Energy-consistent shear coefficients for beams with circular cross sections and radially inhomogeneous materials

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Energy-consistent shear coefficients for beams with circular cross sections and radially inhomogeneous materials

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Abstract

An exact computational method for the shear stiffness of beams with circular cross sections and arbitrarily radially inhomogeneous Young's modulus is presented. We derive the displacement and stress field of a cantilever beam according to 3D theory of elasticity, which requires to solve just a 1D linear boundary value problem. The shear stiffness is obtained by setting the shear strain energy from the exact solution equal to that from technical beam theory. Results and closed analytical formulae are given for several functionally graded and layered cross sections.

Keywords: Shear correction factor, Shear stiffness, Shear deformation, Functionally graded materials, Circular cross section, Radial inhomogeneity, Saint-Venant solution, Analytical solutions, Timoshenko beam

1. Introduction

If beams with a low slenderness ratio are loaded by transverse forces, shear deformations may significantly contribute to their overall flexure. Such structures and load cases can be analyzed by means of Timoshenko's beam theory (Timoshenko, 1921, 1922), which requires the computation of shear coefficients. Various computational methods have been proposed for this purpose. However, there is still no consensus on the most accurate way of computing shear coefficients, especially if the beams are made of inhomogeneous materials.

The use of the finite element method to compute shear coefficients of general arbitrarily shaped or inhomogeneous cross sections is common practice (cf. the literature overview given in Section 2). The finite element method is a general numerical approach and its far-reaching applicability is unquestioned but for specific cross sections it may be possible and reasonable to use alternative methods. In the following, four good reasons are given why it is desirable to compute shear coefficients by means of either analytical methods or numerical approaches that are computationally less expensive than the finite element method:

- Closed-form analytical solutions can be used as benchmark results, e. g., for verifying finite element codes.
- Analytical solution methods do not require meshing and analysis of the discretization error that may reveal the need for grid refinement.
- An analytical solution process usually provides deeper insight into the nature of the respective problem than application of black-box numerical methods.
- In real-time applications like control, computer power may still be a limiting factor, which requires the application of tailored, highly efficient mathematical models.

In the current paper, we explore whether circular cross sections allow a closed-form or at least a simplified solution for the computation of the shear stiffness. Our aims are as follows:

- A tractable general method to compute the shear stiffness of circular cross sections is to be developed.
- The method should be applicable to cross sections with arbitrarily radially inhomogeneous isotropic materials.
- It should be applicable to both solid and hollow cross sections.
- The method should not rely on first-order beam theory, the computation of mean displacements, the assumption that certain components of the stress tensor vanish, or other restrictive assumptions concerning the deformation of cross sections.
- The method is to be verified by comparing the results with shear coefficients available in the literature.

Our research is motivated by an application in shape control of rolling mills (Ginzburg, 2009). The deflection of the rolls can be conveniently computed based on Timoshenko's beam theory. Typical rolls, especially the back-up rolls of four-high mills, have a layered circular cross section. As indicated in Fig. 1, the core is rather soft whereas the shell is made of hard and wear-resistant steel. The diameter of the rolls decreases over the time because of wear and regular machining with grinding wheels. More-

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Nomenclature

Latin symbols

A area, usually cross-sectional area (m^2)
$B(r)$ function that adds to the warping displacement along the direction x_3 (m^2)
$\boldsymbol{b}_0$, $\boldsymbol{b}_1$ vectors used to map between boundary values
b_0, b_1 integration constants in an expression for $B(r)$
C^0 space of continuous functions
$E(r)$ Young's modulus ($\mathrm{N/m}^2$)
$\bar{E}$ reference value of Young's modulus used for normalization ($\mathrm{N/m}^2$)
E_i homogeneous Young's modulus of layer i in the range (r_{i-1}, r_i) ($\mathrm{N/m}^2$)
e_0, e_1 parameters used for defining the shape of $E(r)$
F force at the end of the beam along the direction x_1 (N)
$G(r)$ shear modulus ($\mathrm{N/m}^2$)
i index, either in the range $1, 2, 3$ or $1, \ldots, N$ ($-$)
j index, usually in the range $1, 2, 3$ ($-$)
K_B bending stiffness (Nm^2)
K_S shear stiffness (N)
$\bar{K}_S$ reference value of shear stiffness (N)
k_0, k_1 constants used for defining the characteristic structure of K_S ($-$)
m, m_0, m_1 ratios of radii ($-$)
N number of layers ($-$)
n ratio of Young's moduli ($-$)
r radius (m)
r_0 inner radius of a hollow cross section (m)
r_i outer radius of layer i (m)
r_N outer radius of a cross section (m)
r, θ, x_3 cylindrical coordinates
t thickness of a thin-walled annular beam (m)
u_i displacement along the Cartesian direction x_i (m)
x_1, x_2, x_3 Cartesian coordinates (m)

Greek symbols

ε_{ij} strain with $i, j = 1, 2, 3$ or $i, j = r, \theta, 3$ ($-$)
θ angle (rad)
ν Poisson's ratio ($-$)
σ_{ij} stress with $i, j = 1, 2, 3$ or $i, j = r, \theta, 3$ ($\mathrm{N/m}^2$)
Ω cross section
$\xi(r)$ abbreviation for $E'(r)/E(r)$ (1/m)

Operators

$(\bullet)'$ total derivative with respect to r (1/m)
$(\bullet)_{,i}$ partial derivative with respect to a spatial direction i with $i = 1, 2, 3$ or $i = r, \theta, 3$

Superscripts

$+$ short notation for a right-hand limit
$-$ short notation for a left-hand limit

over, the temperature distribution in the roll is generally inhomogeneous (Guerrero et al., 1999), which may also influence the local material parameters.

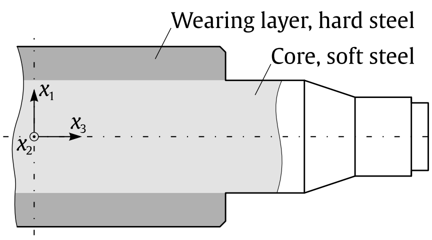

Figure 1: Back-up roll of a rolling mill.

This paper is organized as follows: Section 2 provides a brief literature overview. In Section 3, the 3D Saint-Venant flexure problem is solved for cylindrical beams of radially inhomogeneous circular cross sections. In Section 4, we describe the analysis of shear coefficients based on equal strain energies and apply the method to various cross sections.

2. Background

Due to the large number of existing publications in the field, a comprehensive overview of the relevant literature would exceed the scope of this paper. Hence, the following outline of the extensive body of available knowledge should merely be viewed as a possible starting point for further exploration.

The basic beam theory that takes into account shear deformations is attributed to Timoshenko (1921, 1922). According to this theory, the shear deformation can be characterized by an angular rotation of the cross section and so-called shear coefficients, which are also referred to as shear correction or shear deformation factors. Meanwhile, many refinements of Timoshenko's shear deformation theory have been published, e. g., the consideration of out-of-plane displacements (warping) in addition to a rotation of the cross section. However, there is still no consensus in the literature on the most appropriate calculation method of shear coefficients (cf. for instance, Kaneko, 1975; Renton, 1991; Hutchinson, 2001; Stephen, 2001; Dong et al., 2010; Kennedy et al., 2011; Mentrasti, 2012). As outlined in the following, there are several methods of computing such coefficients. Some definitions of shear coefficients inherently depend on the specific problem formulation (Men-

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trasti, 2012), e. g., whether a static or a dynamic problem is considered (Dong et al., 2010).

In technical beam theory, a beam is formulated as a 1D Cosserat continuum (Cosserat and Cosserat, 1909), i. e., each point of the continuum is characterized by translational degrees of freedom and it may have additional (aggregate) degrees of freedom like rotation or warping of the local cross section. Most authors use shear coefficients to reconcile results from technical beam theory, often first-order beam theory, with full 3D theory-of-elasticity solutions. The derivation of shear coefficients thus requires to match in terms of some gross response characteristics the solution from technical beam theory with the solution from the theory of elasticity. Typically matching criteria are

- equal average displacement values (Cowper, 1966; Stephen and Levinson, 1979),
- equal natural frequencies (Timoshenko, 1922; Kaneko, 1975; Hutchinson, 1981), or
- equal shear strain energies (Bach and Baumann, 1924; Renton, 1991; Pilkey, 2002; Mentrasti, 2012).

Cowper (1966) solved the 3D theory-of-elasticity problem for a homogenous, isotropic, tip-loaded or uniformly loaded cantilever. To describe the out-of-plane displacement of a shear-loaded cross section, Cowper (1966) used classical flexure functions reported for standard cross sections, for instance, in (Love, 1944; Sokolnikoff, 1956). As most authors in this field, (Cowper, 1966) assumed that the shear force varies only continuously along the beam. Cowper (1966) derived a shear coefficient by matching shear rotation angles from displacement averages with shear rotation angles from technical beam theory. For rectangular cross sections, this approach yields a shear coefficient that is independent of the aspect ratio of the rectangle. Dharmarajan and McCutchen Jr. (1973) extended Cowper's (1966) method for homogeneous orthotropic beams.

Based on a second-order beam theory and average displacement values, Stephen and Levinson (1979) derived two analytical shear coefficients. In their analysis, Stephen and Levinson (1979) considered gravity loading and stresses from classical flexure problems solved by Love (1944). Stephen and Levinson (1979) also computed natural frequencies and argued that a good agreement of frequencies does not automatically ensure accurate displacement and stress values. Therefore, Stephen and Levinson (1979) suggested further analyses based on the use of average displacements. Stephen (1980) computed shear coefficients by comparing the curvature of the average displacements of a gravity-loaded beam with the center-line curvature according to Timoshenko's beam theory. For a circular cross section, Stephen (1980) obtained the same shear coefficient as Timoshenko (1922), which was experimentally identified by Kaneko (1975) to be the most accurate expression for shear coefficients.

Hutchinson (1981) computed shear coefficients for a circular cross section by means of a series solution and the frequency matching approach. From a comparison with other published shear coefficients, Hutchinson (1981) inferred that Timoshenko's (1921) shear coefficient is usually the most accurate.

Based on stress functions from classical flexure problems, Renton (1991) computed a 3D theory-of-elasticity solution for homogeneous isotropic beams. To compute shear coefficients, he matched the shear strain energy with the work done by the shear force according to technical beam theory. For some typical cross sections, Renton (1991) derived analytical solutions for the shear coefficient. From the structure of the attained solutions, Renton (1991) concluded that the shear stiffness K_S of homogeneous isotropic cross sections based on equal strain energies has the general form

$$K_S = \frac{GA}{k_0 + k_1\left(\dfrac{\nu}{1+\nu}\right)^2} = \frac{1}{2}\frac{EA(1+\nu)}{k_0 + 2k_0\nu + (k_0+k_1)\nu^2}, \tag{1}$$

where $G = E/(2(1+\nu))$ is the shear modulus, E denotes Young's modulus, A is the cross-sectional area, ν is Poisson's ratio, and the constants $k_0 \geq 1$ and $k_1 \geq 0$ depend on the shape of the cross section. Renton (1997) proved this result for simply connected homogeneous cross sections.

Pai and Schulz (1999) argued that the physical meaning of both shear rotation angles and shear coefficients are not well defined in the literature. They derived shear coefficients for homogeneous isotropic beams analytically by explicit computation of shear warping functions and by introduction of four different shear rotation angles: one defined at the centroid of the cross-sectional area, one associated with displacement averages of shear strains, another one with energy averages of shear strains, and yet another one associated with coupled energy averages of shear strains. Pai and Schulz (1999) calculated shear coefficients by matching the shear strain energy from exact theory-of-elasticity solutions with the shear strain energy from technical beam theory. Therefore, Pai and Schulz (1999) considered their shear coefficients as *energy-consistent*. Apart from a more general formulation, the approach of Pai and Schulz (1999) is the same as that of Renton (1991). For circular cross sections, they obtained the same results for the shear coefficient.

Hutchinson (2001) assumed a displacement field where the cross sections of the beam remain plane. He used stresses from the classical flexure solutions of Love (1944) and employed the dynamic form of the Hellinger-Reissner principle (Reissner, 1950) to overcome the incompatibility of the assumed displacements and the stress field. Finally, Hutchinson (2001) computed a shear deflection coefficient by matching the vibration frequencies with that of Timoshenko's formulation. Stephen (2001) demonstrated that Hutchinson's (2001) shear coefficient is equivalent to that of Stephen (1980).

Egretzberger et al. (2007) used average displacement values for computing shear coefficients of a rectangular

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beam made of a homogeneous, orthotropic material. They computed the stress field based on classical flexure functions reported in (Love, 1944).

Based on first-order beam theory and the equivalence of shear strain energies, Madabhusi-Raman and Davalos (1996) analytically computed shear coefficients for layered orthotropic beams with rectangular cross sections. They neglected the transverse shear stress σ_{23} (cf. Fig. 1). In the tradition of laminar plates and shells, Noor and Peters (1989) analyzed laminated orthotropic cylindrical shells. They estimated shear coefficients for multilayered cylindrical shells by means of first-order theory, a predictor-corrector approach, and equivalent shear strain energies. The accuracy of these estimates depends also on the thickness-to-radius ratio of the hollow cylinder.

Li (2008) analyzed functionally graded and layered isotropic beams of rectangular cross section. Li (2008) assumed that beam cross sections remain plane and neglected displacements, strains, and stresses along the direction x_2 (cf. Fig. 1). That is, Li (2008) conducted a 2D analysis in the x_1x_3-plane. Consequently, the obtained results do not depend on Poisson's ratio and are invariant with respect to the width of the beam along the direction x_2.

Reddy (2011) developed a theory for bending, vibration, and buckling of inhomogeneous rectangular beams with a through-thickness power-law variation of the material. By means of a modified couple-stress theory, the method takes into consideration microstructural effects, i. e., size effects of the material. Reddy (2011) also neglected the displacements along the direction x_2 (cf. Fig. 1).

Chan et al. (2011) reported a frequency matching method for computing shear coefficients of homogeneous isotropic beams with arbitrary cross sections. They assumed that beam cross sections always remain plane, derived a truncated series solution of the elastodynamics equations, and computed natural frequencies based on the Rayleigh quotient.

Favata et al. (2010) proved that 1 is a strict upper bound for shear coefficients of homogeneous isotropic cross sections. Mentrasti (2012) confirmed this result and added a rather conservative lower bound based on the notion of a residual stress field. For solid circular cross sections, Mentrasti (2012) suggested $1/2$ as the lower bound.

By matching average displacement values, Kennedy et al. (2011) computed analytical shear coefficients for orthotropic layered beams of rectangular cross section. They used average displacement quantities and, similar to Li (2008), they considered a plane-stress state with stresses occurring only in the x_1x_3-plane. It is not clear how the implicit assumption $\sigma_{23} = 0$ influences the accuracy of the computed shear coefficient. Using the 2D finite element method, Kennedy and Martins (2012) abandoned the assumption of a plane-stress state and numerically computed shear coefficients for anisotropic layered beams.

A host of publications show how finite element analyses can serve as numerical vehicles for computing shear coefficients. The references (Schramm et al., 1994; Pilkey, 2002; Dong et al., 2010) and the references given therein are potential points of departure for further exploring this strand of research. Wörndle (1982); Wörndle and Mang (1984); Gruttmann and Wagner (2001); Dong et al. (2001); Kosmatka et al. (2001); and Dong et al. (2010) used 2D finite element methods for computing out-of-plane displacements (warping functions) along the axis of arbitrarily shaped cross sections. Gruttmann and Wagner (2001) considered homogeneous isotropic beams and computed shear coefficients based on equal strain energies. Dong et al. (2010) also analyzed homogeneous isotropic beams but computed the shear coefficients based on average displacements. Wörndle (1982) and Wörndle and Mang (1984) developed a method that works for inhomogeneous orthotropic beams; it is also based on average displacements. Dong et al. (2001) analyzed inhomogeneous, anisotropic beams and computed cross-sectional stiffness matrices (Dong et al., 2001) and properties (Kosmatka et al., 2001). Liu and Taciroglu (2008) extended these results to piezoelectric materials and used a meshfree discretization scheme based on shape functions spanning the whole cross section.

Schramm et al. (1994) demonstrated that for nonsymmetrical cross sections the computation of shear coefficients based on average displacement values can result in a non-symmetrical matrix of shear coefficients. This problem does not occur if the shear strain energy is used as a matching criterion. Therefore, Schramm et al. (1994) and Pilkey (2002) advocated this approach and used it in 2D finite element analyses of homogeneous isotropic beams. Later, Dong et al. (2010) made the following recommendations for a correct computation of shear coefficients of non-symmetrical cross sections: First, a coordinate system that corresponds to the principal bending directions should be used. Second, two individual calculations, each with a single transverse force along a principal bending direction, should be performed to compute shear coefficients associated with the principal bending directions. The shear coefficients obtained in this manner have the properties of a second-rank tensor and can thus be easily transformed to other coordinate systems.

3. Flexure problem for radially inhomogeneous circular cross sections

The 3D flexure problem of the tip-loaded, circular cylindrical, radially inhomogeneous cantilever beam shown in Fig. 2 will be treated. We consider a static scenario without body forces and without surface tractions along the radial surface. Young's modulus $E(r)$ may be radially inhomogeneous whereas Poisson's ratio ν is taken as constant, which is a reasonable assumption for most metallic materials. The cross section Ω may be hollow and its centroid defines the beam axis. The length of the beam is not relevant for the present investigation. We make use of Saint-Venant's principle, i. e., the boundary conditions

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at the ends of the beam are satisfied in an integral sense rather than point by point.

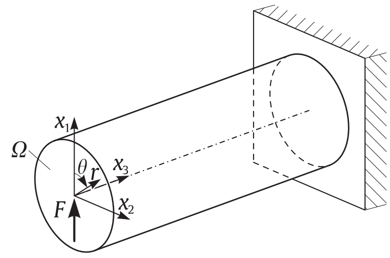

Figure 2: Cantilever beam.

We will use Cartesian coordinates x_1, x_2, and x_3 as well as cylindrical coordinates r, θ, and x_3. Obviously, the axes x_1 and x_2 are principal bending and principal shear axes. Because of rotational symmetry, it suffices to study the case of a single transversal load F passing through the centroid of the cross section along the direction x_1. Non-centered loads would additionally induce torsion, which is trivial for the considered circular cross sections.

3.1. Displacements and equilibrium conditions

Inspired by the classical solution of the Saint-Venant flexure problem (cf. Love, 1944; Sokolnikoff, 1956; Ieşan, 2009), the displacements of the tip-loaded beam along the directions x_1, x_2, and x_3 can be formulated as

$$u_1 = \frac{Fx_3}{2K_B}\Big(\frac{x_3^2}{3} + \nu(x_1^2 - x_2^2)\Big) - \frac{Fx_3}{K_S} \tag{2a}$$

$$u_2 = \frac{F}{K_B}\nu x_1 x_2 x_3 \tag{2b}$$

$$u_3 = \frac{Fx_1}{2K_B}\Big(\frac{x_1^2 + x_2^2}{2} - x_3^2 + \Big(\frac{3}{2} + \nu\Big)B(r)\Big) + \frac{Fx_1}{K_S}, \tag{2c}$$

respectively, with the unknown expression $B(r) \in C^0$, which generally has negative values, the radius $r = \sqrt{x_1^2 + x_2^2}$, the bending stiffness

$$K_B = \int_\Omega E(r) x_1^2 \mathrm{d}A, \tag{3}$$

and the yet unknown shear stiffness K_S of the cross section. Equ. (2) defines the displacements modulo some rigid body motions, which are irrelevant for the current analysis. The following derivation will show, as a byproduct, that (2) is indeed a solution of the flexure problem.

Specialization of (2) for $x_1 = x_2 = 0$ gives the displacements of the beam axis. $B(r)$ adds to the warping displacement along the direction x_3. Later, we will see that $B(r)$ depends only on the cross section and the distribution of $E(r)$ but neither on the load F nor on Poisson's ratio ν. To find $B(r)$, we first compute the strains $\varepsilon_{ij} = {}^1\!/\!{}_2(u_{i,j} + u_{j,i})$ with $i, j = 1, 2, 3$ and $(\bullet)_{,i} = \partial(\bullet)/\partial x_i$. They follow as

$$\begin{aligned}
\varepsilon_{11} &= \varepsilon_{22} = \frac{F}{K_B}\nu x_1 x_3 \\
\varepsilon_{33} &= -\frac{F}{K_B} x_1 x_3 \\
\varepsilon_{23} &= \frac{F}{2K_B} x_1 x_2 \Big(\frac{1}{2} + \nu + \Big(\frac{3}{2} + \nu\Big)\frac{B'(r)}{2r}\Big) \\
\varepsilon_{13} &= \frac{F}{4K_B}\Big(\Big(\frac{1}{2} - \nu\Big)x_2^2 \\
&\qquad + \Big(\frac{3}{2} + \nu\Big)\Big(x_1^2 + B(r) + \frac{B'(r)}{r}x_1^2\Big)\Big) \\
\varepsilon_{12} &= 0,
\end{aligned}$$

where $(\bullet)' = \mathrm{d}(\bullet)/\mathrm{d}r$, and their transformation to cylindrical coordinates yields

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = \frac{F}{K_B}\nu r\cos(\theta) x_3 \tag{4a}$$

$$\varepsilon_{33} = -\frac{F}{K_B} r\cos(\theta) x_3 \tag{4b}$$

$$\varepsilon_{\theta 3} = -\frac{F\sin(\theta)}{4K_B}\Big(\Big(\frac{1}{2} - \nu\Big)r^2 + \Big(\frac{3}{2} + \nu\Big)B(r)\Big) \tag{4c}$$

$$\varepsilon_{r3} = \frac{F\cos(\theta)}{4K_B}\Big(\frac{3}{2} + \nu\Big)(r^2 + B'(r)r + B(r)) \tag{4d}$$

$$\varepsilon_{r\theta} = 0. \tag{4e}$$

As this strain field has been derived from the displacements (2), it satisfies the compatibility conditions. With Hooke's law for isotropic material, the stresses follow as

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{r\theta} = 0 \tag{5a}$$

$$\sigma_{33} = -\frac{E(r)F}{K_B} r\cos(\theta) x_3 \tag{5b}$$

$$\sigma_{\theta 3} = -\frac{E(r)F\sin(\theta)}{4K_B(1+\nu)}\Big(\Big(\frac{1}{2} - \nu\Big)r^2 + \Big(\frac{3}{2} + \nu\Big)B(r)\Big) \tag{5c}$$

$$\sigma_{r3} = \frac{E(r)F\cos(\theta)}{4K_B(1+\nu)}\Big(\frac{3}{2} + \nu\Big)(r^2 + B'(r)r + B(r)). \tag{5d}$$

This stress field trivially satisfies the equilibrium conditions along the directions r and θ. It remains to be shown that the equilibrium condition along the direction x_3, i. e.,

$$\sigma_{r3,r} + \frac{\sigma_{r3}}{r} + \frac{\sigma_{\theta 3,\theta}}{r} + \sigma_{33,3} = 0, \tag{6}$$

is also satisfied and that surface tractions vanish. These conditions will determine the unknown function $B(r)$.

Assuming for the time being that $E(r) \in C^0$, i. e., that Young's modulus is continuous, and substituting (5) into (6) yields

$$r^2\xi(r) + \xi(r)B(r) + (3 + r\xi(r))B'(r) + rB''(r) = 0 \tag{7}$$

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with the abbreviation

$$\xi(r) = \frac{E'(r)}{E(r)} = \frac{\mathrm{d}}{\mathrm{d}r}\ln(E(r)).$$

In Sections 3.2 to 3.4, we supplement this linear ordinary differential equation with boundary conditions.

It is not obvious and it is one of the key findings of this work that the 3D flexure problem of a beam with an inhomogeneous cross section reduces to a 1D linear boundary value problem. The reason for this remarkable reduction of the number of dimensions is that we have considered inhomogeneities only in radial direction. In the case of more general inhomogeneities, the Saint-Venant flexure problem results in a 2D boundary value problem (cf. Wörndle, 1982; Ieşan and Quintanilla, 2007), which is considerably more difficult to solve than (7).

In Section 4, we will explore various scenarios where (7) can be analytically solved. At the end of the current section, we will show that general scenarios which do not permit an analytical solution of (7) can easily be treated numerically.

3.2. Free surface

Consider that the circular cross section has a free surface at the radius $r = r_s$. No matter whether this is an inner or an outer surface, the surface tractions must vanish, which implies

$$\sigma_{rr}|_{r=r_s} = 0$$
$$\sigma_{r\theta}|_{r=r_s} = 0$$
$$\sigma_{r3}|_{r=r_s} = 0.$$

Based on (5d), we thus get

$$r_s^2 + B'(r_s)r_s + B(r_s) = 0 \tag{8a}$$

for any free surface.

3.3. Center

In case of a solid cross section, (8a) is only applicable at the outer surface and generally $\sigma_{r3}|_{r=0} \neq 0$. The required additional boundary condition is found by evaluating (7) for $r = 0$. This yields

$$\xi(0)B(0) + 3B'(0) = 0. \tag{8b}$$

3.4. Discontinuous Young's modulus

Finally, we abandon the assumption $E(r) \in C^0$ and allow for discontinuities of $E(r)$. Consider that $E(r)$ is discontinuous at $r = r_i$, i.e., $E(r_i^-) \neq E(r_i^+)$ with the short notation $E(r_i^\pm) = \lim_{\rho\to 0^+} E(r_i \pm \rho)$. We may think of r_i as a perfectly bonded material interface.

Because of (2c) and the condition $B(r) \in C^0$, we get the continuity condition

$$B(r_i^-) = B(r_i^+) = B(r_i). \tag{8c}$$

The second required boundary condition is found from the continuity conditions of the stress field. They imply

$$\lim_{r\to r_i^-} \sigma_{rr} = \lim_{r\to r_i^+} \sigma_{rr}$$
$$\lim_{r\to r_i^-} \sigma_{r\theta} = \lim_{r\to r_i^+} \sigma_{r\theta}$$
$$\lim_{r\to r_i^-} \sigma_{r3} = \lim_{r\to r_i^+} \sigma_{r3}.$$

Based on (5d), we thus get

$$\begin{aligned} &E(r_i^-)((r_i)^2 + B'(r_i^-)r_i + B(r_i)) \\ &\quad = E(r_i^+)((r_i)^2 + B'(r_i^+)r_i + B(r_i)). \end{aligned} \tag{8d}$$

The same result is obtained if (7) is multiplied by $E(r)$ and integrated in the range $(r_i - \rho, r_i + \rho)$ with $\rho \to 0^+$.

3.5. Solution of the boundary value problem

We can now solve (7) with the appropriate boundary conditions (8) depending on the respective cross section. As expected, (7) and (8) are independent of the load F and of Poisson's ratio ν. Consider a scenario as shown in Fig. 3. $E(r)$ is arbitrarily inhomogeneous and has several discontinuities at the radii r_i with $i = 1, \ldots, N-1$. The inner and the outer radius of the cross section is r_0 and r_N, respectively. In case of a solid cross section, $r_0 = 0$.

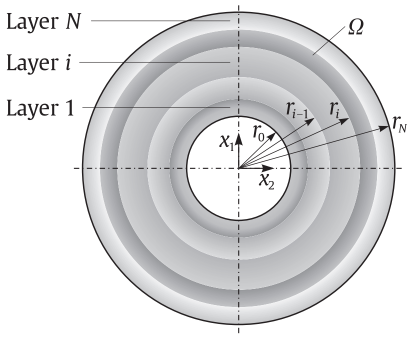

Figure 3: Inhomogeneous circular cross section.

To compute the unknown function $B(r) \in C^0$ from the linear multi-point boundary value problem (7) and (8), we can conveniently use the single shooting method (cf. Stoer and Bulirsch, 2002). The calculation proceeds as follows:

a) Make an arbitrary initial guess $B(r_0)$ and compute

$$B'(r_0) = \begin{cases} -r_0 - \frac{B(r_0)}{r_0} & \text{if } r_0 > 0 \\ -\frac{\xi(r_0)}{3}B(r_0) & \text{if } r_0 = 0 \end{cases}$$

according to (8a) and (8b).

b) With known values $B(r_0)$ and $B'(r_0)$, (7) constitutes an initial value problem. Integrate it, e.g., numerically, and use

$$B(r_i^-) = B(r_i^+) = B(r_i)$$

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$$B'(r_i^+) = \frac{E(r_i^-)}{E(r_i^+)} B'(r_i^-) + \Big(\frac{E(r_i^-)}{E(r_i^+)} - 1\Big)\Big(r_i + \frac{B(r_i)}{r_i}\Big)$$

according to (8c) and (8d) at the interface positions r_i with $i = 1, \ldots, N-1$.

c) Repeat steps a) and b) for a different arbitrary initial guess $B(r_0)$ and from the resulting two triplets $(B(r_0), B(r_N), B'(r_N))$ compute the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_0$ of the affine function

$$\begin{bmatrix} B(r_N) \\ B'(r_N) \end{bmatrix} = \boldsymbol{b}_1 B(r_0) + \boldsymbol{b}_0. \tag{9}$$

d) Compute the correct values $B(r_0)$, $B(r_N)$, and $B'(r_N)$ from the linear system (9) and the boundary condition

$$r_N^2 + B'(r_N) r_N + B(r_N) = 0$$

(cf. (8a)). With the correct initial value $B(r_0)$, repeat steps a) and b) once more to compute the solution $B(r)$.

This procedure yields $B(r)$ for general cross sections with arbitrarily varying $E(r)$. Therefore, the 3D flexure problem is solved. The result $B(r)$ is *exact* up to numerical accuracy, which may, for instance, be limited when solving (7) by some numerical integrator.

4. Shear stiffness of radially inhomogeneous circular cross sections

We compute the shear stiffness K_S for an inhomogeneous cross section like that shown in Fig. 3. For homogeneous cross sections, the calculation of K_S is tantamount to the computation of a shear coefficient $K_S/(GA)$. As mentioned in Section 2, there are several possible criteria for matching the gross response characteristics according to technical beam theory with 3D theory-of-elasticity solutions like the one computed in the previous section. In this paper, we adopt the idea of equal shear strain energies (Bach and Baumann 1924; Renton 1991; Schramm et al. 1994; Madabhusi-Raman and Davalos 1996; Pai and Schulz 1999; Gruttmann and Wagner 2001; Pilkey 2002; Favata et al. 2010; Mentrasti 2012). The motivation for this choice is that we consider a static problem, where it is known that the exact solution of the Saint-Venant flexure problem minimizes the total strain energy (Sternberg and Knowles, 1966; Ericksen, 1980; Ieşan, 2009). It is thus a reasonable conjecture that parameterizing the shear stiffness such that the same total strain energy is obtained will result in an accurate technical beam formulation.

Implicit to this conjecture is the assumption of independence of the strain energies from bending and torsion. This assumption is attributed to Trefftz (1935), and it is naturally satisfied for symmetrical cross sections.

Another assumption of our approach is that the shear stiffness obtained for the cross section of a tip-loaded cantilever remains valid also for other load cases, e. g., distributed loads in the form of body forces or surface tractions. This assumption rests on the observation that the distribution of the shear stresses within the cross section is the same for a tip-loaded beam and a beam with uniformly distributed load (Love, 1944). The tenability of this assumption was, for instance, discussed by Cowper (1966); Stephen and Levinson (1979); Stephen (1980); and Dong et al. (2010).

4.1. Arbitrary radial inhomogeneity

In Timoshenko's beam theory, the elastic deformation energy per unit length x_3 caused by the shear force F is

$$\frac{F^2}{2K_S}. \tag{10}$$

This follows directly from (2a) because the extra displacement along the direction x_1 per unit length x_3 induced by shear is F/K_S. The shear strain energy from the 3D theory-of-elasticity solution is

$$\int_\Omega (\sigma_{\theta 3}\varepsilon_{\theta 3} + \sigma_{r3}\varepsilon_{r3}) \mathrm{d}A. \tag{11}$$

Setting (10) equal to (11) and using (4), (5), (7), and (8a) yields

$$K_S = \frac{-\frac{4}{\pi} K_B^2 (1+\nu)}{\int_{r_0}^{r_N} E(r) r^3 \Big((1+2\nu) r^2 + \Big(3 + 6\nu + \frac{8}{3}\nu^2\Big) B(r)\Big) \mathrm{d}r} \tag{12}$$

with

$$K_B = \pi \int_{r_0}^{r_N} E(r) r^3 \mathrm{d}r \tag{13}$$

according to (3). The derivation of (12) is described in more detail in the Appendix.

K_S according to (12) has the structure given by (1), i. e., in the denominator, the coefficient of ν is twice as large as the coefficient of ν^0. This shows that the shear stiffness of an isotropic, radially inhomogeneous circular cross section as shown in Fig. 3 has always the characteristic form (1), which was found by Renton (1991, 1997) for homogeneous cross sections.

This concludes the computation of the shear stiffness of a general circular cross section. In the following, we analyze some special cross sections.

4.2. Several homogeneous layers

Consider a circular cross section with layers of homogeneous materials. Fig. 3 is still applicable but Young's modulus has a constant value $E(r) = E_i$ in each layer (r_{i-1}, r_i) with $i = 1, \ldots, N$. According to (13), we thus get

$$K_B = \frac{\pi}{4} \sum_{i=1}^{N} E_i (r_i^4 - r_{i-1}^4).$$

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Moreover, (7) can be analytically integrated, which gives

$$B(r) = B(r_{i-1}) + \frac{r_{i-1}}{2}\Big(1 - \frac{r_{i-1}^2}{r^2}\Big)B'(r_{i-1}^+) \tag{14a}$$

$$B'(r) = \frac{r_{i-1}^3}{r^3}B'(r_{i-1}^+) \tag{14b}$$

for the layer i, i.e., for $r \in (r_{i-1}, r_i)$. If $r_{i-1} = 0$ (innermost layer of a solid cross section), the second term in (14a) vanishes and $B'(r) = 0$ holds for this layer. Hence, the integration in step b) of the algorithm given in Section 3.5 can be carried out analytically. Finally, evaluation of (14) for $r = r_i$ with $i = 1, \ldots, N$ and consideration of (8) yields a linear equation for all unknown boundary values.

To compute the shear stiffness K_S, note that the integral in (12) can be evaluated separately for each layer $i = 1, \ldots, N$. Substitution of (14a) into (12) yields

$$K_S = -\frac{4}{\pi}K_B^2(1+\nu)\Bigg(\sum_{i=1}^{N} E_i\bigg((1+2\nu)\frac{r_i^6 - r_{i-1}^6}{6} + \Big(3+6\nu + \frac{8}{3}\nu^2\Big)\big(B(r_{i-1})r_{i-1}^2 + B(r_i)r_i^2\big)\frac{r_i^2 - r_{i-1}^2}{4}\bigg)\Bigg)^{-1}. \tag{15}$$

4.3. Other cross sections permitting an analytical solution

By analogy to the previous section, closed-form analytical expressions for the shear stiffness of cross sections with inhomogeneous layers can be computed if an analytical solution of (7) is available for each individual layer. That is, the inhomogeneity may be different in each layer, which implies that the integrals in (12) and (13) are evaluated separately for each layer. As the principle is analogous to what was shown in the previous section, we omit the expressions for K_S and K_B and discuss just two simple types of inhomogeneities.

Consider that Young's modulus of a layer is

$$E(r) = e_0 e_1^r,$$

where $e_0 > 0$ and $e_1 > 0$ are arbitrary constants. Such an exponential inhomogeneity has also been studied by Lekhnitskii (1981) and Ieşan and Quintanilla (2007). If $e_1 = 1$, we have the situation analyzed in Section 4.2 with the solution (14a). For $e_1 \neq 1$, since $\xi(r) = \ln(e_1)$, (7) has the straightforward solution

$$B(r) = b_0\frac{r\ln(e_1) - 1}{r^2} + b_1\frac{1}{e_1^r r^2} - \frac{8}{r^2\ln(e_1)^4} + \frac{8}{r\ln(e_1)^3} - \frac{4r}{\ln(e_1)^2} + \frac{4}{3\ln(e_1)} - \frac{r^2}{3}. \tag{16}$$

The integration constants b_0 and b_1 are determined by the boundary conditions of the respective layer. If the inner boundary of the layer is $r = 0$, there are additional constraints on b_0 and b_1 to avoid singularities.

As a second example of an inhomogeneity allowing an analytical solution, consider that Young's modulus of a layer is

$$E(r) = e_0 r^{e_1},$$

where $e_0 > 0$ and e_1 are arbitrary constants. Clearly, for $e_1 > 0$, the inner boundary of the layer cannot be $r = 0$. Since $\xi(r) = e_1/r$, (7) has the straightforward solution

$$B(r) = \frac{b_0 r^{-\sqrt{1/4e_1^2+1}} + b_1 r^{\sqrt{1/4e_1^2+1}}}{r^{1/2e_1+1}} - \frac{e_1 r^2}{8+3e_1}, \tag{17}$$

where b_0 and b_1 are again determined by boundary conditions. The solution for a homogeneous layer is obtained by setting $e_1 = 0$. Note that both (16) and (17) are independent of e_0 because only the ratio $\xi(r) = E'(r)/E(r)$ enters (7). We may generally infer that $B(r)$ depends only on the shape of $E(r)$ but not its absolute value.

4.4. Homogeneous circle

If we apply the method described in Section 4.2 to a homogeneous solid circular cross section with the radius r_1 ($N = 1$, $r_0 = 0$), we obtain

$$B(r) = -r_1^2$$

and

$$K_S = \frac{3AE(1+\nu)}{7 + 14\nu + 8\nu^2} \tag{18}$$

with the cross sectional area $A = r_1^2\pi$. This result was also reported by Renton (1991) and Pai and Schulz (1999).

4.5. Homogeneous annulus

If we apply the method outlined in Section 4.2 to a homogeneous annular cross section with the inner radius r_0 and the outer radius r_1 ($N = 1$, $r_0 > 0$), we obtain

$$B(r) = -r_0^2 - r_1^2 - \frac{r_0^2 r_1^2}{r^2}$$

and thus

$$K_S = \frac{3AE(1+\nu)(1+m^2)^2}{(1+m^2)^2(7+14\nu+8\nu^2) + 4m^2(5+10\nu+4\nu^2)} \tag{19}$$

with the cross sectional area $A = (r_1^2 - r_0^2)\pi$ and $m = r_1/r_0$ or $m = r_0/r_1$. This result was also reported by Renton (1997) and Ladevèze et al. (2002). Evaluation of (19) for $m = 0$ or $m \to \infty$ yields (18).

4.6. Circle with two homogeneous layers

If we apply the method outlined in Section 4.2 to a solid circular cross section with two homogeneous layers ($N = 2$,

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$r_0 = 0$), we obtain

$$K_S = 3AE_2(1+\nu)(n(1+m^2)+1-m^2)((n-1)m^4+1)^2 \Big((n^2m^6(1+m^2)+n(1+2m^4-m^6-2m^8)+(1 - m^4)^2)(7+14\nu+8\nu^2)+4(1-n)m^2(1-m^2)^2 (5+10\nu+4\nu^2)\Big)^{-1} \tag{20}$$

with the cross sectional area $A = r_2^2\pi$ and the ratios $m = r_1/r_2$ and $n = E_1/E_2$. For $n = 0$, we recover (19). For $n = 1$, $n \to \infty$, $m = 0$, and also for $m = 1$, we recover (18).

4.7. Annulus with two homogeneous layers

If we apply the method outlined in Section 4.2 to an annular cross section with two homogeneous layers ($N = 2$, $r_0 > 0$), we obtain

$$K_S = \frac{3AE_2(1+\nu)}{1-m_0^2}(n(m_1^2-m_0^2)(1+m_1^2)+(m_0^2+m_1^2) (1-m_1^2))(n(m_1^4-m_0^4)+1-m_1^4)^2\Big((n^2(m_0^4-m_1^4)^2 (1+m_1^2)+n(m_0^2-m_1^2)(m_1^2-1)(m_0^6+m_0^4m_1^2 + 4m_0^2m_1^2+3m_0^2m_1^4+1+m_1^2+3m_1^4+2m_1^6) + (m_0^2+m_1^2)(1-m_1^4)^2)(7+14\nu+8\nu^2) + 4(n^2m_0^2m_1^2(m_0^2-m_1^2)^2(1+m_1^2)+nm_1^2(m_0^2 - m_1^2)(1-m_1^2)(m_0^4-4m_0^2-m_0^2m_1^2-m_1^2+1) + m_1^2(m_0^2+m_1^2)(1-m_1^2)^2)(5+10\nu+4\nu^2)\Big)^{-1}$$

with the cross sectional area $A = (r_2^2 - r_0^2)\pi$ and the ratios $m_0 = r_0/r_2$, $m_1 = r_1/r_2$, and $n = E_1/E_2$. For $m_0 = 0$, we recover (20). For $n = 0$, $n = 1$, $n \to \infty$, $m_0 = m_1$, and also for $m_1 = 1$, we recover (19).

4.8. Circle with linear inhomogeneity

Consider a solid circular cross section with radius r_1 and Young's modulus

$$E(r) = e_0 + e_1 r \tag{21}$$

with the constants $e_0 > 0$ and e_1. We will study the influence of the slope e_1 on the shear stiffness. To make the results comparable, we choose e_0 such that the bending stiffness has always the value $K_B = \bar{E}r_1^4\pi/4$, where $\bar{E} > 0$ is a constant.

In case of a linear inhomogeneity like (21), the boundary value problem (7) and (8) does not have a concise analytical solution; in fact, the solution would involve hypergeometric series. Therefore, we solve the problem numerically as described in Section 3.5. The results are shown in Fig. 4, where the range of profiles $E(r)$ plotted in the left part of the figure corresponds to the range along the abscissa of the right plot. In Fig. 4, the shear stiffness K_S is normalized with respect to

$$\bar{K}_S = \frac{3r_1^2\pi\bar{E}(1+\nu)}{7+14\nu+8\nu^2}. \tag{22}$$

Therefore, the presented results are independent of the actual values of $\bar{E}$ and r_1.

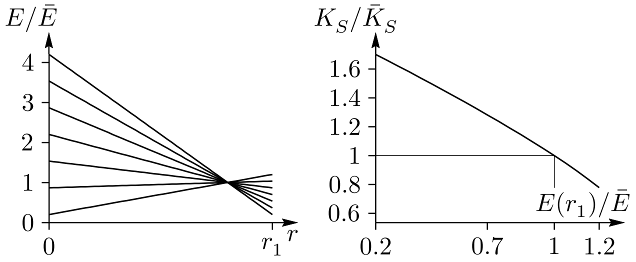

Figure 4: Shear stiffness of a circle with linear inhomogeneity.

The results indicate that for solid circular cross sections with equal bending stiffness, the shear stiffness increases if Young's modulus is increasing in the core and decreasing at the surface. The main reason for this effect is that the core contributes relatively more to the shear stiffness than to the bending stiffness (cf. the third power in (13)). As a consequence of this effect, the shear deformation of circular cylindrical beams with a soft core is higher than that of homogeneous beams with the same radius and equal bending stiffness. For our example of back-up rolls for rolling mills in Section 1, this implies that consideration of shear deformations and accurate computation of shear coefficients is all the more important.

4.9. Circle with parabolic inhomogeneity

We repeat this numerical experiment for a solid circular cross section with radius r_1 and Young's modulus

$$E(r) = e_0 + e_1\Big(r - \frac{r_1}{2}\Big)^2,$$

i.e., a parabolic inhomogeneity where $E(0) = E(r_1)$. Again the parameters e_0 and e_1 are chosen such that the bending stiffness has the value $K_B = \bar{E}r_1^4\pi/4$ with $\bar{E} > 0$. An analytical solution of the boundary value problem (7) and (8) would involve a series of Heun's equations. Hence, the boundary value problem is numerically solved.

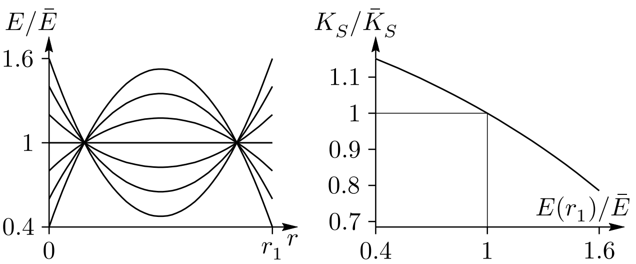

Figure 5: Shear stiffness of a circle with parabolic inhomogeneity.

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The results are shown in Fig. 5, where K_S is again normalized with respect to $\bar{K}_S$ from (22). Like in the previous example, transferring stiffness from the surface towards the core of the cross section increases the shear stiffness. However, the sensitivity of $K_S/\bar{K}_S$ with respect to $E(r_1)/\bar{E}$ is now smaller because the center ($r = 0$) is forced to have the same Young's modulus as the surface.

4.10. Annulus with linear inhomogeneity

Consider an annular cross section with the inner radius r_0, the outer radius $r_1 = 2r_0$, and Young's modulus

$$E(r) = e_0 + e_1 r,$$

where the parameters e_0 and e_1 are tuned such that the bending stiffness has always the value $K_B = \bar{E}(r_1^4 - r_0^4)\pi/4$ with $\bar{E} > 0$. The numerically obtained solution for this scenario is shown in Fig. 6, where K_S is normalized with respect to

$$\bar{K}_S = \frac{3(r_1^2 - r_0^2)\pi\bar{E}(1+\nu)(1+m^2)^2}{(1+m^2)^2(7+14\nu+8\nu^2)+4m^2(5+10\nu+4\nu^2)} \tag{23}$$

with $m = r_0/r_1 = 1/2$.

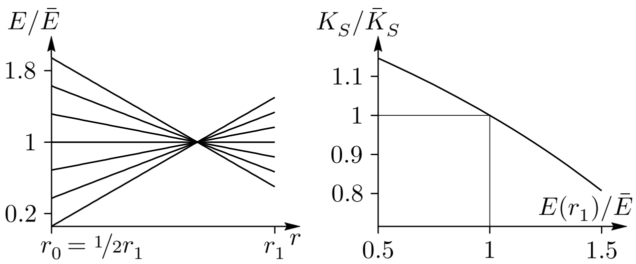

Figure 6: Shear stiffness of an annulus with linear inhomogeneity.

The principal observation that transferring stiffness from the outer surface towards the inner surface (towards the core) increases the shear stiffness is in line with the previous two examples. A comparison of Fig. 4 and Fig. 6 shows that the sensitivity of $K_S/\bar{K}_S$ with respect to $E(r_1)/\bar{E}$ is now smaller. We may conceive of the hollow cross sections as a solid one having a core with zero stiffness.

4.11. Annulus with parabolic inhomogeneity

At first sight, it may seem uninteresting to repeat the previous numerical exercise also for an annular cross section with a quadratically distributed Young's modulus. However, the following will reveal that there is a qualitative difference compared to the previous three examples. Consider an annular cross section with the inner radius r_0, the outer radius $r_1 = 2r_0$, and Young's modulus

$$E(r) = e_0 + e_1\left(r - \frac{r_0 + r_1}{2}\right)^2,$$

where the parameters e_0 and e_1 are tuned such that $K_B = \bar{E}(r_1^4 - r_0^4)\pi/4$ with $\bar{E} > 0$. Fig. 7 shows the numerically obtained value K_S normalized with respect to $\bar{K}_S$ from (23).

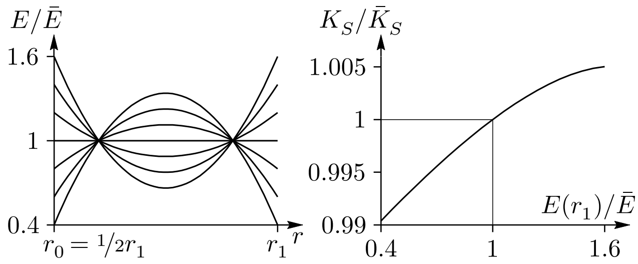

Figure 7: Shear stiffness of an annulus with parabolic inhomogeneity.

In contrast to the previous results, $K_S/\bar{K}_S$ is now increasing for growing values of $E(r_1)/\bar{E}$. Fig. 7 indicates only a weak dependence but this is merely a question of the ratio $m = r_0/r_1$. For decreasing m, the curve in the right plot of Fig. 7 rotates clockwise. For increasing m, it rotates moderately counterclockwise. Therefore, the shear stiffness of (thin) annular cross sections increases if stiffness is symmetrically transferred towards the near-surface region. This result may also be interesting as regards sandwich structures of circular shells, where layers with larger Young's modulus are typically arranged closer to the surfaces of the beam, whereas softer material, e. g., low-density foam, is concentrated in the core.

Generally, the slope of $K_S/\bar{K}_S$ will remain rather small, especially if m approaches its maximum value 1. We may conclude that for thin circular shells the distribution of $E(r)$ has a smaller influence on both the bending and the shear stiffness than for thick shells and solid cross sections.

4.12. Annulus with thin wall

Consider an annular beam with the mean radius $r > 0$ and a thin wall that has the thickness $t \ll r$. Based on (7) and (8a), it is easy to show that in this case

$$B = -3r^2 \tag{24}$$

and thus

$$K_S = \frac{AE}{4(1+\nu)} \tag{25}$$

with the cross sectional area $A = 2rt\pi$. K_S from (25) is identical to the result of Ligarò and Barsotti (2012) and corresponds to the lower bound given by Mentrasti (2012) for thin-walled circular cross sections. The stress field according to (5) with B from (24) is equivalent to the standard textbook result in form of Jourawski's formula (Jourawski, 1856; Beer et al., 2011).

5. Conclusions

The original results and findings of this work are the following:

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1. A 3D flexure problem of a beam with a circular cross section and arbitrarily radially inhomogeneous Young's modulus reduces in the most general case to a 1D linear multi-point boundary value problem. Its exact solution is straightforward.
2. Based on this solution, a simple general algorithm for computing the energy-consistent shear stiffness of such cross sections has been proposed. Closed-form analytical expressions for the shear stiffness are available for layered cross sections where Young's modulus of each layer is a power or an exponential function of the radius.
3. The shear stiffness of circular cross sections with radially inhomogeneous Young's modulus has the same characteristic form that was found by Renton (1991, 1997) for homogeneous cross sections. A plausible conjecture is that functionally graded non-circular cross sections also have this characteristic form.
4. For radially inhomogeneous circular cross sections with equal bending stiffness, the shear stiffness increases if stiffness is transferred from the outer surface towards the core. For thin annular cross sections of equal bending stiffness, the shear stiffness can also be raised by simultaneously increasing Young's modulus at the inner and the outer surface. However, this effect is weak.
5. For thin circular shells, the influence of the distribution of Young's modulus on the shear stiffness is smaller than for thick shells and solid cross sections.

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Appendix

We derive the expression (12) for the shear stiffness K_S. Setting (10) equal to (11) and using (4) and (5) yields

$$K_S = -\frac{4K_B^2(1+\nu)}{\pi D}$$

with the abbreviation

$$D = -\frac{1}{8}\int_{r_0}^{r_N} E(r)r\Big(\big((1-2\nu)r^2 + (3+2\nu)B(r)\big)^2 + (3+2\nu)^2\big(r^2 + B'(r)r + B(r)\big)^2\Big)\mathrm{d}r. \tag{A.1}$$

We will now show that D is indeed the denominator of (12). Rearranging some terms in (A.1) gives

$$D = \int_{r_0}^{r_N} E(r)r^3\Big((1+2\nu)r^2 + \Big(3+6\nu + \frac{8}{3}\nu^2\Big)B(r)\Big)\mathrm{d}r - \frac{(3+2\nu)^2}{8}R \tag{A.2}$$

with the abbreviation

$$\begin{aligned} R &= \int_{r_0}^{r_N} E(r)r\Big(2r^4 + 2B'(r)r^3 + (B'(r))^2r^2 + \frac{16}{3}B(r)r^2 + 2B'(r)B(r)r + 2B^2(r)\Big)\mathrm{d}r \\ &= \int_{r_0}^{r_N} E(r)r^3\Big(\frac{2}{3}r^2 + \frac{2}{3}B'(r)r + (B'(r))^2 + 2B(r)\Big)\mathrm{d}r \\ &\quad + \int_{r_0}^{r_N} 2E(r)r\Big(\frac{2}{3}r^2 + B(r)\Big)(r^2 + B'(r)r + B(r))\mathrm{d}r. \end{aligned}$$

Integration by parts of the last integral and consideration of (8a) for both r_0 and r_N (hollow cross section) or consideration of $r_0 = 0$ and (8a) only for r_N (solid cross section) yields

$$\begin{aligned} R &= \int_{r_0}^{r_N} E(r)r^3\Big(\frac{2}{3}r^2 + \frac{2}{3}B'(r)r + (B'(r))^2 + 2B(r)\Big)\mathrm{d}r \\ &\quad + \underbrace{\Big[E(r)r^2\Big(\frac{1}{3}r^2 + B(r)\Big)(r^2 + B'(r)r + B(r))\Big]_{r_0}^{r_N}}_{=0} \\ &\quad - \int_{r_0}^{r_N} r^2\Big(\frac{1}{3}r^2 + B(r)\Big)\{E'(r)(r^2 + B'(r)r \\ &\quad + B(r)) + E(r)(2r + 2B'(r) + B''(r)r)\} \\ &\quad + E(r)B'(r)r^2(r^2 + B'(r)r + B(r))\mathrm{d}r. \end{aligned}$$

Substitution of (7) into the curly brackets gives

$$\begin{aligned} R &= \int_{r_0}^{r_N} E(r)r^3\Big(\frac{2}{3}r^2 + \frac{2}{3}B'(r)r + (B'(r))^2 + 2B(r)\Big)\mathrm{d}r \\ &\quad - \int_{r_0}^{r_N} r^2\Big(\frac{1}{3}r^2 + B(r)\Big)\{E(r)(2r - B'(r))\} \\ &\quad + E(r)B'(r)r^2(r^2 + B'(r)r + B(r))\mathrm{d}r \\ &= 0. \end{aligned}$$

Using this result in (A.2) concludes the derivation of (12).

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