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## Hamilton's Principle for Material and Nonmaterial Control Volumes Using Lagrangian and Eulerian Description of Motion

authored by **Andreas Steinboeck, Martin Saxinger, and Andreas Kugi**  
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# Hamilton's Principle for Material and Non-Material Control Volumes Using Lagrangian and Eulerian Description of Motion

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*The standard form of Hamilton's principle is only applicable to material control volumes. There exist specialized formulations of Hamilton's principle that are tailored to non-material (open) control volumes. In case of continuous mechanical systems, these formulations contain extra terms for the virtual shift of kinetic energy and the net transport of a product of the virtual displacement and the momentum across the system boundaries. This raises the theoretically and practically relevant question whether there is also a virtual shift of potential energy across the boundary of open systems.*

*To answer this question from a theoretical perspective, we derive various formulations of Hamilton's principle applicable to material and non-material control volumes. We explore the roots and consequences of (virtual) transport terms if non-material control volumes are considered and show that these transport terms can be derived by Reynold's transport theorem. The equations are deduced for both the Lagrangian and the Eulerian description of the particle motion. This reveals that the (virtual) transport terms have a different form depending on the respective description of the particle motion. To demonstrate the practical relevance of these results, we solve an example problem where the ob-*

*tained formulations of Hamilton's principle are used to deduce the equations of motion of an axially moving elastic tension bar.*

## Keywords

*Hamilton's principle, principle of virtual work, non-material control volume, open system, (virtual) transport terms, Lagrangian and Eulerian description of motion*

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4.1	Total Time Derivative and Variation of Densities, Reynold’s Transport Theorem . . . . .	8	In continuum mechanics, Hamilton’s principle is commonly used with a Lagrangian description of the motion of the particles, i. e., with material coordinates. Moreover, in its standard form, Hamilton’s principle is only applicable to material control volumes, i. e., the control volume must move with the material or (equivalently) material particles must not enter or leave the control volume [1]. However, the use of an Eulerian description of the particle motion [2, 3] or the consideration of different control volumes (arbitrary non-material control volumes, i. e., systems with changing mass and open systems) [4, 5] can be useful in many applications. Countless examples for such systems can be found, e. g., in fluid dynamics, where generally an Eulerian description of the particle motion is preferred, or in continuous production processes where material continuously enters and leaves the control volume.	
4.2	Hamilton’s Principle for a Non-Material Control Volume . . . . .	8	Dost and Tabarrok [6] demonstrated that Hamilton’s principle can be applied using both a Lagrangian and an Eulerian description of the particle motion. For this purpose, Dost and Tabarrok [6] distinguished between Lagrangian and Eulerian variations of the particle position. However, they consistently used a control volume that is material, i. e., they considered a fixed aggregate of particles and allowed only variations that hold the mass in the control volume constant. They argued that Hamilton’s principle is essentially concerned with the motion and configuration of material points and concluded that Hamilton’s principle is Lagrangian in character. Similarly, Penfield [7] claimed that the variational expressions in Hamilton’s principle vanish on the surface of the control volume.	
<b>5</b>	<b>Relations for a Non-Material Control Volume Using Eulerian Spatial Coordinates</b>	<b>9</b>	Bampi and Morro [8] elaborated on the nexus between Lagrangians formulated with the Eulerian and the Lagrangian description of the particle motion. Their results are also applicable to Hamilton’s principle. Their main finding is that Lagrangians do exist and can be equivalently formulated in both the Eulerian and the Lagrangian description whenever there exists a homeomorphic transformation between these two descriptions. Bampi and Morro [8] tacitly assumed a material control volume. They qualified boundary terms that emerge in variational expressions as a consequence of Reynold’s transport theorem [3] as inessential. In contrast, these boundary terms (transport terms) will be of central interest in the current paper.	
5.1	Total Time Derivative and Variation of Densities, Reynold’s Transport Theorem . . . . .	9	Benjamin [9] and McIver [10] realized that the standard form of Hamilton’s principle cannot be used for systems with non-material control volumes. They were among the first who extended the standard form of Hamilton’s principle to scenarios with non-material control volumes by adding a term that accounts for the (virtual) momentum transport across open surfaces. They used essentially an Eulerian description of the particle motion.	
5.2	Hamilton’s Principle for a Non-Material Control Volume . . . . .	10	Irschik and Holl [11] used the notion of fictitious particles introduced by Trusdell and Toupin [1] to derive the equations of Lagrange for systems that can be defined by	
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a finite number of generalized coordinates. Irschik and Holl [11] formulated the equations of Lagrange for a non-material volume using an Eulerian description of the particle motion. In [12], they extended these results to a Lagrangian description of the particle motion.

Casetta and Pesce [13] derived a generalized version of Hamilton's principle where additionally the flux of kinetic energy associated with relative virtual displacements between the material particles and the control volume are included. They considered non-material control volumes and tacitly used an Eulerian description of the particle motion. Moreover, they considered a finite number of generalized coordinates and demonstrated that Lagrange's equation for a non-material volume as reported in [11] follow from their version of Hamilton's principle. Casetta and Pesce [13] showed that the formulation of Hamilton's principle as presented by McIver [10] is a special case of their results. In fact, the difference between these two versions of Hamilton's principle for non-material volumes vanishes if the local surface normal of the control volume is orthogonal to the local difference between the virtual displacement of the surface of the considered control volume and the virtual displacement of the local material particles. This was demonstrated by Kheiri and Païdoussis [14], who analyzed a flexible pipe that conveys a fluid.

In essence, the state of the art is that Hamilton's principle for non-material control volumes contains two extra transport terms compared to the standard form of Hamilton's principle. These two terms capture the transport of the product of the virtual displacement and the momentum as well as the virtual shift of the kinetic energy across the system boundary. Typically, Hamilton's principle for non-material control volumes is formulated using an Eulerian description of the particle motion. This state of the art raises the following research questions:

- Can virtual displacements of the material and the control volume also entail a virtual shift of potential energy across system boundaries? How can Hamilton's principle be formulated to capture such virtual shifts of potential energy across system boundaries?
- How can Hamilton's principle for non-material control volumes be formulated using a Lagrangian description of the particle motion? How does this differ from Hamilton's principle for non-material control volumes formulated using an Eulerian description of the particle motion?

The central objective of this paper is to answer these theoretically and practically relevant questions. The paper is organized as follows: In Section 2, we briefly reiterate the derivation of Hamilton's principle for a material control volume based on the balance of linear momentum formulated with a Lagrangian description of motion. We repeat this derivation with an Eulerian description of motion in Section 3. In Section 4, we use Reynold's transport theorem to derive Hamilton's principle for a non-material volume based on the Lagrangian description of motion. We repeat this derivation with an Eulerian description of motion in Section 5. Table 1 gives an overview of the formulations of Hamilton's prin-

ciple derived in this paper. Section 6 contains a discussion of the theoretical findings of this paper. To demonstrate the practical relevance of the above stated questions, we apply these findings to derive the equations of motion of an axially moving elastic tension bar in Section 7. Conclusions are drawn in Section 8.

**Table 1. Formulations of Hamilton's principle derived in this paper**

Control volume	Lagrangian description	Eulerian description
material	Section 2, eq. (14)	Section 3, eq. (24)
arbitrary non-material	Section 4, eq. (30)	Section 5, eq. (40)

In this paper, we consider only holonomic systems.<sup>1</sup> Moreover, we concentrate the discussion on continuum and fluid mechanics problems. Generally, we consider the three-dimensional Euclidean space with Cartesian coordinates only. Hence, there is no need to distinguish between co- and contravariant tensors. For matrices and vectors, we use boldface letters. In favor of a compact notation, arguments like the spatial coordinates  $\mathbf{x}$  or  $\mathbf{X}$  and the time  $t$  are omitted whenever confusion is ruled out.

A word of caution concerning the terminology concludes this introduction. McIver [10] denoted his results as an *extended form of Hamilton's principle*. This is however different to *Hamilton's extended principle*. The latter term is described in [16] and refers to a form of Hamilton's principle which contains the virtual work associated with non-conservative fields or thermal effects. Meirovitch [15] also denoted a form of Hamilton's principle containing the virtual work of external forces as *extended Hamilton's principle*. Atanacković et al. [17] formulated Hamilton's principle with a Lagrangian that can contain fractional derivatives and called it *generalized Hamilton's principle*. Although dealing with a completely different problem, i. e., the application of Hamilton's principle to a non-material volume, Casetta and Pesce [13] reported their findings also by using the expression *generalized Hamilton's principle*. To avoid further confusion, we simply use the name *Hamilton's principle* in this paper.

## 2 Relations for a Material Control Volume Using Lagrangian Spatial Coordinates

As a preparation, we briefly summarize the derivation of Hamilton's principle for a material control volume using a Lagrangian description of the particle motion and based on the local balance of linear momentum. Let  $V_{m0}$  be the considered material control volume in the reference configuration.

<sup>1</sup>See [15] for an explanation why the restriction to holonomic systems is reasonable.

Assume that the configuration (e. g., position, velocity, acceleration) of all particles in the control volume is uniquely defined by generalized coordinates  $\mathbf{q}$  and their time derivatives. The coordinates  $\mathbf{q}$  (or some of them) may be distributed parameters, i. e., they may be functions of the time  $t$  and the spatial coordinates  $\mathbf{x}$  or  $\mathbf{X}$ . The main purpose of Hamilton's principle is the derivation of a differential equation (maybe including spatial boundary conditions) for  $\mathbf{q}$ , which uniquely defines the dynamical behavior of the system. Insofar, Hamilton's principle is a tool to formulate the problems of mechanics based on the variation of a scalar integral rather than a solution method for these problems [15].

## 2.1 Balance of Linear Momentum

The local balance of linear momentum can be formulated as

$$\rho_0 \mathbf{a} - \nabla_{\mathbf{X}} \cdot (\mathbf{T}^{PI})^T - \rho_0 \mathbf{f} = \mathbf{0} \quad (1)$$

[2, 18–20] for an infinitesimal material particle at the current position  $\mathbf{p}(\mathbf{X}, t)$ , where  $\mathbf{X}$  is its position in the reference configuration (Lagrangian representation). Generally,  $\mathbf{X}$  are Lagrangian spatial coordinates. Moreover,  $\mathbf{a}(\mathbf{X}, t)$  is the acceleration of a material particle (second material derivative),  $\rho_0$  is the mass density in the reference configuration,  $\nabla_{\mathbf{X}}$  gives the spatial derivative with respect to Lagrangian spatial coordinates  $\mathbf{X}$ ,  $\mathbf{T}^{PI}$  is the first Piola-Kirchhoff stress tensor (force per unit area defined in the reference configuration), and  $\mathbf{f}$  is a specific volume force (external load, force per unit mass). The operator  $\cdot$  represents the inner product and is defined in Appendix A.1.

Let

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{p}(\mathbf{X}, t) - \mathbf{X}$$

be the total displacement of a material particle (Lagrangian representation). Then we obtain

$$\mathbf{v}(\mathbf{X}, t) = \frac{\partial \mathbf{p}}{\partial t} = \frac{\partial \mathbf{u}}{\partial t}$$

for its velocity and

$$\mathbf{a}(\mathbf{X}, t) = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 \mathbf{p}}{\partial t^2} = \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

for its acceleration.

**Remark** *The quantities  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{a}$  can be expressed as functions of the generalized coordinates  $\mathbf{q}$  and their time derivatives. This is why the coordinates  $\mathbf{q}$  are just tacitly used but do generally not explicitly appear in the following derivations.*

Multiplying (1) by an arbitrary admissible virtual dis-

placement  $\delta \mathbf{u}$  yields

$$\delta \mathbf{u} \cdot (\rho_0 \mathbf{a} - \nabla_{\mathbf{X}} \cdot (\mathbf{T}^{PI})^T - \rho_0 \mathbf{f}) = 0. \quad (2)$$

This may be interpreted as virtual work expression (work density, work per unit reference volume) for a specific material particle [cf. 15]. It is emphasized that the variation  $\delta$  is defined as a material (Lagrangian) variation throughout this paper<sup>2</sup>. This implies that the reference position  $\mathbf{X}$  of a material particle does not change if the particle undergoes the virtual displacement  $\delta \mathbf{u}$ , i. e.,  $\delta \mathbf{X} = \mathbf{0}$  and  $\delta \mathbf{u} = \delta \mathbf{p}$ . The first two terms of (2) can be rewritten in the form

$$\begin{aligned} \rho_0 \delta \mathbf{u} \cdot \mathbf{a} &= \rho_0 \delta \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} = \rho_0 \left( \frac{\partial(\delta \mathbf{u} \cdot \mathbf{v})}{\partial t} - \frac{\partial \delta \mathbf{u}}{\partial t} \cdot \mathbf{v} \right) \\ &= \rho_0 \left( \frac{\partial(\delta \mathbf{u} \cdot \mathbf{v})}{\partial t} - \delta \mathbf{v} \cdot \mathbf{v} \right) \\ &= \rho_0 \left( \frac{\partial(\delta \mathbf{u} \cdot \mathbf{v})}{\partial t} - \frac{1}{2} \delta(\mathbf{v} \cdot \mathbf{v}) \right) \end{aligned} \quad (3)$$

and

$$\begin{aligned} \delta \mathbf{u} \cdot (\nabla_{\mathbf{X}} \cdot (\mathbf{T}^{PI})^T) &= \nabla_{\mathbf{X}} \cdot ((\mathbf{T}^{PI})^T \cdot \delta \mathbf{u}) - (\nabla_{\mathbf{X}} \delta \mathbf{u}) : (\mathbf{T}^{PI})^T \\ &= \nabla_{\mathbf{X}} \cdot ((\mathbf{T}^{PI})^T \cdot \delta \mathbf{u}) - \delta \mathbf{F} : \mathbf{T}^{PI}. \end{aligned} \quad (4)$$

Here, we used the identity

$$\delta \mathbf{F} = (\nabla_{\mathbf{X}} \delta \mathbf{u})^T = (\nabla_{\mathbf{X}} \delta \mathbf{p})^T, \quad (5)$$

which follows from the definition of the deformation gradient  $\mathbf{F} = (\nabla_{\mathbf{X}} \mathbf{p})^T$ . The operator  $:$  represents the double inner product and is defined in Appendix A.1. Insertion of (3) and (4) into (2) and integration over the considered control volume, i. e., the material volume  $V_{m0}$  in the reference configuration, give

$$\begin{aligned} \int_{V_{m0}} \rho_0 \left[ \left( \frac{\partial(\delta \mathbf{u} \cdot \mathbf{v})}{\partial t} - \frac{1}{2} \delta(\mathbf{v} \cdot \mathbf{v}) \right) - \nabla_{\mathbf{X}} \cdot ((\mathbf{T}^{PI})^T \cdot \delta \mathbf{u}) \right. \\ \left. + \delta \mathbf{F} : \mathbf{T}^{PI} - \rho_0 \delta \mathbf{u} \cdot \mathbf{f} \right] dV_0 = 0. \end{aligned} \quad (6)$$

The term  $\delta \mathbf{F} : \mathbf{T}^{PI}$  represents the variation of the strain energy density (strain energy per unit reference volume) [2, 19]. However, it is more convenient to use the expression

$$\delta \pi = \frac{1}{\rho_0} \delta \mathbf{F} : \mathbf{T}^{PI} \quad (7)$$

<sup>2</sup>The distinction between Lagrangian and Eulerian variations is discussed and utilized in [6].

for the variation of the specific strain energy (per unit mass) [2, 19]. This is because  $\delta\pi$  is independent of the considered coordinate system (Eulerian or Lagrangian description). Appendix A.2 shows how the general variation  $\delta\pi$  can be formulated using various deformation (strain) and stress tensors. For the special case of a conservative elastic material behavior (cf. Appendix A.3), there exists a scalar-valued function  $\pi$  representing the specific potential strain energy (per unit mass).

## 2.2 Time Derivative and Variation of Specific Quantities

Consider that  $\chi(\mathbf{X}, t)$  is a specific quantity (per unit mass). The time derivative of its integral over the mass contained in the control volume  $V_{m0}$  follows in the form

$$\frac{d}{dt} \int_{V_{m0}} \rho_0 \chi dV_0 = \int_{V_{m0}} \rho_0 \frac{\partial \chi}{\partial t} dV_0 \quad (8)$$

because  $V_{m0}$  and  $\rho_0$  do not depend on  $t$ . In a similar way, we obtain the identity

$$\delta \int_{V_{m0}} \rho_0 \chi dV_0 = \int_{V_{m0}} \rho_0 \delta \chi dV_0. \quad (9)$$

Because the variation  $\delta$  is to be understood as material, neither  $V_{m0}$  nor  $\rho_0$  changes upon virtual displacements  $\delta\mathbf{u}$  of the material particles in the current configuration.

## 2.3 Hamilton's Principle for a Material Control Volume

Using (7), (8), (9), and the divergence theorem, (6) can be rewritten in the form

$$\begin{aligned} & \frac{d}{dt} \int_{V_{m0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{v} dV_0 - \delta \underbrace{\int_{V_{m0}} \rho_0 \frac{1}{2} \mathbf{v} \cdot \mathbf{v} dV_0}_{= T} \\ & + \underbrace{\int_{V_{m0}} \rho_0 \delta \pi dV_0}_{= \delta \Pi} \\ & - \underbrace{\left( \int_{\partial V_{m0}} \delta \mathbf{u} \cdot \mathbf{T}^{PI} \cdot \mathbf{n}_0 dA_0 + \int_{V_{m0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{f} dV_0 \right)}_{= \delta W_e} = 0. \end{aligned} \quad (10)$$

Here,  $T$  is the total kinetic energy,  $\delta\Pi$  is the variation of the total strain energy,  $\partial V_{m0}$  is the bounding surface of  $V_{m0}$  in the reference configuration,  $\mathbf{n}_0$  is the outward unit normal vector on  $\partial V_{m0}$  in the reference configuration, and  $\delta W_e$  is the virtual work performed by external forces acting on the material in the control volume  $V_{m0}$  or on its bounding surface  $\partial V_{m0}$  (positive if work is supplied to  $V_{m0}$ ). The normal vector  $\mathbf{n}_0$  is only defined on  $\partial V_{m0}$ .

Henceforth, we assume a conservative elastic material behavior (cf. Appendix A.3), i. e.,  $\pi$  is the specific potential

strain energy (per unit mass) and  $\Pi$  is the total potential strain energy. The virtual work expression  $\delta W_e$  may contain conservative and non-conservative parts. In most formulations of Hamilton's principle, the virtual work parts of conservative and non-conservative (external and internal) forces are assembled in separate terms. This can simplify the application of Hamilton's principle because the virtual work of conservative forces is the variation of a typically well known scalar-valued potential function. This is why we split the external surface traction (force per unit reference surface, Piola-Kirchhoff traction vector)

$$\mathbf{T}^{PI} \cdot \mathbf{n}_0 = \mathbf{t}_0^c + \mathbf{t}_0^{nc}$$

into a conservative part  $\mathbf{t}_0^c$  and a non-conservative part  $\mathbf{t}_0^{nc}$ . These are pseudo traction vectors insofar as they act on the bounding surface  $\partial V_m$  in the current configuration whereas they are defined on the bounding surface  $\partial V_{m0}$  in the reference configuration [2, 19]. Similarly, we split the external specific volume force in the form

$$\mathbf{f} = \mathbf{f}^c + \mathbf{f}^{nc}, \quad (11)$$

where  $\mathbf{f}^c$  accommodates the conservative parts and  $\mathbf{f}^{nc}$  the non-conservative parts. A typical example for  $\mathbf{f}^c$  is gravity. Assume that  $\varphi$  is the specific potential energy (per unit mass) associated with  $\mathbf{f}^c$ , i. e.,  $\mathbf{f}^c = -\partial\varphi/\partial\mathbf{u}$  and  $\delta\varphi = -\delta\mathbf{u} \cdot \mathbf{f}^c$ . Using these stipulations and (9), we rewrite (10) in the form

$$\begin{aligned} & \frac{d}{dt} \int_{V_{m0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{v} dV_0 - \delta \underbrace{\int_{V_{m0}} \rho_0 \frac{1}{2} \mathbf{v} \cdot \mathbf{v} dV_0}_{= T} \\ & + \delta \underbrace{\int_{V_{m0}} \rho_0 (\pi + \varphi) dV_0}_{= E_V} + \underbrace{\int_{\partial V_{m0}} -\delta \mathbf{u} \cdot \mathbf{t}_0^c dA_0}_{= \delta E_S} \\ & - \underbrace{\left( \int_{\partial V_{m0}} \delta \mathbf{u} \cdot \mathbf{t}_0^{nc} dA_0 + \int_{V_{m0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{f}^{nc} dV_0 \right)}_{= \delta W_{nc}} = 0, \end{aligned} \quad (12)$$

where  $e_V$  is the specific potential energy (per unit mass) of conservative strains and volume forces,  $E_V$  is the total potential energy of conservative strains and volume forces, and  $\delta E_S$  is the variation of the total potential energy of conservative surface tractions. Hence,  $-\delta E_V - \delta E_S$  is the virtual work of all conservative forces. In contrast,  $\delta W_{nc}$  represents the virtual work of all non-conservative forces (surface tractions and volume forces).

Consider an arbitrary time interval  $[t_1, t_2]$ . Hamilton's principle is a variational tool to derive the equations of motion for the generalized coordinates  $\mathbf{q}$  based on a stationarity condition of a scalar integral [cf. 15], e. g., the integral of (12) over the time interval  $[t_1, t_2]$ , and the assumption of pre-

scribed configurations  $\mathbf{q}$  at the times  $t_1$  and  $t_2$ . Among all possible paths  $\mathbf{q} + \delta\mathbf{q}$  which are defined on the time interval  $[t_1, t_2]$  and which transform the system from a prescribed initial state at the time  $t_1$  to a prescribed final state at the time  $t_2$ , Hamilton's principle thus determines the path  $\mathbf{q}$  (the so-called true path) that renders a scalar integral extremal. This explains why the identity

$$\delta\mathbf{q}|_{t=t_1} = \delta\mathbf{q}|_{t=t_2} = \mathbf{0}$$

[cf. 15] must be satisfied<sup>3</sup>. It implies

$$\delta\mathbf{u}|_{t=t_1} = \left( \frac{\partial\mathbf{u}}{\partial\mathbf{q}} \delta\mathbf{q} \right) \Big|_{t=t_1} = \mathbf{0}, \quad \delta\mathbf{u}|_{t=t_2} = \left( \frac{\partial\mathbf{u}}{\partial\mathbf{q}} \delta\mathbf{q} \right) \Big|_{t=t_2} = \mathbf{0}$$

and consequently

$$\begin{aligned} & \int_{t_1}^{t_2} \left( \frac{d}{dt} \int_{V_{m0}} \rho_0 \delta\mathbf{u} \cdot \mathbf{v} dV_0 \right) dt \\ & = \left[ \int_{V_{m0}} \rho_0 \delta\mathbf{u} \cdot \mathbf{v} dV_0 \right]_{t_1}^{t_2} = 0. \end{aligned} \quad (13)$$

We integrate (12) along the time interval  $[t_1, t_2]$  and insert (13) to get

$$\int_{t_1}^{t_2} (-\delta T + \delta E_V + \delta E_S - \delta W_{nc}) dt = 0. \quad (14)$$

This is a standard form of Hamilton's principle which uses a material control volume and a Lagrangian description of the particle motion. This fact may lead us into thinking that Hamilton's principle was Lagrangian in character [cf. 6, 7], i. e., that it required a material control volume and a Lagrangian description of the particle motion. In the derivation of (14), the chosen control volume was  $V_{m0}$  throughout.

### 3 Relations for a Material Control Volume Using Eulerian Spatial Coordinates

Based on the results of Section 2, we briefly summarize the derivation of Hamilton's principle for a material control volume and an Eulerian description of the particle motion. Let  $V_m$  be the considered material control volume in the current configuration. Henceforth, the diacritic  $\bar{\cdot}$  marks variables that depend on  $\mathbf{x}$  (and  $t$ ), i. e., variables formulated using an Eulerian description of the particle motion.

<sup>3</sup>The equations of motion obtained by Hamilton's principle contain time derivatives of  $\mathbf{q}$  up to the order two and thus require two boundary conditions in the time domain. In this sense, the assumption of a prescribed configuration at both boundaries  $t_1$  and  $t_2$  is consistent. A generalization for situations where this assumption is not made is reported in [21].

### 3.1 Balance of Linear Momentum

In an Eulerian representation, the counterpart of (2) has the form

$$\delta\bar{\mathbf{u}} \cdot (\rho\bar{\mathbf{a}} - \nabla_{\mathbf{x}} \cdot \mathbf{T}^C - \rho\mathbf{f}) = 0 \quad (15)$$

[2, 18–20]. Here,  $\rho$  is the mass density in the current configuration,  $\mathbf{T}^C$  is the Cauchy stress tensor (force per unit current area), and  $\nabla_{\mathbf{x}}$  gives the spatial derivative with respect to Eulerian spatial coordinates  $\mathbf{x}$ . The relation (15) may be interpreted as virtual work expression (work density, work per unit current volume) for an infinitesimal particle at the current position  $\mathbf{x} = \mathbf{p}(\mathbf{X}, t)$ . Generally,  $\mathbf{x}$  are Eulerian spatial coordinates. Moreover, let  $\mathbf{X} = \mathbf{P}(\mathbf{x}, t)$  be the reference position of the material particle that is currently at the spatial point  $\mathbf{x}$ . Using an Eulerian description of the particle motion, we abbreviate the material derivative in the form

$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + \nabla_{\mathbf{x}}(\cdot) \cdot \bar{\mathbf{v}}$$

with the velocity  $\bar{\mathbf{v}}(\mathbf{x}, t)$ . Moreover, the quantities that correspond to the total displacement  $\mathbf{u}(\mathbf{X}, t)$ , the velocity  $\mathbf{v}(\mathbf{X}, t)$ , and the acceleration  $\mathbf{a}(\mathbf{X}, t)$  are

$$\bar{\mathbf{u}}(\mathbf{x}, t) = \mathbf{x} - \mathbf{P}(\mathbf{x}, t) = \mathbf{u}(\mathbf{P}(\mathbf{x}, t), t)$$

$$\bar{\mathbf{v}}(\mathbf{x}, t) = \frac{D\bar{\mathbf{u}}}{Dt} = \mathbf{v}(\mathbf{P}(\mathbf{x}, t), t)$$

and

$$\bar{\mathbf{a}}(\mathbf{x}, t) = \frac{D^2\bar{\mathbf{u}}}{Dt^2} = \frac{D\bar{\mathbf{v}}}{Dt} = \mathbf{a}(\mathbf{P}(\mathbf{x}, t), t),$$

respectively. The variables  $\bar{\mathbf{u}}$ ,  $\bar{\mathbf{v}}$ , and  $\bar{\mathbf{a}}$  depend on  $\mathbf{x}$  (and  $t$ ), i. e., they represent an Eulerian description of the particle motion. The variables  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{a}$  have the same values but depend on  $\mathbf{X}$  (and  $t$ ), i. e., they represent a Lagrangian description of the particle motion.

In this paper, the variation  $\delta$  is defined as material, even if an Eulerian description of the particle motion is used. Therefore,  $\delta$  commutes with the material derivative  $\frac{D}{Dt}$  [cf. 6] and the identity

$$\frac{D\delta\bar{\mathbf{u}}}{Dt} = \frac{\partial\delta\mathbf{u}}{\partial t} = \delta \frac{\partial\mathbf{u}}{\partial t} = \delta\mathbf{v} = \delta\bar{\mathbf{v}} \quad (16)$$

holds. This relation is used to rewrite the first term of (15) in

the form

$$\begin{aligned} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{a}} &= \rho \delta \bar{\mathbf{u}} \cdot \frac{D\bar{\mathbf{v}}}{Dt} = \rho \left( \frac{D(\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}})}{Dt} - \frac{D\delta \bar{\mathbf{u}}}{Dt} \cdot \bar{\mathbf{v}} \right) \\ &= \rho \left( \frac{D(\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}})}{Dt} - \delta \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} \right) = \rho \left( \frac{D(\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}})}{Dt} - \frac{1}{2} \delta(\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}) \right). \end{aligned} \quad (17)$$

Substitution of (17) and

$$\delta \bar{\mathbf{u}} \cdot (\nabla_{\mathbf{x}} \cdot \mathbf{T}^C) = \nabla_{\mathbf{x}} \cdot (\mathbf{T}^C \cdot \delta \bar{\mathbf{u}}) - (\nabla_{\mathbf{x}} \delta \bar{\mathbf{u}}) : \mathbf{T}^C$$

into (15) and integration over the considered control volume, i. e., the material volume  $V_m$  in the current configuration, give

$$\begin{aligned} \int_{V_m} \left[ \rho \left( \frac{D(\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}})}{Dt} - \frac{1}{2} \delta(\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}) \right) - \nabla_{\mathbf{x}} \cdot (\mathbf{T}^C \cdot \delta \bar{\mathbf{u}}) \right. \\ \left. + \nabla_{\mathbf{x}} \delta \bar{\mathbf{u}} : \mathbf{T}^C - \rho \delta \bar{\mathbf{u}} \cdot \mathbf{f} \right] dV = 0. \end{aligned} \quad (18)$$

As can be inferred from (71) in Appendix A.2, the term

$$\nabla_{\mathbf{x}} \delta \bar{\mathbf{u}} : \mathbf{T}^C = \rho \delta \pi \quad (19)$$

represents the variation of the potential strain energy density (per unit current volume).

### 3.2 Time Derivative and Variation of Specific Quantities

Consider that  $\bar{\chi}(\mathbf{x}, t)$  is a specific quantity (per unit mass). Because of

$$\frac{d}{dt} \int_M \bar{\chi} dm = \int_M \frac{D\bar{\chi}}{Dt} dm$$

with the total mass  $M = \int_{V_m} \rho dV$  inside  $V_m$  and the mass  $dm = \rho dV$  of the respective infinitesimal particle, we get the identity

$$\frac{d}{dt} \int_{V_m} \rho \bar{\chi} dV = \int_{V_m} \rho \frac{D\bar{\chi}}{Dt} dV. \quad (20)$$

In the same way, the relation

$$\delta \int_M \bar{\chi} dm = \int_M \delta \bar{\chi} dm$$

yields the identity

$$\delta \int_{V_m} \rho \bar{\chi} dV = \int_{V_m} \rho \delta \bar{\chi} dV. \quad (21)$$

### 3.3 Hamilton's Principle for a Material Control Volume

Using (19), (20), (21), and the divergence theorem, (18) can be rewritten in the form

$$\begin{aligned} \frac{d}{dt} \int_{V_m} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV - \delta \underbrace{\int_{V_m} \rho \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} dV}_T + \underbrace{\int_{V_m} \rho \delta \pi dV}_{\delta \Pi} \\ - \left( \underbrace{\int_{\partial V_m} \delta \bar{\mathbf{u}} \cdot \mathbf{T}^C \cdot \mathbf{n} dA + \int_{V_m} \rho \delta \bar{\mathbf{u}} \cdot \mathbf{f} dV}_{\delta W_e} \right) = 0. \end{aligned} \quad (22)$$

Here,  $\partial V_m$  is the bounding surface of  $V_m$  in the current configuration and  $\mathbf{n}$  is the outward unit normal vector on  $\partial V_m$  in the current configuration. This vector is only defined on  $\partial V_m$ .

We split the external surface traction (force per unit current surface, Cauchy traction vector)

$$\mathbf{T}^C \cdot \mathbf{n} = \mathbf{t}^c + \mathbf{t}^{nc}$$

into a conservative part  $\mathbf{t}^c$  and a non-conservative part  $\mathbf{t}^{nc}$ . Using this splitting together with (11) and (21), we rewrite (22) in the form

$$\begin{aligned} \frac{d}{dt} \int_{V_m} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV - \delta \underbrace{\int_{V_m} \rho \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} dV}_T \\ + \delta \underbrace{\int_{V_m} \rho (\pi + \varphi) dV}_{E_V} + \underbrace{\int_{\partial V_m} -\delta \bar{\mathbf{u}} \cdot \mathbf{t}^c dA}_{\delta E_S} \\ - \left( \underbrace{\int_{\partial V_m} \delta \bar{\mathbf{u}} \cdot \mathbf{t}^{nc} dA + \int_{V_m} \rho \delta \bar{\mathbf{u}} \cdot \mathbf{f}^{nc} dV}_{\delta W_{nc}} \right) = 0. \end{aligned} \quad (23)$$

The quantities  $T$ ,  $e_V$ ,  $E_V$ ,  $\delta E_S$ , and  $\delta W_{nc}$  have the same meaning as in (12). They have the same value as in (12) if  $V_{m0}$  is the material volume in the reference configuration that corresponds to  $V_m$  in the current configuration.

We integrate (23) along the time interval  $[t_1, t_2]$  and consider the identity  $\delta \bar{\mathbf{u}}|_{t=t_1} = \delta \bar{\mathbf{u}}|_{t=t_2} = \mathbf{0}$ , which implies

$$\int_{t_1}^{t_2} \left( \frac{d}{dt} \int_{V_m} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV \right) dt = \left[ \int_{V_m} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV \right]_{t_1}^{t_2} = 0,$$

to get

$$\int_{t_1}^{t_2} (-\delta T + \delta E_V + \delta E_S - \delta W_{nc}) dt = 0. \quad (24)$$

This is a standard form of Hamilton's principle which uses



a material control volume and an Eulerian description of the particle motion. In the derivation of (24), the chosen control volume was  $V_m$  throughout. Equation (24) looks identical to (14) because, in this form, it is no longer apparent whether a Lagrangian or an Eulerian description of the particle motion is used for the computation of  $\delta T$ ,  $\delta E_V$ ,  $\delta E_S$ , and  $\delta W_{nc}$ . The important point is that both (14) and (24) use a material control volume ( $V_{m0}$  and  $V_m$ , respectively). This simplified the analysis in Sections 2 and 3 (see especially (8), (9), (20), and (21)) and will be different in the following sections.

#### 4 Relations for a Non-Material Control Volume Using Lagrangian Spatial Coordinates

We extend the derivation from Section 2 using an arbitrary, generally non-material (open) control volume  $V_{c0}$  in the reference configuration. In contrast to  $V_{m0}$ ,  $V_{c0}$  may depend on the time  $t$ .  $V_{c0}$  is assumed to coincide with  $V_{m0}$  at the current time  $t$  but, because it is non-material, it may be different from  $V_{m0}$  at other times. Note that all derivations in Section 2 require the consideration of  $V_{m0}$  only. Especially the steps from (6) to (10) and from (10) to (12) rely on material derivatives and variations with respect to  $V_{m0}$ . To obtain a formulation of Hamilton's principle suitable for non-material control volumes like  $V_{c0}$ , we have to correctly replace these derivatives and variations in the current section.

##### 4.1 Total Time Derivative and Variation of Densities, Reynold's Transport Theorem

Consider a quantity  $\psi_0(X, t)$  defined per unit reference volume (density in the reference configuration). The time derivative of its integral over the control volume  $V_{m0}$  follows in the form

$$\frac{d}{dt} \int_{V_{m0}} \psi_0 dV_0 = \int_{V_{m0}} \frac{\partial \psi_0}{\partial t} dV_0 \quad (25)$$

because  $V_{m0}$  does not depend on  $t$ . Reynold's transport theorem [2, 3] applied to the arbitrary control volume  $V_{c0}$  reads as

$$\frac{d}{dt} \int_{V_{c0}} \psi_0 dV_0 = \int_{V_{c0}} \frac{\partial \psi_0}{\partial t} dV_0 + \int_{\partial V_{c0}} \psi_0 \mathbf{v}_{c0} \cdot \mathbf{n}_{c0} dA_0. \quad (26)$$

Here,  $\partial V_{c0}$  is the bounding surface of  $V_{c0}$ ,  $\mathbf{n}_{c0}$  is the outward unit normal vector on  $\partial V_{c0}$ ,  $\mathbf{u}_{c0}$  is the displacement of  $\partial V_{c0}$ , and  $\mathbf{v}_{c0} = \frac{\partial \mathbf{u}_{c0}}{\partial t}$  is the local velocity of  $\partial V_{c0}$ . All these quantities are defined in the reference configuration. The vectors  $\mathbf{n}_{c0}$ ,  $\mathbf{u}_{c0}$ , and  $\mathbf{v}_{c0}$  are only defined on  $\partial V_{c0}$ .

**Remark**  $\partial V_{c0}$  and thus also  $\mathbf{n}_{c0}$ ,  $\mathbf{u}_{c0}$ , and  $\mathbf{v}_{c0}$  may depend on the generalized coordinates  $\mathbf{q}$ . Therefore, in the most general case, the displacements of both the material and the bounding surface  $\partial V_{c0}$  depend on  $\mathbf{q}$

At the current time  $t$ ,  $V_{m0} = V_{c0}$  and consequently  $\partial V_{m0} = \partial V_{c0}$  and  $\mathbf{n}_0 = \mathbf{n}_{c0}$  hold. It follows thus from (25)

and (26) that

$$\begin{aligned} \frac{d}{dt} \int_{V_{m0}} \psi_0 dV_0 &= \int_{V_{m0}} \frac{\partial}{\partial t} \psi_0 dV_0 \\ &= \frac{d}{dt} \int_{V_{c0}} \psi_0 dV_0 - \int_{\partial V_{c0}} \psi_0 \mathbf{v}_{c0} \cdot \mathbf{n}_{c0} dA_0. \end{aligned} \quad (27)$$

By the same line of reasoning, we obtain

$$\begin{aligned} \delta \int_{V_{m0}} \psi_0 dV_0 &= \int_{V_{m0}} \delta \psi_0 dV_0 \\ &= \delta_c \int_{V_{c0}} \psi_0 dV_0 - \int_{\partial V_{c0}} \psi_0 \delta_c \mathbf{u}_{c0} \cdot \mathbf{n}_{c0} dA_0, \end{aligned} \quad (28)$$

where  $\delta_c \mathbf{u}_{c0}$  is the virtual displacement of the bounding surface  $\partial V_{c0}$  in the reference configuration. The variational operator  $\delta_c$  as defined in (28) is to be evaluated with respect to the arbitrary control volume  $V_{c0}$  and may thus be non-material. In contrast, the variation  $\delta$  is defined as material, i. e., it concerns the particles in the material control volume  $V_{m0}$ . The left parts of (27) and (28) are in line with (8) and (9) because of  $\frac{\partial \rho_0}{\partial t} = 0$  and  $\delta \rho_0 = 0$ .

##### 4.2 Hamilton's Principle for a Non-Material Control Volume

We can now utilize (27) and (28) in (12) to eliminate time derivatives and variations, respectively, of integrals over  $V_{m0}$ . This yields

$$\begin{aligned} \frac{d}{dt} \int_{V_{c0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{v} dV_0 - \int_{\partial V_{c0}} \rho_0 (\delta \mathbf{u} \cdot \mathbf{v}) (\mathbf{v}_{c0} \cdot \mathbf{n}_{c0}) dA_0 \\ - \delta_c \underbrace{\int_{V_{c0}} \rho_0 \frac{1}{2} \mathbf{v} \cdot \mathbf{v} dV_0}_T + \delta_c \underbrace{\int_{V_{c0}} \rho_0 e_V dV_0}_{E_V} \\ + \int_{\partial V_{c0}} \rho_0 \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} - e_V \right) \delta_c \mathbf{u}_{c0} \cdot \mathbf{n}_{c0} dA_0 \\ + \underbrace{\int_{\partial V_{c0}} -\delta \mathbf{u} \cdot \mathbf{t}_0^c dA_0}_{\delta_c E_S} \\ - \underbrace{\left( \int_{\partial V_{c0}} \delta \mathbf{u} \cdot \mathbf{t}_0^{nc} dA_0 + \int_{V_{c0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{f}^{nc} dV_0 \right)}_{\delta_c W_{nc}} = 0. \end{aligned} \quad (29)$$

Integration of (29) along the time interval  $[t_1, t_2]$  and consideration of the identity  $\delta \mathbf{u}|_{t=t_1} = \delta \mathbf{u}|_{t=t_2} = \mathbf{0}$ , which implies

$$\int_{t_1}^{t_2} \left( \frac{d}{dt} \int_{V_{c0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{v} dV_0 \right) dt = \left[ \int_{V_{c0}} \rho_0 \delta \mathbf{u} \cdot \mathbf{v} dV_0 \right]_{t_1}^{t_2} = 0,$$

yield Hamilton's principle for a non-material control volume

$$\int_{t_1}^{t_2} \left[ - \int_{\partial V_{c0}} \rho_0 (\delta \mathbf{u} \cdot \mathbf{v}) (\mathbf{v}_{c0} \cdot \mathbf{n}_{c0}) dA_0 - \delta_c T + \delta_c E_V + \int_{\partial V_{c0}} \rho_0 \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} - e_V \right) \delta_c \mathbf{u}_{c0} \cdot \mathbf{n}_{c0} dA_0 + \delta_c E_S - \delta_c W_{nc} \right] dt = 0. \quad (30)$$

This form of Hamilton's principle uses an arbitrary control volume and a Lagrangian description of the particle motion.

## 5 Relations for a Non-Material Control Volume Using Eulerian Spatial Coordinates

Based on the results of Section 3 and similar to Section 4, we briefly summarize the derivation of Hamilton's principle for a non-material (open) control volume and an Eulerian description of the particle motion. Let  $V_c$  be the considered arbitrary, generally non-material control volume in the current configuration.  $V_c$  is assumed to coincide with  $V_m$  at the current time  $t$  but, because it is non-material, it may evolve differently from  $V_m$ .

### 5.1 Total Time Derivative and Variation of Densities, Reynold's Transport Theorem

Considering a specific quantity  $\psi(\mathbf{x}, t)$  per unit current volume (density), Reynold's transport theorem [2, 3] applied to the material volume  $V_m$  reads as

$$\frac{d}{dt} \int_{V_m} \psi dV = \int_{V_m} \frac{\partial \psi}{\partial t} dV + \int_{\partial V_m} \psi \bar{\mathbf{v}} \cdot \mathbf{n} dA. \quad (31)$$

This relation is in line with (20), which can be shown based on the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \bar{\mathbf{v}}) = \frac{D\rho}{Dt} + \rho \nabla_{\mathbf{x}} \cdot \bar{\mathbf{v}} = 0 \quad (32)$$

and the divergence theorem. Reynold's transport theorem applied to the arbitrary control volume  $V_c$  reads as

$$\frac{d}{dt} \int_{V_c} \psi dV = \int_{V_c} \frac{\partial \psi}{\partial t} dV + \int_{\partial V_c} \psi \mathbf{v}_c \cdot \mathbf{n}_c dA. \quad (33)$$

Here,  $\partial V_c$  is the bounding surface of  $V_c$ ,  $\mathbf{n}_c$  is the outward unit normal vector on  $\partial V_c$ ,  $\mathbf{u}_c$  is the displacement of  $\partial V_c$ , and  $\mathbf{v}_c = \frac{\partial \mathbf{u}_c}{\partial t}$  is the local velocity of  $\partial V_c$ . All these quantities are defined in the current configuration. The vectors  $\mathbf{n}_c$ ,  $\mathbf{u}_c$ , and  $\mathbf{v}_c$  are only defined on  $\partial V_c$ .

**Remark**  $\partial V_c$  and thus also  $\mathbf{n}_c$ ,  $\mathbf{u}_c$ , and  $\mathbf{v}_c$  may depend on  $\mathbf{q}$ . Therefore, in the most general case, the displacements of

both the material and the bounding surface  $\partial V_c$  depend on  $\mathbf{q}$  [11, 13].

At the current time  $t$ ,  $V_m = V_c$  and consequently  $\partial V_m = \partial V_c$  and  $\mathbf{n} = \mathbf{n}_c$  hold. It follows thus from (31) and (33) that

$$\begin{aligned} \frac{d}{dt} \int_{V_m} \psi dV &= \int_{V_m} \frac{\partial \psi}{\partial t} dV + \int_{\partial V_m} \psi \bar{\mathbf{v}} \cdot \mathbf{n} dA \\ &= \frac{d}{dt} \int_{V_c} \psi dV + \int_{\partial V_c} \psi (\bar{\mathbf{v}} - \mathbf{v}_c) \cdot \mathbf{n}_c dA. \end{aligned} \quad (34)$$

By analogy to (32), conservation of mass implies

$$\delta \rho + \rho \nabla_{\mathbf{x}} \cdot \delta \bar{\mathbf{u}} = 0. \quad (35)$$

Based on (35) and the divergence theorem, we can rewrite (21) into

$$\begin{aligned} \delta \int_{V_m} \psi dV &= \int_{V_m} [\delta \psi - \delta \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \psi] dV \\ &\quad + \int_{\partial V_m} \psi \delta \bar{\mathbf{u}} \cdot \mathbf{n} dA, \end{aligned} \quad (36)$$

which can be considered as a variational form of Reynold's transport theorem applied to the material control volume  $V_m$ . The integrand  $\delta \psi - \delta \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \psi$  in (36) represents the Eulerian (non-material) variation of  $\psi$ . Consequently, the variational form of Reynold's transport theorem applied to the arbitrary control volume  $V_c$  reads as

$$\begin{aligned} \delta_c \int_{V_c} \psi dV &= \int_{V_c} [\delta \psi - \delta \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \psi] dV \\ &\quad + \int_{\partial V_c} \psi \delta_c \mathbf{u}_c \cdot \mathbf{n}_c dA. \end{aligned} \quad (37)$$

where  $\delta_c \mathbf{u}_c$  is the virtual displacement of the bounding surface  $\partial V_c$  in the current configuration. The variational operator  $\delta_c$  as defined in (37) (see also [14]) is to be evaluated with respect to the arbitrary control volume  $V_c$  and may thus be non-material. In contrast, the variation  $\delta$  is defined as material, i. e., it concerns the particles in the material control volume  $V_m$ . We consider again that  $V_m = V_c$ ,  $\partial V_m = \partial V_c$ , and  $\mathbf{n} = \mathbf{n}_c$  hold at the current time  $t$ . It follows thus from (36) and (37) that

$$\begin{aligned} \delta \int_{V_m} \psi dV &= \int_{V_m} [\delta \psi - \delta \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \psi] dV \\ &\quad + \int_{\partial V_m} \psi \delta \bar{\mathbf{u}} \cdot \mathbf{n} dA \\ &= \delta_c \int_{V_c} \psi dV + \int_{\partial V_c} \psi (\delta \bar{\mathbf{u}} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA. \end{aligned} \quad (38)$$

## 5.2 Hamilton's Principle for a Non-Material Control Volume

We can now utilize (34) and (38) in (23) to eliminate total time derivatives and variations, respectively, of integrals over  $V_m$ . This yields

$$\begin{aligned} & \frac{d}{dt} \int_{V_c} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV + \int_{\partial V_c} \rho (\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}}) (\bar{\mathbf{v}} - \mathbf{v}_c) \cdot \mathbf{n}_c dA \\ & - \delta_c \int_{V_c} \rho \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} dV + \delta_c \int_{V_c} \rho e_V dV \\ & - \int_{\partial V_c} \rho \left( \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - e_V \right) (\delta \bar{\mathbf{u}} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA \\ & + \int_{\partial V_c} -\delta \bar{\mathbf{u}} \cdot \mathbf{t}^c dA \\ & - \left( \int_{\partial V_c} \delta \bar{\mathbf{u}} \cdot \mathbf{t}^{nc} dA + \int_{V_c} \rho \delta \bar{\mathbf{u}} \cdot \mathbf{f}^{nc} dV \right) = 0. \end{aligned} \quad (39)$$

Integration of (39) along the time interval  $[t_1, t_2]$  and consideration of the identity  $\delta \bar{\mathbf{u}}|_{t=t_1} = \delta \bar{\mathbf{u}}|_{t=t_2} = \mathbf{0}$ , which implies

$$\int_{t_1}^{t_2} \left( \frac{d}{dt} \int_{V_c} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV \right) dt = \left[ \int_{V_c} \rho \delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} dV \right]_{t_1}^{t_2} = 0,$$

yield Hamilton's principle for a non-material control volume

$$\begin{aligned} & \int_{t_1}^{t_2} \left[ \int_{\partial V_c} \rho (\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}}) (\bar{\mathbf{v}} - \mathbf{v}_c) \cdot \mathbf{n}_c dA - \delta_c T + \delta_c E_V \right. \\ & \quad - \int_{\partial V_c} \rho \left( \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - e_V \right) (\delta \bar{\mathbf{u}} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA \\ & \quad \left. + \delta_c E_S - \delta_c W_{nc} \right] dt = 0. \end{aligned} \quad (40)$$

This form of Hamilton's principle uses an arbitrary control volume and an Eulerian description of the particle motion.

## 6 Discussion

The main results of the Sections 2 through 5 are various formulations of Hamilton's principle as indicated in Table 1.

For a material control volume, the standard form of Hamilton's principle can be used. It is given in (14) and (24) for a Lagrangian and an Eulerian description of the particle motion, respectively. These two equations look identical. Their difference, however, is that the expressions  $\delta T$ ,  $\delta E_V$ ,  $\delta E_S$ , and  $\delta W_{nc}$  are computed in the reference configuration (control volume  $V_{m0}$ ) and the current configuration (control volume  $V_m$ ), respectively.

Hamilton's principle for an arbitrary control volume and

a Lagrangian description of the particle motion is given in (30). A comparison of (14) and (30) reveals several differences:

- The variations of  $T$  and  $E_V$  in (14) are to be evaluated with respect to the material control volume  $V_{m0}$  whereas the variations of  $T$  and  $E_V$  in (30) are to be evaluated with respect to the arbitrary control volume  $V_{c0}$ . Note that the expressions  $\delta E_S$  and  $\delta_c E_S$  are identical and the subscript  $c$  is written here only for notational consistency. The same applies to  $\delta W_{nc}$  and  $\delta_c W_{nc}$ .
- Compared to (14), (30) contains the extra term  $-\int_{\partial V_{c0}} \rho_0 (\delta \mathbf{u} \cdot \mathbf{v}) (\mathbf{v}_{c0} \cdot \mathbf{n}_{c0}) dA_0$  representing the net transport of the product of the virtual displacement  $\delta \mathbf{u}$  and the momentum  $\rho_0 \mathbf{v}$  across the bounding surface  $\partial V_{c0}$ .
- Compared to (14), (30) contains the extra term  $\int_{\partial V_{c0}} \rho_0 \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} - e_V \right) \delta_c \mathbf{u}_{c0} \cdot \mathbf{n}_{c0} dA_0$  representing the net virtual shift of kinetic energy minus potential energy associated with conservative strains and volume forces across  $\partial V_{c0}$ .

Hamilton's principle for an arbitrary control volume and an Eulerian description of the particle motion is given in (40). A comparison of (24) and (40) reveals several differences:

- The variations of  $T$  and  $E_V$  in (24) are to be evaluated with respect to the material control volume  $V_m$  whereas the variations of  $T$  and  $E_V$  in (40) are to be evaluated with respect to the arbitrary control volume  $V_c$ . Again, the expressions  $\delta E_S$  and  $\delta_c E_S$  as well as  $\delta W_{nc}$  and  $\delta_c W_{nc}$  are identical.
- Compared to (24), (40) contains the extra term  $\int_{\partial V_c} \rho (\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{v}}) (\bar{\mathbf{v}} - \mathbf{v}_c) \cdot \mathbf{n}_c dA$  representing the net transport of the product of the virtual displacement  $\delta \bar{\mathbf{u}}$  and the momentum  $\rho \bar{\mathbf{v}}$  across the bounding surface  $\partial V_c$ .
- Compared to (24), (40) contains the extra term  $-\int_{\partial V_c} \rho \left( \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - e_V \right) (\delta \bar{\mathbf{u}} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA$  representing the net virtual shift of kinetic energy minus potential energy associated with conservative strains and volume forces across  $\partial V_c$ .

A comparison of (30) and (40) shows that the extra terms accounting for the net transport of the product of the virtual displacement and the momentum as well as the virtual shift of kinetic minus potential energy across the bounding surface have a different form in the Lagrangian and Eulerian description of the particle motion. In the current configuration, these terms contain the relative normal velocity  $(\bar{\mathbf{v}} - \mathbf{v}_c) \cdot \mathbf{n}_c$  and the relative normal virtual displacement  $(\delta \bar{\mathbf{u}} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c$ . In the reference configuration, the material velocity  $\mathbf{v}_0$  and the virtual displacement  $\delta \mathbf{u}_0$  are zero by definition. This explains why in (30) only the normal velocity  $-\mathbf{v}_{c0} \cdot \mathbf{n}_{c0}$  and the normal virtual displacement  $-\delta_c \mathbf{u}_{c0} \cdot \mathbf{n}_{c0}$  of the bounding surface remain.

How do these results compare to the state of the art?

- The first surface integral in (40) representing the net transport of the product of the virtual displacement  $\delta \bar{\mathbf{u}}$  and the momentum  $\rho \bar{\mathbf{v}}$  across the bounding surface  $\partial V_c$  was introduced in the same form by McIver [10]. However, he denoted this term as virtual momentum transport and Casetta and Pesce [13] denoted this term as flux

of momentum.

- The net virtual shift of kinetic energy, i. e., the term  $-\int_{\partial V_c} \rho \frac{1}{2} \bar{v} \cdot \bar{v} (\delta \bar{u} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA$  in (40) was introduced in this form by Casetta and Pesce [13]. However, they denoted this term as flux of kinetic energy.
- An essentially new term found in the current work is the net virtual shift of potential energy associated with conservative strains and volume forces, i. e., the term  $\int_{\partial V_c} \rho e_V (\delta \bar{u} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA$  in (40). Casetta and Pesce [13] effectively included the terms  $\delta_c E_V + \int_{\partial V_c} \rho e_V (\delta \bar{u} - \delta_c \mathbf{u}_c) \cdot \mathbf{n}_c dA + \delta_c E_S - \delta_c W_{nc}$  in a single general virtual work expression  $\delta W$ . Using this compact notation, it may be more difficult to correctly capture the virtual work and net virtual shift of potential energy associated with conservative strains and volume forces.
- McIver [10] as well as Casetta and Pesce [13] used an Eulerian description of the particle motion. Insofar, the current paper extends the state of the art by stating also the formulations of Hamilton's principle applicable if a Lagrangian description of the particle motion is used.

The derivations in Sections 2 and 3 demonstrate that the standard form of Hamilton's principle essentially relies on the evaluation of total time derivatives and variations with respect to a material control volume  $V_{m0}$  or  $V_m$ . The Sections 4.1 and 5.1 show how Reynold's transport theorem helps to replace these total time derivatives and variations by total time derivatives and variations with respect to an arbitrary control volume  $V_{c0}$  or  $V_c$ , respectively.

A transfer of the obtained results to other physical domains, e. g., heat transfer, thermodynamics, or chemistry [7, 16, 22], is possible with moderate effort. That is, by analogy to the presented approach, it can be shown that other energy terms (including entropy and chemical potentials) appearing in the Lagrangian may also be subject to a net virtual shift across the system boundary.

### 7 Example Problem: Axially Moving Elastic Tension Bar

To highlight the practical relevance of the obtained theoretical results, we apply them in this example problem to derive the equations of motion of an axially moving elastic tension bar for a material and a spatially fixed (non-material) control volume as shown in Figs. 1 and 2, respectively. We solve the problem based on Hamilton's principle using both the Lagrangian and the Eulerian description of the particle motion. For validation, we additionally derive the equations of motion in Appendix B based on the momentum balance and based on the energy balance. Table 2 gives an overview of the applied derivation methods and the subsections where they can be found.

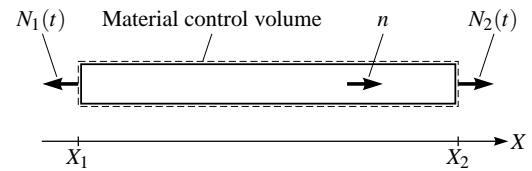


Fig. 1. Axially moving elastic tension bar with a material control volume

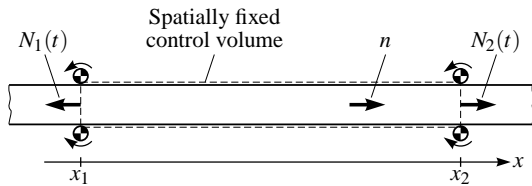


Fig. 2. Axially moving elastic tension bar with a spatially fixed control volume

Table 2. Methods to derive the equations of motion of an axially moving elastic tension bar

Derivation method	Control volume	Lagrangian description	Eulerian description
Hamilton's principle	material	Section 7.1.1	Section 7.2.1
Hamilton's principle	spatially fixed	Section 7.1.2	Section 7.2.2
momentum balance	-	Appendix B.1.1	Appendix B.2.1
energy balance	material	Appendix B.1.2	Appendix B.2.2
energy balance	spatially fixed	Appendix B.1.3	Appendix B.2.3

The Eulerian spatial coordinate is  $x$ ; the Lagrangian spatial coordinate is  $X$ . The material of the elastic bar moves along the direction  $x$  with the local velocity  $v(X, t) = \bar{v}(x, t)$ . The properties of the bar are uniform along the direction  $x$ . Its mass per unit length is  $M'_0$  in the unloaded state (reference configuration) and  $M'(x, t)$  in the loaded state (current configuration). The tension bar is linear elastic and its uniform tensional stiffness is  $K$ . Let  $\varepsilon$  be the engineering strain in the bar along the direction  $x$  and  $N$  the total cross sectional tension force. Clearly,  $\varepsilon$  and  $N$  may depend on  $x$  (or  $X$ ) and  $t$ . They are coupled by the constitutive law

$$N = K\varepsilon. \tag{41}$$

We consider both a material control volume that ranges from  $X = X_1$  to  $X = X_2$  and a spatially fixed (non-material) control volume that ranges from  $x = x_1$  to  $x = x_2$ . The bar is loaded by the concentrated generally non-conservative external forces  $N_1(t)$  and  $N_2(t)$  at the respective system boundaries. Within the control volume, the bar is subject to the distributed external force  $n$ , which is defined per unit current length and which may be non-conservative. In case of the spatially fixed control volume, we assume that the bar exceeds the control volume. Moreover, we assume in this case that some external devices (e. g., the driven rolls indicated in Fig. 2) control the total cross sectional forces  $N_1(t)$  and  $N_2(t)$  at the system boundaries. We thus consider pure force boundary conditions for both types of control volumes (material and non-material).

Throughout this section, we use the abbreviations

$$(\dot{\cdot}) = \frac{\partial(\cdot)}{\partial t}, \quad (\cdot)_x = \frac{\partial(\cdot)}{\partial x}, \quad (\cdot)_X = \frac{\partial(\cdot)}{\partial X}$$

for partial derivatives.

### 7.1 Lagrangian Description of the Particle Motion

Consider a cross section of the bar characterized by the Lagrangian spatial coordinate (original position)  $X = p(X, 0)$ . Let  $u(X, t)$  denote its current total displacement along the spatial direction  $x$  in the Lagrangian representation. Hence, the cross section has the current position

$$p(X, t) = X + u(X, t) \tag{42}$$

measured in Eulerian spatial coordinates. In the Lagrangian description, the displacement  $u(X, t)$  serves as a generalized coordinate. The (material) velocity of the cross section is

$$\dot{p} = v = \dot{u}, \tag{43}$$

and its acceleration is

$$\ddot{p} = \dot{v} = \ddot{u}. \tag{44}$$

The differential of (42) follows in the form

$$dp(X, t) = (1 + u_X)dX, \tag{45}$$

which gives the engineering strain

$$\varepsilon = \frac{dp(X, t)}{dX} - 1 = u_X. \tag{46}$$

#### 7.1.1 Hamilton's Principle for a Material Control Volume

We consider a material control volume ranging from  $X_1$  to  $X_2$ . The kinetic energy density (kinetic energy per unit reference length) is

$$T'_0 = M'_0 \frac{v^2}{2} = M'_0 \frac{\dot{u}^2}{2}, \tag{47}$$

which gives the total kinetic energy

$$T = \int_{X_1}^{X_2} M'_0 \frac{\dot{u}^2}{2} dX \tag{48}$$

in the control volume. The strain energy density (strain energy per unit reference length) is

$$E'_{V0} = M'_0 e_V = K \frac{\varepsilon^2}{2} = K \frac{u_X^2}{2}, \tag{49}$$

which gives the total strain energy

$$E_V = \int_{X_1}^{X_2} K \frac{u_X^2}{2} dX. \tag{50}$$

The distributed force  $n$  is defined per unit current length. Based on (45), the corresponding quantity defined per unit reference length follows in the form  $n(1 + u_X)$ . Hence, the virtual work of the non-conservative loads reads as

$$\delta W_{nc} = \delta u(X_2, t)N_2(t) - \delta u(X_1, t)N_1(t) + \int_{X_1}^{X_2} \delta un(1 + u_X)dX. \tag{51}$$

Insertion of these results into (14) gives

$$\int_{t_1}^{t_2} \left( \int_{X_1}^{X_2} (-M'_0 \dot{u} \delta \dot{u} + K u_X \delta u_X) dX - \delta u(X_2, t)N_2(t) + \delta u(X_1, t)N_1(t) - \int_{X_1}^{X_2} \delta un(1 + u_X)dX \right) dt = 0.$$

Integration by parts, consideration of  $\delta u(X, t_1) = \delta u(X, t_2) = 0$  and (43), and application of the fundamental lemma of calculus of variations yield the differential equation

$$M'_0 \ddot{u} = K u_{XX} + n(1 + u_X) \tag{52a}$$

and the boundary conditions

$$N_1(t) = N(X_1, t) = K u_X(X_1, t) \tag{52b}$$

$$N_2(t) = N(X_2, t) = Ku_X(X_2, t). \quad (52c)$$

This system of equations has to be supplemented by appropriate initial conditions.

### 7.1.2 Hamilton's Principle for a Spatially Fixed Control Volume

We consider a spatially fixed (non-material) control volume ranging from  $x_1$  to  $x_2$ . These points correspond to  $X_1 = P(x_1, t)$  and  $X_2 = P(x_2, t)$ , respectively, in the reference configuration. Utilization of the identity  $\delta_c W_{nc} = \delta W_{nc}$ , insertion of (48) through (51) into (30), and consideration of (9), (28), and (47) give

$$\begin{aligned} & \int_{t_1}^{t_2} \left( - [M'_0 \delta uvv_{c0}]_{X_1}^{X_2} - \delta_c T + \delta_c E_V \right. \\ & \quad \left. + [M'_0 \left( \frac{v^2}{2} - e_V \right) \delta_c u_{c0}]_{X_1}^{X_2} - \delta_c W_{nc} \right) dt \\ &= \int_{t_1}^{t_2} \left( - [M'_0 \delta uvv_{c0}]_{X_1}^{X_2} + \int_{X_1}^{X_2} \delta (-T'_0 + E'_{V0}) dX \right. \\ & \quad \left. + \underbrace{[(-T'_0 + E'_{V0}) \delta_c u_{c0}]_{X_1}^{X_2} + [M'_0 \left( \frac{v^2}{2} - e_V \right) \delta_c u_{c0}]_{X_1}^{X_2}}_{=0} \right) \\ & \quad - \delta_c W_{nc} \Big) dt \\ &= \int_{t_1}^{t_2} \left( - [M'_0 \delta uuv_{c0}]_{X_1}^{X_2} \right. \\ & \quad + \int_{X_1}^{X_2} (-M'_0 \dot{u} \delta u + Ku_X \delta u_X) dX - \delta u(X_2, t) N_2(t) \\ & \quad \left. + \delta u(X_1, t) N_1(t) - \int_{X_1}^{X_2} \delta un(1 + u_X) dX \right) dt = 0. \end{aligned} \quad (53)$$

For the terms  $\delta_c T$  and  $\delta_c E_V$  in (53), a correct evaluation of the (generally non-material) variational operator  $\delta_c$  according to (9) and (28) is essential. Integration by parts converts (53) into

$$\begin{aligned} & \int_{t_1}^{t_2} \left[ \underbrace{- [M'_0 \delta uuv_{c0}]_{X_1}^{X_2} - (-M'_0 \dot{u}(X_2, t) \delta u(X_2, t) \dot{X}_2}_{=0} \right. \\ & \quad \left. + M'_0 \dot{u}(X_1, t) \delta u(X_1, t) \dot{X}_1 \right] dt \\ & \quad + \left[ \int_{X_1}^{X_2} -M'_0 \dot{u} \delta u dX \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \int_{X_1}^{X_2} M'_0 \ddot{u} \delta u dX \right. \\ & \quad + [Ku_X \delta u]_{X_1}^{X_2} - \int_{X_1}^{X_2} Ku_{XX} \delta u dX - \delta u(X_2, t) N_2(t) \\ & \quad \left. + \delta u(X_1, t) N_1(t) - \int_{X_1}^{X_2} \delta un(1 + u_X) dX \right) dt = 0. \end{aligned} \quad (54)$$

In this derivation, correct integration by parts with respect to the time  $t$  requires to take into account that  $X_1$  and  $X_2$  depend on  $t$ . This yields terms which exactly compensate the net transport of the product of the virtual displacement and the momentum across the bounding surface. Consequently, the first line in (54) vanishes. Insertion of  $\delta u(X, t_1) = \delta u(X, t_2) = 0$  into (54), consideration of (43), and application of the fundamental lemma of calculus of variations yield again (52).

### 7.2 Eulerian Description of the Particle Motion

Consider a cross section of the bar that is currently at the position  $x = p(X, t)$ . Let  $\bar{u}(x, t)$  denote its current total displacement along the direction  $x$  in the Eulerian representation. Hence, we have

$$x = p(X, t) = X + \bar{u}(p(X, t), t). \quad (55)$$

In the Eulerian description, the displacement  $\bar{u}(x, t)$  serves as a generalized coordinate. The (material) velocity of the cross section is

$$\frac{D\bar{u}}{Dt} = \dot{p} = \bar{v}(x, t) = \dot{\bar{u}} + \bar{u}_x \dot{p} = \frac{\ddot{\bar{u}}}{1 - \bar{u}_x}, \quad (56)$$

and its acceleration is

$$\begin{aligned} \frac{D^2 \bar{u}}{Dt^2} &= \dot{\bar{p}} = \frac{D\bar{v}}{Dt} = \ddot{\bar{u}} + 2\ddot{\bar{u}}_x \dot{p} + \bar{u}_{xx} \dot{p}^2 \\ &= \frac{\ddot{\bar{u}}(1 - \bar{u}_x)^2 + 2\ddot{\bar{u}}_x \dot{\bar{u}}(1 - \bar{u}_x) + \dot{\bar{u}}^2 \bar{u}_{xx}}{(1 - \bar{u}_x)^3}. \end{aligned} \quad (57)$$

The differential of (55) follows in the form

$$dx = dp(X, t) = dX + \bar{u}_x dp(X, t) = \frac{dX}{1 - \bar{u}_x}, \quad (58)$$

which gives

$$\varepsilon = \frac{dp(X, t)}{dX} - 1 = \frac{\bar{u}_x}{1 - \bar{u}_x} \quad (59)$$

for the engineering strain. From (58) and the conservation of mass written in the form  $M'_0 dX = M' dx$ , we obtain the relation

$$M' = (1 - \bar{u}_x) M'_0. \quad (60)$$

A comparison of (45) and (58) shows that

$$1 + u_X = \frac{1}{1 - \bar{u}_x}. \quad (61)$$

**7.2.1 Hamilton's Principle for a Material Control Volume** and

We consider a material control volume ranging from  $X_1$  to  $X_2$ . These points correspond to  $x_1 = p(X_1, t)$  and  $x_2 = p(X_2, t)$ , respectively, in the current configuration. Based on (56) and (60), the kinetic energy density (kinetic energy per unit current length) is

$$T' = M' \frac{\bar{v}^2}{2} = M'_0 \frac{\dot{\bar{u}}^2}{2(1 - \bar{u}_x)}, \quad (62)$$

which gives the total kinetic energy

$$T = \int_{x_1}^{x_2} M' \frac{\bar{v}^2}{2} dx \quad (63)$$

in the control volume. Based on (49) and (60), the strain energy density (strain energy per unit current length) is

$$E'_V = M' e_V = \frac{M'}{M'_0} K \frac{\varepsilon^2}{2} = K \frac{\bar{u}_x^2}{2(1 - \bar{u}_x)}, \quad (64)$$

which gives the total strain energy

$$E_V = \int_{x_1}^{x_2} \frac{M'}{M'_0} K \frac{\varepsilon^2}{2} dx. \quad (65)$$

The variational operator  $\delta$  is material by definition, i. e., it is a Lagrangian variation [cf. 6]. If we use an Eulerian description of the particle motion, the variation  $\delta$  thus commutes with the corresponding derivatives  $\frac{D}{Dt}$  (see also (16)) and  $\frac{\partial}{\partial X}$ . Consequently, these derivative operators themselves are not subject to the variation  $\delta$ . For derivatives of the virtual displacement  $\delta u = \delta \bar{u}$ , this implies

$$\begin{aligned} \delta \bar{v} &= \delta v = \delta \dot{\bar{u}} = \frac{\partial \delta \bar{u}}{\partial t} = \frac{D \delta \bar{u}}{Dt} \\ &= \delta \dot{\bar{u}} + \bar{v} \delta \bar{u}_x = \delta \dot{\bar{u}} + \frac{\dot{\bar{u}}}{1 - \bar{u}_x} \delta \bar{u}_x \end{aligned} \quad (66a)$$

$$\delta \varepsilon = \delta u_X = \frac{\partial \delta u}{\partial X} = \frac{1}{1 - \bar{u}_x} \frac{\partial \delta \bar{u}}{\partial x} = \frac{\delta \bar{u}_x}{1 - \bar{u}_x}, \quad (66b)$$

where the relations (56) and (58) have been used. Based on (21), (56), (59), (60), and (66), we obtain the variations of (63) and (65) in the form

$$\begin{aligned} \delta T &= \delta \int_{x_1}^{x_2} M' \frac{\bar{v}^2}{2} dx = \int_{x_1}^{x_2} M' \bar{v} \delta \bar{v} dx \\ &= \int_{x_1}^{x_2} M'_0 \dot{\bar{u}} \left( \delta \dot{\bar{u}} + \frac{\dot{\bar{u}}}{1 - \bar{u}_x} \delta \bar{u}_x \right) dx \end{aligned} \quad (67a)$$

$$\begin{aligned} \delta E_V &= \delta \int_{x_1}^{x_2} \frac{M'}{M'_0} K \frac{\varepsilon^2}{2} dx = \int_{x_1}^{x_2} \frac{M'}{M'_0} K \varepsilon \delta \varepsilon dx \\ &= \int_{x_1}^{x_2} K \frac{\bar{u}_x}{1 - \bar{u}_x} \delta \bar{u}_x dx, \end{aligned} \quad (67b)$$

respectively. The virtual work of the non-conservative loads reads as

$$\delta W_{nc} = \delta \bar{u}(x_2, t) N_2(t) - \delta \bar{u}(x_1, t) N_1(t) + \int_{x_1}^{x_2} \delta \bar{u} n dx. \quad (68)$$

Insertion of these results into (24) gives

$$\begin{aligned} \int_{t_1}^{t_2} \left( \int_{x_1}^{x_2} \left[ -M'_0 \dot{\bar{u}} \left( \delta \dot{\bar{u}} + \frac{\dot{\bar{u}}}{1 - \bar{u}_x} \delta \bar{u}_x \right) + K \frac{\bar{u}_x}{1 - \bar{u}_x} \delta \bar{u}_x \right] dx \right. \\ \left. - \delta \bar{u}(x_2, t) N_2(t) + \delta \bar{u}(x_1, t) N_1(t) - \int_{x_1}^{x_2} \delta \bar{u} n dx \right) dt = 0. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} & - \left[ \int_{x_1}^{x_2} M'_0 \dot{\bar{u}} \delta \bar{u} dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[ \left[ M'_0 \dot{\bar{u}} \delta \bar{u} \bar{v} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} M'_0 \ddot{\bar{u}} \delta \bar{u} dx \right. \\ & - \left[ M'_0 \dot{\bar{u}} \frac{\dot{\bar{u}}}{1 - \bar{u}_x} \delta \bar{u} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} M'_0 \left( \frac{2\dot{\bar{u}}\dot{\bar{u}}_x}{1 - \bar{u}_x} + \frac{\dot{\bar{u}}^2 \bar{u}_{xx}}{(1 - \bar{u}_x)^2} \right) \delta \bar{u} dx \\ & + \left[ K \frac{\bar{u}_x}{1 - \bar{u}_x} \delta \bar{u} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} K \left( \frac{\bar{u}_{xx}}{1 - \bar{u}_x} + \frac{\bar{u}_x \bar{u}_{xx}}{(1 - \bar{u}_x)^2} \right) \delta \bar{u} dx \\ & \left. - \delta \bar{u}(x_2, t) N_2(t) + \delta \bar{u}(x_1, t) N_1(t) - \int_{x_1}^{x_2} \delta \bar{u} n dx \right] dt \\ & = - \left[ \int_{x_1}^{x_2} M'_0 \dot{\bar{u}} \delta \bar{u} dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \int_{x_1}^{x_2} \left[ M'_0 \left( \ddot{\bar{u}} + \frac{2\dot{\bar{u}}\dot{\bar{u}}_x}{1 - \bar{u}_x} \right. \right. \right. \\ & \left. \left. + \frac{\dot{\bar{u}}^2 \bar{u}_{xx}}{(1 - \bar{u}_x)^2} \right) \delta \bar{u} - K \frac{\bar{u}_{xx}}{(1 - \bar{u}_x)^2} \delta \bar{u} - n \delta \bar{u} \right] dx \\ & \left. + \left[ K \frac{\bar{u}_x}{1 - \bar{u}_x} \delta \bar{u} \right]_{x_1}^{x_2} - \delta \bar{u}(x_2, t) N_2(t) + \delta \bar{u}(x_1, t) N_1(t) \right) dt = 0. \end{aligned}$$

For correct integration by parts with respect to the time  $t$ , we have to take into account that  $x_1$  and  $x_2$  depend on  $t$ . Insertion of  $\delta \bar{u}(x, t_1) = \delta \bar{u}(x, t_2) = 0$ , consideration of (56), and application of the fundamental lemma of calculus of variations yield the differential equation

$$M'_0 \left( \ddot{\bar{u}} + \frac{2\dot{\bar{u}}_x \dot{\bar{u}}}{1 - \bar{u}_x} + \frac{\dot{\bar{u}}^2 \bar{u}_{xx}}{(1 - \bar{u}_x)^2} \right) = K \frac{\bar{u}_{xx}}{(1 - \bar{u}_x)^2} + n \quad (69a)$$

and the boundary conditions

$$N_1(t) = N(x_1, t) = K \frac{\bar{u}_x(x_1, t)}{1 - \bar{u}_x(x_1, t)} \quad (69b)$$

$$N_2(t) = N(x_2, t) = K \frac{\bar{u}_x(x_2, t)}{1 - \bar{u}_x(x_2, t)}. \quad (69c)$$

This system of equations has to be supplemented by appropriate initial conditions. Considering (46), (57), (58), (59), (61), and

$$u_{XX} = \varepsilon_X = \frac{1}{1 - \bar{u}_x} \varepsilon_x = \frac{1}{1 - \bar{u}_x} \frac{\partial}{\partial x} \left( \frac{\bar{u}_x}{1 - \bar{u}_x} \right) = \frac{\bar{u}_{xx}}{(1 - \bar{u}_x)^3},$$

it follows that (52) and (69) are equivalent.

### 7.2.2 Hamilton's Principle for a Spatially Fixed Control Volume

We consider a spatially fixed (non-material) control volume ranging from  $x_1$  to  $x_2$ . Utilization of the identity  $\delta_c W_{nc} = \delta W_{nc}$ , insertion of (63), (64), (65), and (68) into (40), and consideration of (21), (38), (60), and (66) give

$$\begin{aligned} & \int_{t_1}^{t_2} \left( [M' \delta \bar{u} \bar{v}^2]_{x_1}^{x_2} - \delta_c T + \delta_c E_V \right. \\ & \left. - [M' \left( \frac{\bar{v}^2}{2} - e_V \right) \delta \bar{u}]_{x_1}^{x_2} - \delta_c W_{nc} \right) dt \\ = & \int_{t_1}^{t_2} \left( [M' \delta \bar{u} \bar{v}^2]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[ -M' \bar{v} \delta \bar{v} + \frac{M'}{M_0} K \varepsilon \delta \varepsilon \right] dx \right. \\ & \left. - [(-T' + E'_V) \delta \bar{u}]_{x_1}^{x_2} - [M' \left( \frac{\bar{v}^2}{2} - e_V \right) \delta \bar{u}]_{x_1}^{x_2} \right) dt \\ & \underbrace{\hspace{10em}}_{=0} \\ & - \delta_c W_{nc} \Big) dt \\ = & \int_{t_1}^{t_2} \left( [M'_0 \delta \bar{u} \dot{\bar{u}} \bar{v}]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[ -M'_0 \dot{\bar{u}} (\delta \bar{u} + \bar{v} \delta \bar{u}_x) \right. \right. \\ & \left. \left. + K \bar{u}_x \frac{\delta \bar{u}_x}{1 - \bar{u}_x} \right] dx - \delta \bar{u}(x_2, t) N_2(t) + \delta \bar{u}(x_1, t) N_1(t) \right. \\ & \left. - \int_{x_1}^{x_2} \delta \bar{u} n dx \right) dt = 0. \end{aligned} \quad (70)$$

For the terms  $\delta_c T$  and  $\delta_c E_V$  in (70), a correct evaluation of the (generally non-material) variational operator  $\delta_c$  according to (21) and (38) is essential. Integration by parts, insertion of  $\delta \bar{u}(x, t_1) = \delta \bar{u}(x, t_2) = 0$ , consideration of (56), and application of the fundamental lemma of calculus of variations yield again (69).

## 8 Conclusions

The original results, main findings, and conclusions of this work are as follows:

- Hamilton's principle was derived for material and non-material (open) control volumes using both a Lagrangian and an Eulerian description of the particle motion. This yielded four different formulations of Hamilton's principle.
- In Hamilton's principle, variations of energy and work terms are to be evaluated with respect to the considered control volume. Special care must be taken in case of non-material control volumes because the variational operator  $\delta$  is material per definition.
- The standard form of Hamilton's principle uses a material control volume, i. e., total time derivatives and variations are evaluated with respect to a material control volume. Reynold's transport theorem is a helpful tool to convert them into total time derivatives and variations with respect to an arbitrary control volume.
- If a non-material control volume is considered, Hamilton's principle contains an extra term to capture the net transport of the product of the virtual displacement and the momentum across the system boundary. This extra term has a different form for the Lagrangian and the Eulerian description of the particle motion.
- If a non-material control volume is considered, Hamilton's principle contains an extra term to capture the net virtual shift of kinetic energy minus potential energy associated with conservative strains and volume forces across the system boundary. This extra term has a different form for the Lagrangian and the Eulerian description of the particle motion.
- In an example problem, the equations of motion (including boundary conditions) of an axially moving elastic tension bar were deduced using all four derived formulations of Hamilton's principle. As shown in Appendix B, the same equations of motion follow from the momentum balance and the energy balance.

In essence, this work showed how Hamilton's principle can be formulated for material and non-material (open) control volumes using both a Lagrangian and an Eulerian description of the particle motion. We hope that the obtained results and findings will be supportive for both theorists and practitioners who derive equations of motion by means of Hamilton's principle.

## A Some Mathematical and Mechanical Relations

This appendix summarizes a few mathematical and mechanical relations used in the main part of the paper.

### A.1 Inner Products

The operator  $\cdot$  represents the inner product (tensor contraction) defined in the form

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = a_i b_i$$



$$\begin{aligned} \mathbf{a} \cdot \mathbf{B} &= \mathbf{B}^T \cdot \mathbf{a} = [a_i B_{ij}] \\ \mathbf{A} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{A}^T = [A_{ij} b_j] \\ \mathbf{A} \cdot \mathbf{B} &= (\mathbf{B}^T \cdot \mathbf{A}^T)^T = [A_{ij} B_{jk}] \end{aligned}$$

for arbitrary vectors  $\mathbf{a}$  and  $\mathbf{b}$  (first-order tensors) and arbitrary matrices  $\mathbf{A}$  and  $\mathbf{B}$  (second-order tensors), where the length of  $\mathbf{a}$ , the length of  $\mathbf{b}$ , the number of columns of  $\mathbf{A}$ , and the number of rows of  $\mathbf{B}$  must be equal. The operator  $:$  represents the double inner product defined in the form

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \mathbf{A}^T : \mathbf{B}^T = \text{tr}(\mathbf{A}^T \cdot \mathbf{B}) = A_{ij} B_{ij}$$

for two arbitrary matrices  $\mathbf{A}$  and  $\mathbf{B}$  of equal size. Based on these definitions, it follows that

$$\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}^T) : \mathbf{B}$$

holds for arbitrary matrices  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , where the number of rows of  $\mathbf{A}$  must equal the number of rows of  $\mathbf{B}$ , the number of columns of  $\mathbf{A}$  must equal the number of columns of  $\mathbf{C}$ , and the number of columns of  $\mathbf{B}$  must equal the number of rows of  $\mathbf{C}$ .

## A.2 Variation of Specific Strain Energy

Consider the Eulerian spatial coordinates  $\mathbf{x}$ , the total displacement  $\bar{\mathbf{u}}$  formulated with an Eulerian description of the particle motion, the Cauchy stress tensor  $\mathbf{T}^C$  (force per unit current area), and the mass density  $\rho$  in the current configuration. Based on the definition  $\mathbf{T}^{PI} = \det(\mathbf{F})\mathbf{T}^C \cdot \mathbf{F}^{-T}$  of the first Piola-Kirchhoff stress tensor [2, 19], the rules for the double inner product from Appendix A.1, and the identities  $\rho_0 = \rho \det(\mathbf{F})$ , (5) and  $\nabla_{\mathbf{x}} = \mathbf{F}^T \cdot \nabla_{\mathbf{X}}$ , (7) can be rewritten in the form

$$\begin{aligned} \delta\pi &= \frac{1}{\rho_0} \delta\mathbf{F} : (\det(\mathbf{F})\mathbf{T}^C \cdot \mathbf{F}^{-T}) = \frac{1}{\rho} (\delta\mathbf{F} \cdot \mathbf{F}^{-1}) : \mathbf{T}^C \\ &= \frac{1}{\rho} (\nabla_{\mathbf{x}} \delta \bar{\mathbf{u}})^T : \mathbf{T}^C = \frac{1}{\rho} (\nabla_{\mathbf{x}} \delta \bar{\mathbf{u}}) : \mathbf{T}^C. \end{aligned} \quad (71)$$

Based on the definition  $\mathbf{T}^{PII} = \mathbf{F}^{-1} \cdot \mathbf{T}^{PI} = \det(\mathbf{F})\mathbf{F}^{-1} \cdot \mathbf{T}^C \cdot \mathbf{F}^{-T}$  of the second Piola-Kirchhoff stress tensor [2, 19], the rules for the double inner product from Appendix A.1, and the definition  $\mathbf{G} = (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I})/2$  of the Green-Lagrangian strain tensor [2, 19], (7) can be rewritten in the form

$$\begin{aligned} \delta\pi &= \frac{1}{\rho_0} \delta\mathbf{F} : (\mathbf{F} \cdot \mathbf{T}^{PII}) = \frac{1}{\rho_0} (\mathbf{F}^T \cdot \delta\mathbf{F}) : \mathbf{T}^{PII} \\ &= \frac{1}{2\rho_0} (\delta\mathbf{F}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \delta\mathbf{F}) : \mathbf{T}^{PII} = \frac{1}{\rho_0} \delta\mathbf{G} : \mathbf{T}^{PII}. \end{aligned} \quad (72)$$

To summarize, (7), (71), and (72) define the variation of the specific strain energy (per unit mass)

$$\delta\pi = \frac{1}{\rho_0} \delta\mathbf{F} : \mathbf{T}^{PI} = \frac{1}{\rho} (\nabla_{\mathbf{x}} \delta \bar{\mathbf{u}}) : \mathbf{T}^C = \frac{1}{\rho_0} \delta\mathbf{G} : \mathbf{T}^{PII}.$$

This shows that  $\delta\pi$  can be expressed as the double inner product of the variation of a deformation (strain) tensor and a stress tensor. Moreover, this expression shows when the factor  $1/\rho$  or  $1/\rho_0$  appears in  $\delta\pi$ .

## A.3 Hyperelastic Material

For materials with conservative elastic behavior, which are also known as hyperelastic or Green elastic materials [2, 18–20], there exists a scalar-valued specific potential strain energy  $\pi$  (per unit mass) that has the following properties. The value of  $\pi$  depends only on the current local deformation state (strain) defined, e. g., by  $\mathbf{F}$  or  $\mathbf{G}$ . The stresses follow from partial derivatives of  $\pi$  in the form

$$\mathbf{T}^{PI} = \rho_0 \frac{\partial \pi}{\partial \mathbf{F}}, \quad \mathbf{T}^C = \rho \frac{\partial \pi}{\partial \mathbf{F}} \cdot \mathbf{F}^T, \quad \mathbf{T}^{PII} = \rho_0 \frac{\partial \pi}{\partial \mathbf{G}}.$$

In the context of Hamilton’s principle, the existence of  $\pi$  is of interest because its integral over the considered mass appears as potential energy term in the Lagrangian.

## B Alternative Solution Methods for the Example Problem

In this appendix, solution methods other than Hamilton’s principle are used to derive the equations of motion of the axially moving elastic tension bar considered in Section 7. The purpose of this appendix is to validate the results of the example problem found in Section 7.

### B.1 Lagrangian Description of the Particle Motion

#### B.1.1 Momentum Balance

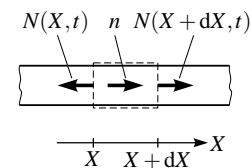


Fig. 3. Slice of the axially moving elastic tension bar in the reference configuration

We formulate the momentum balance for an (infinitesimal) slice of the considered bar. The slice has the thickness  $dX$  along the direction  $X$  in the reference configuration as outlined in Fig. 3. The linear momentum of the slice is  $M'_0 v dX$  and it is loaded by the forces  $-N(X, t)$ ,  $N(X + dX, t)$ ,

and  $ndp(X, t)$ . Note that  $n$  is a distributed force per unit current length. Insertion into the momentum balance and consideration of (45) gives

$$M'_0 \dot{v} dX = N(X + dX, t) - N(X, t) + n(1 + u_X) dX.$$

Consideration of (41), (43), (44), and (46), division by  $dX$ , and computation of the limit  $dX \rightarrow 0$  yield again (52a).

### B.1.2 Energy Balance for a Material Control Volume

We consider an arbitrary material control volume ranging from  $X_1$  to  $X_2$ . Adding up (47) and (49), we obtain the total energy density (total energy per unit reference length)

$$E'_0 = M'_0 \frac{\dot{u}^2}{2} + K \frac{u_X^2}{2}. \quad (73)$$

The power of external loads acting on the control volume is

$$P_{ex} = v(X_2, t)N(X_2, t) - v(X_1, t)N(X_1, t) + \int_{X_1}^{X_2} vn(1 + u_X) dX. \quad (74)$$

Using (41) and (46), the energy balance thus gives

$$\begin{aligned} \frac{d}{dt} \int_{X_1}^{X_2} E'_0 dX - P_{ex} &= \int_{X_1}^{X_2} \frac{\partial}{\partial t} E'_0 dX - P_{ex} \\ &= \int_{X_1}^{X_2} (M'_0 \ddot{u} + Ku_X \dot{u}_X) dX - [vKu_X]_{X_1}^{X_2} \\ &\quad - \int_{X_1}^{X_2} vn(1 + u_X) dX \\ &= \int_{X_1}^{X_2} v (M'_0 \ddot{u} - Ku_{XX} - n(1 + u_X)) dX = 0, \end{aligned} \quad (75)$$

where integration by parts was applied. Because (75) must hold for arbitrary (material) control volumes  $[X_1, X_2]$ , we obtain again (52a).

### B.1.3 Energy Balance for a Spatially Fixed Control Volume

We consider an arbitrary spatially fixed (non-material) control volume ranging from  $x_1$  to  $x_2$ . This corresponds to the range  $X_1 = P(x_1, t)$  to  $X_2 = P(x_2, t)$  in the reference configuration. For this scenario, the terms  $E'_0$  and  $P_{ex}$  are identical to (73) and (74), respectively. Using (41), (46), and Reynold's transport theorem and considering also the energy transfer associated with the mass flow across the system

boundary, the energy balance gives

$$\begin{aligned} \frac{d}{dt} \int_{X_1}^{X_2} E'_0 dX - E'_0(X_2, t)\dot{X}_2 + E'_0(X_1, t)\dot{X}_1 - P_{ex} \\ = \int_{X_1}^{X_2} \frac{\partial}{\partial t} E'_0 dX - P_{ex} = 0. \end{aligned}$$

Because this expression is equivalent to (75), we obtain again (52a).

## B.2 Eulerian Description of the Particle Motion

### B.2.1 Momentum Balance

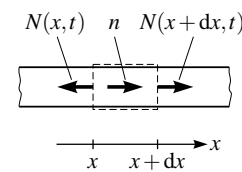


Fig. 4. Slice of the axially moving elastic tension bar in the current configuration

We formulate the momentum balance for an (infinitesimal) slice of the considered bar. The slice has the thickness  $dx$  along the direction  $x$  in the current configuration as outlined in Fig. 4. The linear momentum of the slice is  $M' \bar{v} dx$  and it is loaded by the forces  $-N(x, t)$ ,  $N(x + dx, t)$ , and  $ndx$ . Insertion into the momentum balance gives

$$M' \frac{D\bar{v}}{Dt} dx = N(x + dx, t) - N(x, t) + ndx.$$

Consideration of (41), (56), (57), (59), and (60), division by  $dx$ , and computation of the limit  $dx \rightarrow 0$  yield again (69a).

### B.2.2 Energy Balance for a Material Control Volume

We consider an arbitrary material control volume ranging from  $X_1$  to  $X_2$ . This corresponds to the range  $x_1 = p(X_1, t)$  to  $x_2 = p(X_2, t)$  in the current configuration. Adding up (62) and (64), we obtain the total energy density (total energy per unit current length)

$$E' = M'_0 \frac{\dot{u}^2}{2(1 - \bar{u}_x)} + K \frac{\bar{u}_x^2}{2(1 - \bar{u}_x)}. \quad (76)$$

The power of external loads acting on the control volume is

$$P_{ex} = \bar{v}(x_2, t)N(x_2, t) - \bar{v}(x_1, t)N(x_1, t) + \int_{x_1}^{x_2} \bar{v} ndx. \quad (77)$$

Using (41), (56), (59), and Reynold's transport theorem, the energy balance thus gives

$$\begin{aligned}
 \frac{d}{dt} \int_{x_1}^{x_2} E' dx - P_{ex} &= \int_{x_1}^{x_2} \frac{\partial E'}{\partial t} dx + [E' \bar{v}]_{x_1}^{x_2} - P_{ex} \\
 &= \int_{x_1}^{x_2} \left[ M'_0 \left( \frac{\dot{u}\ddot{u}}{1 - \bar{u}_x} + \frac{\dot{u}^2 \dot{\bar{u}}_x}{2(1 - \bar{u}_x)^2} \right) \right. \\
 &\quad \left. + K \left( \frac{\dot{u}_x \bar{u}_x}{1 - \bar{u}_x} + \frac{\dot{u}_x \bar{u}_x^2}{2(1 - \bar{u}_x)^2} \right) \right] dx + \left[ \left( M'_0 \frac{\dot{u}^2}{2(1 - \bar{u}_x)} \right. \right. \\
 &\quad \left. \left. + K \frac{\bar{u}_x^2}{2(1 - \bar{u}_x)} \right) \bar{v} \right]_{x_1}^{x_2} - \left[ \bar{v} K \frac{\bar{u}_x}{1 - \bar{u}_x} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \bar{v} n dx \\
 &= \int_{x_1}^{x_2} \left[ M'_0 \left( \frac{\dot{u}\ddot{u}}{1 - \bar{u}_x} + \frac{\dot{u}^2 \dot{\bar{u}}_x}{2(1 - \bar{u}_x)^2} \right) \right. \\
 &\quad \left. + K \left( \frac{\dot{u}_x \bar{u}_x}{1 - \bar{u}_x} + \frac{\dot{u}_x \bar{u}_x^2}{2(1 - \bar{u}_x)^2} \right) \right. \\
 &\quad \left. + \frac{\partial}{\partial x} \left( M'_0 \frac{\dot{u}^2}{2(1 - \bar{u}_x)} \bar{v} + K \frac{\bar{u}_x^2 - 2\bar{u}_x}{2(1 - \bar{u}_x)} \bar{v} \right) - \bar{v} n \right] dx \\
 &= \int_{x_1}^{x_2} \bar{v} \left[ M'_0 \left( \ddot{u} + \frac{2\dot{u}_x \dot{u}}{1 - \bar{u}_x} + \frac{\dot{u}^2 \bar{u}_{xx}}{(1 - \bar{u}_x)^2} \right) \right. \\
 &\quad \left. - K \frac{\bar{u}_{xx}}{(1 - \bar{u}_x)^2} - n \right] dx = 0.
 \end{aligned} \tag{78}$$

Because (78) must hold for arbitrary (material) control volumes  $[x_1, x_2]$ , we obtain again (69a).

### B.2.3 Energy Balance for a Spatially Fixed Control Volume

We consider an arbitrary spatially fixed (non-material) control volume ranging from  $x_1$  to  $x_2$ . For this scenario, the terms  $E'$  and  $P_{ex}$  are identical to (76) and (77), respectively. Using (41) and (59) and considering also the energy transfer associated with the mass flow across the system boundary, the energy balance gives

$$\begin{aligned}
 \frac{d}{dt} \int_{x_1}^{x_2} E' dx - E'(x_2, t) \bar{v}(x_2, t) + E'(x_1, t) \bar{v}(x_1, t) - P_{ex} \\
 = \int_{x_1}^{x_2} \frac{\partial E'}{\partial t} dx + [E' \bar{v}]_{x_1}^{x_2} - P_{ex} = 0.
 \end{aligned}$$

Because this expression is equivalent to (78), we obtain again (69a).

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### Nomenclature

- $A$  = Area
- $\mathbf{a}$  = Acceleration, second material derivative
- $D$  = Material derivative operator
- $d$  = Total derivative operator
- $\partial$  = Partial derivative operator, prefix to indicate boundary of a domain
- $E_S$  = Potential energy of conservative surface tractions
- $E_V$  = Potential energy of conservative strains and volume forces
- $e_V$  = Specific potential energy (per unit mass) of conservative strains and volume forces
- $\mathbf{F}$  = Deformation gradient
- $\mathbf{f}$  = Specific volume force (per unit mass)
- $\mathbf{G}$  = Green-Lagrangian strain tensor
- $K$  = Tensional stiffness
- $M$  = Total mass
- $m$  = Mass
- $N$  = Tension force
- $N_1, N_2$  = Boundary values of  $N$
- $\mathbf{n}$  = Outward unit normal vector
- $n$  = Distributed external force (per unit length)
- $\mathbf{P}, P$  = Position of a material particle in the reference configuration
- $P_{ex}$  = Power of external loads
- $\mathbf{p}, p$  = Position of a material particle in the current configuration
- $\mathbf{q}$  = Generalized coordinates
- $T$  = Kinetic energy
- $\mathbf{T}^C$  = Cauchy stress tensor
- $\mathbf{T}^{PI}$  = First Piola-Kirchhoff stress tensor
- $\mathbf{T}^{PII}$  = Second Piola-Kirchhoff stress tensor
- $\mathbf{t}$  = Surface traction
- $t$  = Time
- $t_1, t_2$  = Start and end of considered time interval
- $\mathbf{u}, u$  = Total displacement
- $V$  = Volume
- $V_c$  = Arbitrary, generally non-material (open) control volume
- $V_m$  = Material control volume
- $\mathbf{v}, v$  = Velocity, first material derivative
- $W_e$  = Work performed by external forces
- $W_{nc}$  = Work performed by non-conservative forces
- $\mathbf{X}, X$  = Position in the reference configuration, Lagrangian spatial coordinate
- $X_1, X_2$  = Start and end of the control volume in the reference configuration
- $\mathbf{x}, x$  = Position in the current configuration, Eulerian spatial coordinate
- $x_1, x_2$  = Start and end of the control volume in the current configuration
- $\delta$  = Variational operator, prefix to indicate virtual expression
- $\varepsilon$  = Engineering strain
- $\Pi$  = Strain energy
- $\pi$  = Specific strain energy (per unit mass)
- $\rho$  = Mass density

- $\varphi$  = Specific potential energy (per unit mass) associated with conservative external volume forces
- $\nabla_X$  = Gradient operator, spatial derivative with respect to Lagrangian spatial coordinates
- $\nabla_x$  = Gradient operator, spatial derivative with respect to Eulerian spatial coordinates
- $_0$  = Subscript to indicate a quantity in the reference configuration
- $_c$  = Subscript to indicate a quantity that corresponds to the arbitrary control volume  $V_c$
- $_X$  = Subscript to indicate partial derivative with respect to  $X$
- $_x$  = Subscript to indicate partial derivative with respect to  $x$
- $^c$  = Superscript to indicate conservative quantity
- $^{nc}$  = Superscript to indicate non-conservative quantity
- $^{\cdot}$  = Diacritic to indicate a quantity that depends on Eulerian spatial coordinates (Eulerian description of the particle motion)
- $^{\dot{\cdot}}$  = Diacritic to indicate partial time derivative
- $'$  = Mark to indicate quantity per unit length

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