,c}} = p_{\mathrm{sv,o}}$.

With these assumptions together with (13) and (22), we get

$$\alpha_{\mathrm{fmu}} A_{\mathrm{fmu}} \sqrt{2\bar{\varrho}} \sqrt{p_{\mathrm{gp}} - p_{\mathrm{gal}}} = \alpha_{\mathrm{sv}} A_{\mathrm{sv},0} \sqrt{2\bar{\varrho}} \sqrt{p_{\mathrm{gal}} - p_{\mathrm{c}}} \ , \tag{32}$$

which directly results in an expression for the gallery pressure p_{gal} in the form

$$p_{\mathrm{gal}} = \frac{p_{\mathrm{gp}} \alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 + p_{\mathrm{c}} \alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2}{\alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 + \alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2} \ . \tag{33}$$

Substituting (33) into (22) and using this result in (31) yields

$$\bar{\varrho} N_{\mathrm{e}} \omega_{\mathrm{cr}} \frac{V_{\mathrm{ch}}}{2} \sin(N_{\mathrm{e}} \omega_{\mathrm{cr}} t) = \alpha_{\mathrm{sv}} A_{\mathrm{sv},0} \sqrt{2\bar{\varrho}} \sqrt{p_{\mathrm{gal}} - p_{\mathrm{c}}} \tag{34}$$

and with (33) and (34) the cylinder pressure $p_{\mathrm{c}}(t)$ is given by

$$p_{\mathrm{c}}(t) = p_{\mathrm{gp}} - \bar{\varrho} (N_{\mathrm{e}} \omega_{\mathrm{cr}})^2 \frac{V_{\mathrm{ch}}^2}{4} \sin^2(N_{\mathrm{e}} \omega_{\mathrm{cr}} t) \frac{\alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 + \alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2}{2 \alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 \alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2} \ . \tag{35}$$

The time $t = t_2$, when cavitation in the cylinder starts, i. e. $p_{\mathrm{c}} = p_{\mathrm{vap}}$, can be found in the form

$$t_2 = \frac{1}{N_{\mathrm{e}} \omega_{\mathrm{cr}}} \arcsin \left(\sqrt{\frac{2}{\bar{\varrho}}} \frac{2}{V_{\mathrm{ch}}} \frac{1}{N_{\mathrm{e}} \omega_{\mathrm{cr}}} \alpha_{\mathrm{sv}} A_{\mathrm{sv},0} \alpha_{\mathrm{fmu}} A_{\mathrm{fmu}} \sqrt{\frac{p_{\mathrm{gp}} - p_{\mathrm{vap}}}{\alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 + \alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2}} \right) \ . \tag{36}$$

Please note that for almost closed FMU it can happen that $t_2 < t_1$, which means that phase II is not present and cavitation in the cylinder directly begins after phase I.

3.1.3 Phase III: $t_2 < t < t_3$ Cavitation

In the cavitation phase, the pressure p_{c} is equal to p_{vap} and thus constant. This also means that the mass flow $\dot{m}_{\mathrm{sv}}$ is constant. The corresponding constant gallery pressure p_{gal} is given by (33) for constant cylinder pressure $p_{\mathrm{c}} = p_{\mathrm{vap}}$. The mass flow into the cylinder $\dot{m}_{\mathrm{sv}} = \dot{m}_{\mathrm{vap}}$ is then given by (22b) as

$$\dot{m}_{\mathrm{vap}} = \alpha_{\mathrm{sv}} A_{\mathrm{sv},0} \sqrt{2\bar{\varrho}} \alpha_{\mathrm{fmu}} A_{\mathrm{fmu}} \sqrt{\frac{p_{\mathrm{gp}} - p_{\mathrm{vap}}}{\alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 + \alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2}} \ . \tag{37}$$

The cavitation phase III ends when the cylinder volume is completely filled. The corresponding end point $t = t_3$ is given by the (numeric) solution of

$$\begin{aligned} \big(t_3 - \max(t_1, t_2)\big) \dot{m}_{\mathrm{vap}} &= \bar{\varrho} \Big(V_{\mathrm{c}}(t_3) - V_{\mathrm{c}}\big(\max(t_1, t_2)\big) \Big) \\ &= \bar{\varrho} \frac{V_{\mathrm{ch}}}{2} \Big(\cos\big(N_{\mathrm{e}} \omega_{\mathrm{cr}} \max(t_1, t_2)\big) - \cos(N_{\mathrm{e}} \omega_{\mathrm{cr}} t_3) \Big) \ . \end{aligned} \tag{38}$$

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If the end of cavitation t_3 lies after the bottom dead center t_4, the compression phase V directly starts after phase III.

3.1.4 Phase IV: $t_3 < t < t_4$ Cylinder completely filled

If cavitation already ends before the piston has passed the bottom dead center $t = t_4$, again the volume flow into the cylinder is equal to the change in volume. In this case, the inlet valve is closed at the bottom dead center t_4 and diesel compression begins.

3.1.5 Phase V: $\max(t_3, t_4) < t < t_5$ Compression

Equivalent to Phase I the diesel mass in the cylinder is constant. The differential equation (20) with (2) can be solved analytically, using the initial condition $p_\mathrm{c}(\max(t_3, t_4)) = p_\mathrm{vap}$. The opening time t_5 of the outlet valve is reached when $p_\mathrm{c} = p_\mathrm{r} + p_\mathrm{hv,c}$ resulting in

$$t_5 = \frac{1}{N_\mathrm{e}\omega_\mathrm{cr}} \arccos\left(\frac{2V_\mathrm{c}\big(\max(t_3, t_4)\big)}{V_\mathrm{ch}} \left(\frac{\beta_0 + (p_\mathrm{vap} - p_0)b_1}{\beta_0 + (p_\mathrm{r} + p_\mathrm{hv,c} - p_0)b_1} \right)^{\frac{1}{b_1}} + \frac{2V_\mathrm{c0}}{V_\mathrm{ch}} + 1 \right) . \tag{39}$$

3.1.6 Phase VI: $t_5 < t < t_6$ Outlet Valve Open

In this phase, the diesel mass taken in during phases II-IV and compressed in phase V is flowing through the outlet valve into the rail volume. The average mass flow is given by (28), using the delivered mass to the cylinder in phases II-IV.

3.2 *Two cylinders*

To extend the average model of the previous section by the second cylinder of the pump, two cases have to be distinguished:

(i) If the inlet valve of the first cylinder closes before the inlet valve of the second cylinder opens, i. e. $t_3 < t_1 + \frac{T_\mathrm{c}}{2}$ $(T_\mathrm{c} = \frac{2\pi}{N_\mathrm{e}\omega_\mathrm{cr}})$, then the two cylinders of the pump operate independently, cf. Fig. 14. Thus, the overall mass flow $\bar{\dot{m}}_\mathrm{sv}$ of the pump with two cylinders simply doubles such that

$$\bar{\dot{m}}_\mathrm{sv} = 2\bar{\dot{m}}_\mathrm{sv,1}, \tag{40}$$

with $\bar{\dot{m}}_\mathrm{sv,1}$ being the mass flow of one cylinder given by (28).

(ii) If the FMU is almost closed, the inlet valve of the first cylinder closes after the inlet valve of the second cylinder opens, see Fig. 9. In this case, the mass flows of the two cylinders interfere with each other, which alters the overall mass flow of the pump. Taking a look at Fig. 9 it becomes clear that phase I remains unchanged and, as a matter of fact, phase IV does not occur. The time t_2 when cavitation in cylinder 1 starts is changed since the second cylinder already draws mass flow from the FMU. In order to calculate the modified time t_2', the mass balance in the gallery

$$\dot{m}_\mathrm{fmu} = \dot{m}_\mathrm{sv,1} + \dot{m}_\mathrm{sv,2} \tag{41}$$

is considered. The pressure in the second cylinder is given by the vapor pressure $p_\mathrm{c,2} = p_\mathrm{vap}$ and at t_2' also the pressure in the first cylinder reaches $p_\mathrm{c,1} = p_\mathrm{vap}$. Then, $\dot{m}_\mathrm{sv,1}(t_2') = \dot{m}_\mathrm{sv,2}(t_2')$ holds, and with (22), (13) and

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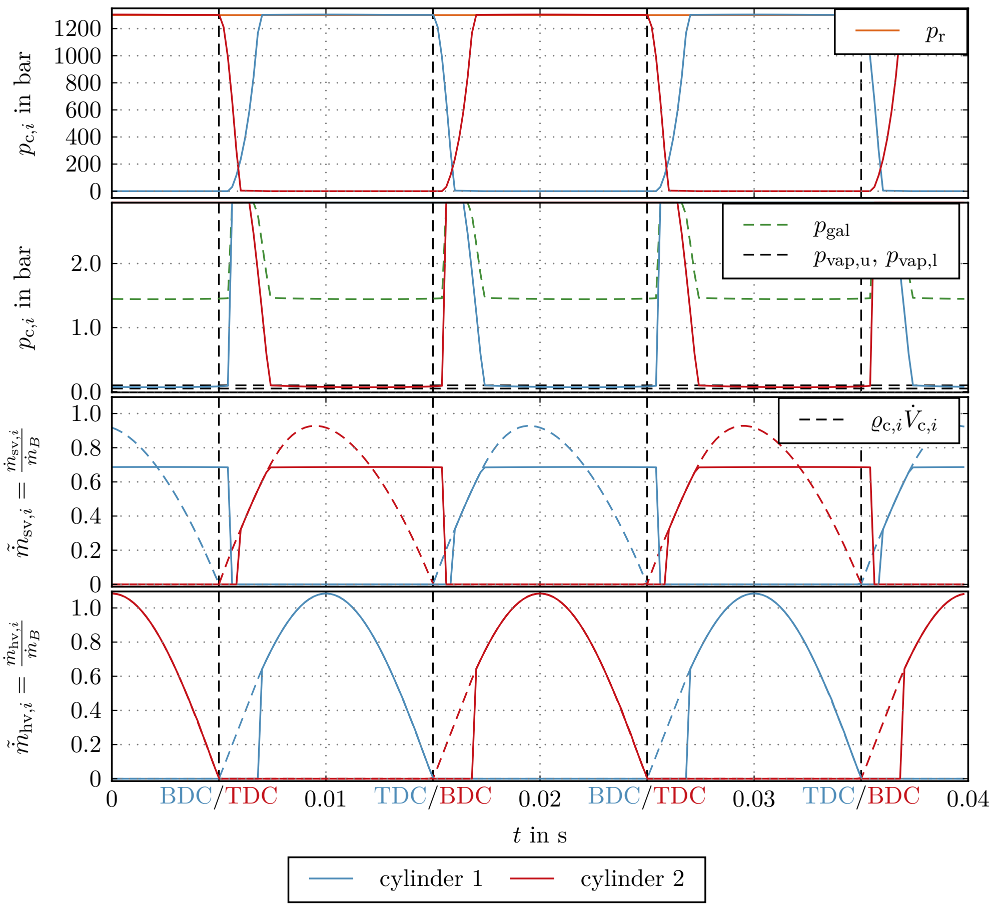

Figure 14. Pressure and mass flow for constant FMU position $\tilde{x}_{fmu} = 0.75$, crank revolution speed $n_{cr} = 1000\,\mathrm{min}^{-1}$ and rail pressure $p_r = 1300\,\mathrm{bar}$.

$\rho_{fmu} = \rho_{sv,i} = \bar{\rho}$ the following equation is fulfilled

$$\alpha_{fmu} A_{fmu} \sqrt{2\bar{\rho}} \sqrt{p_{gp} - p'_{gal}} = 2\alpha_{sv} A_{sv,0} \sqrt{2\bar{\rho}} \sqrt{p'_{gal} - p_{vap}} \,. \tag{42}$$

The gallery pressure p'_{gal} at t'_2 can be obtained as a solution of (42) in the form

$$p'_{gal} = \frac{p_{gp}\alpha^2_{fmu}A^2_{fmu} + 4p_{vap}\alpha^2_{sv}A^2_{sv,0}}{\alpha^2_{fmu}A^2_{fmu} + 4\alpha^2_{sv}A^2_{sv,0}} \,. \tag{43}$$

Using this result and the fact that $\bar{\rho}\dot{V}_{c,1}(t'_2) = \dot{m}_{sv,1}(t'_2)$, finally yields the time t'_2

$$t'_2 = \frac{1}{N_e\omega_{cr}} \arcsin\left(\sqrt{\frac{2}{\bar{\varrho}}}\frac{2}{V_{ch}}\frac{1}{N_e\omega_{cr}}\alpha_{sv}A_{sv,0}\alpha_{fmu}A_{fmu}\sqrt{\frac{p_{gp} - p_{vap}}{\alpha^2_{fmu}A^2_{fmu} + 4\alpha^2_{sv}A^2_{sv,0}}}\right) \,. \tag{44}$$

For the calculation of the modified time t'_3 when the inlet valve of the first cylinder closes, the time span t'_2 to t'_3 is divided into three parts:
(a) From $\max(t_1, t'_2)$ to $(t'_3 - \frac{T_c}{N_c})$ both inlet valves are open and both cylinders

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are in cavitation.[1] The mass flows into the two cylinders are equal in this case and are denoted by $\dot{m}_{\mathrm{vap},2}$. Using (22) with $p_{\mathrm{gal}} = p'_{\mathrm{gal}}$ from (43), the mass flow $\dot{m}_{\mathrm{vap},2}$ results in

$$\dot{m}_{\mathrm{vap},2} = \alpha_{\mathrm{sv}} A_{\mathrm{sv},0} \sqrt{2\bar{\varrho}} \alpha_{\mathrm{fmu}} A_{\mathrm{fmu}} \sqrt{\frac{p_{\mathrm{gp}} - p_{\mathrm{vap}}}{\alpha_{\mathrm{fmu}}^2 A_{\mathrm{fmu}}^2 + 4\alpha_{\mathrm{sv}}^2 A_{\mathrm{sv},0}^2}} \; . \tag{45}$$

(b) At $t = t'_3 - \frac{T_{\mathrm{c}}}{N_{\mathrm{c}}}$ the inlet valve of the second cylinder closes. The mass flow $\dot{m}_{\mathrm{vap}}$ into cylinder 1 is given by (37).

(c) At $t = \max(t_1, t'_2) + \frac{T_{\mathrm{c}}}{N_{\mathrm{c}}}$ the second inlet valve opens again. The mass flow is again given by $\dot{m}_{\mathrm{vap},2}$, see (45).

Integrating the mass flows into the cylinder over the complete time span $\max(t_1, t'_2)$ to t'_3 gives the subsequent equation for t'_3,

$$\begin{aligned} &(t'_3 - \max(t_1, t'_2))\dot{m}_{\mathrm{vap},2} + \left(\frac{2T_{\mathrm{c}}}{N_{\mathrm{c}}} - (t'_3 - \max(t_1, t'_2))\right)(\dot{m}_{\mathrm{vap}} - \dot{m}_{\mathrm{vap},2}) \\ &\qquad = \bar{\varrho}\frac{V_{\mathrm{ch}}}{2}\Big(\cos\big(N_{\mathrm{e}}\omega_{\mathrm{cr}} \max\left(t_1, t'_2\right)\big) - \cos(N_{\mathrm{e}}\omega_{\mathrm{cr}} t'_3)\Big) \; , \end{aligned} \tag{46}$$

which has to be solved numerically. The average mass flow $\bar{\dot{m}}_{\mathrm{sv}}$ is then given by

$$\bar{\dot{m}}_{\mathrm{sv}} = 2\bar{\dot{m}}_{\mathrm{sv},1} \; , \tag{47}$$

using $\bar{\dot{m}}_{\mathrm{sv},1}$ from (28) but replacing t_2 by t'_2 and t_3 by t'_3.

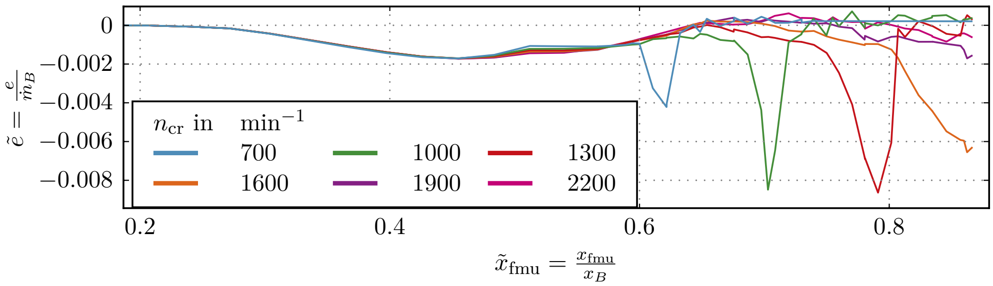

Figure 15. Error between the detailed and average model.

Fig. 15 shows the error between the detailed model derived in Section 2 and the average model of this section for the stationary mass flow into the rail. Here, the rail pressure is constant $p_{\mathrm{r}} = 1300\,\mathrm{bar}$, while six different crank shaft rotational speeds between $n_{\mathrm{cr}} = 700\,\mathrm{min}^{-1}$ and $n_{\mathrm{cr}} = 2200\,\mathrm{min}^{-1}$ are investigated. As already mentioned, a system with a 3-lobe profile camshaft and two pistons, i. e. $N_{\mathrm{c}} = 2$, $N_{\mathrm{e}} = 3$ is considered. It can be seen that the difference between the detailed and the average model is less than 0.01. Thus, it can be concluded that the average model constitutes an accurate approximation of the detailed model and thus serves as a suitable basis for the controller design.

[1] Note that the closing time $t'_3 - \frac{T_{\mathrm{c}}}{N_{\mathrm{c}}}$ of the inlet valve of the second cylinder can be directly obtained from the closing time t'_3 of the inlet valve of the first cylinder by using the fact that the time behavior of the second cylinder is equal to the time behavior of the first shifted by $\frac{T_{\mathrm{c}}}{N_{\mathrm{c}}}$.

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3.3 *Dynamics of mass flow*

The average model developed so far only describes the stationary behavior of the pump, assuming that a change in the position x_{fmu} of the FMU directly leads to a change in the mass flow $\bar{\dot{m}}_{\mathrm{sv}}$. In reality, however, there is a time delay between the intake of the diesel into the cylinder and the delivery into the rail. In order to study the dynamics of the pump, simulation results of the complete model are given in Fig. 16. Here it is assumed that the FMU position x_{fmu} can be directly assigned and the responses due to a step input of the FMU position are discussed. Figure 16 shows the results for the mass flow $\dot{m}_{\mathrm{hv}}$ of the complete model for 5 different steps, which are delayed relative to the top dead center of the first piston. Figure 16 reveals that if the step occurs at the top dead center of a piston, the mass flow into the rail is delayed by $T_{\mathrm{p}} = \frac{2\pi}{N_{\mathrm{c}} N_{\mathrm{e}} \omega_{\mathrm{cr}}}$ in the worst case. The exact value of the dead time depends on the opening of the FMU. If the step occurs between the dead centers then the dead time is reduced.

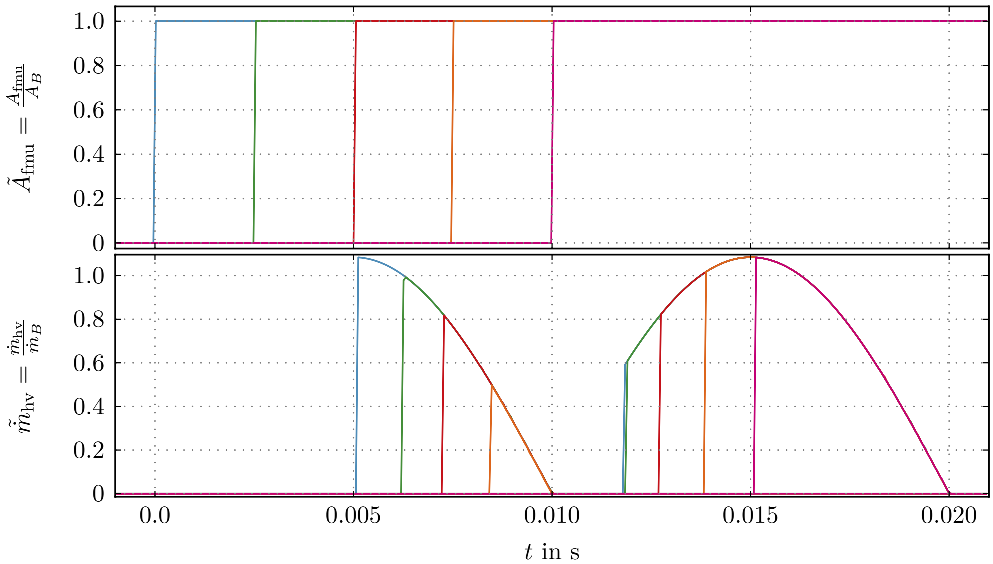

Figure 16. Simulation results of the detailed model for steps in the FMU position x_{fmu} at different times relative to the top dead center of piston 1.

An exact modeling of the dynamics is difficult within the framework of the average model, since the information of the exact piston shaft angular position would be necessary. This information is, however, not available in the real application (and thus for a controller design). So a simple approximation of the dynamics is made. In this work, the dynamic behavior is approximated by a dead time in the form

$$\bar{\dot{m}}_{\mathrm{hv}}(t) = \bar{\dot{m}}_{\mathrm{sv}}(t - T_{\mathrm{p}}) \;, \tag{48}$$

with the dead time $T_{\mathrm{p}} > 0$, $T_{\mathrm{p}} = \frac{2\pi}{N_{\mathrm{c}} N_{\mathrm{e}} \omega_{\mathrm{cr}}}$. Figure 17 depicts the resulting dynamics of the average model for the same step inputs as already used in Fig. 16. It has to be noted that the approximated dynamics represents the worst case in the sense that the real dynamics is not slower than the approximated one.

Dynamic measurements of the system for the validation of the model are very difficult to obtain in the considered application. First, most of the system variables are very difficult to be measured in the real application due to small available space.

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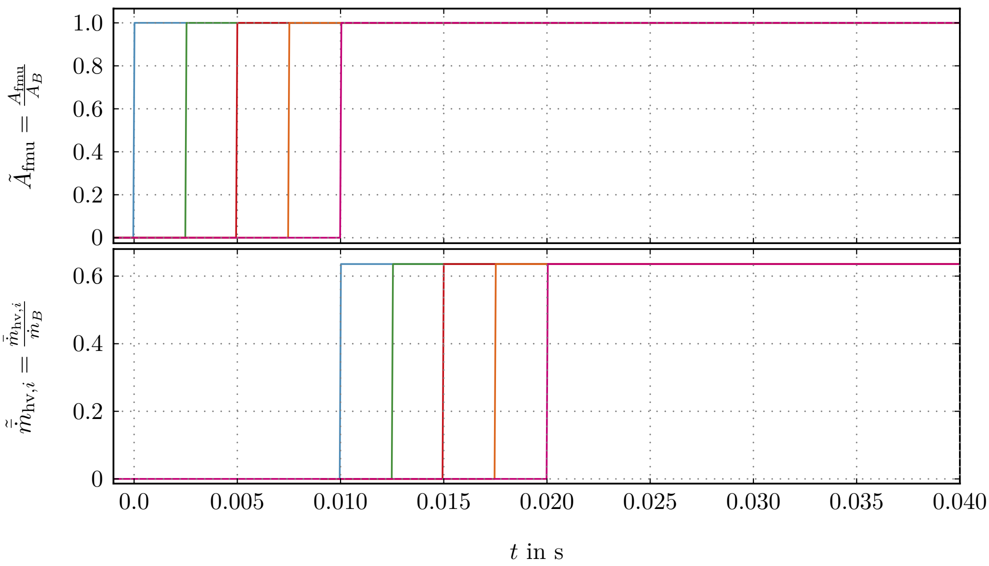

Figure 17. Approximated dynamics of the average model of the radial piston pump.

Thus, e. g. it is not possible to measure the position of the FMU and the cylinder and gallery pressures. Moreover, accurate and high dynamic measurement of mass flows is very difficult and even more complicated because of the high pressures arising in the system. Thus, only the stationary accuracy of the model could be validated by measurements.

3.4 *Overall average model*

The complete average model is given by (17), (15), (48), (25) in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t} f_m = \frac{-f_\mathrm{m} + \chi_\mathrm{m}\left(i_\mathrm{fmu}, x_\mathrm{fmu}\right)}{T_\mathrm{m}} \tag{49a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_\mathrm{fmu} = v_\mathrm{fmu} \tag{49b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\mathrm{fmu} = \frac{1}{m_\mathrm{v}}\left(f_\mathrm{m} - c_\mathrm{v}\left(x_\mathrm{fmu} - x_\mathrm{o}\right) - d_\mathrm{v} v_\mathrm{fmu}\right) \tag{49c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_\mathrm{r} = \frac{\beta_\mathrm{r}}{V_\mathrm{r}} \frac{\bar{\dot{m}}_\mathrm{sv}(t - T_\mathrm{p}) - \bar{\dot{m}}_\mathrm{inj}(t) - \dot{m}_\mathrm{l,r}(t)}{\varrho_\mathrm{r}}\ , \tag{49d}$$

with the average mass flow $\bar{\dot{m}}_\mathrm{sv}$ according to Section 3.2.

4. Conclusions

In this paper, a detailed mathematical model of a common-rail diesel system was derived. A special focus was laid on the accurate modeling of the radial piston pump in combination with the fuel metering unit, where it was shown that cavitation occurs in the cylinders of the pump. This effect is typically not taken into account analytically in mathematical models of the system but has a significant influence on the system's behavior. Based on this detailed model, an average model of considerably reduced complexity was derived. The stationary accuracy of both

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models was proven by means of measurement results of a real common-rail system.

One major advantage of the proposed model is that it can be easily parameterized by means of the geometric parameters of the system, not requiring extensive measurements. Thus, it is also possible to simulate different designs of the system (e. g. different size and number of pistons) and to estimate their performance without relying on extensive measurements. The average model also features these benefits and is, due to the reduced complexity, very well suited for a (nonlinear) controller design.

Current research is dealing with new model based control strategies based on the average model, where already first promising results have been obtained. These control strategies are currently further investigated by our project partner Robert Bosch GmbH.

Acknowledgements

The authors highly appreciate the support provided by Robert Bosch GmbH, in particular Matthias Bitzer, Johannes Nitzsche, Carolina Passenberg, Daniel Seiler-Thull and Adrian Trachte.

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