Trajectory planning for quasilinear parabolic distributed parameter systems based on finite-difference semi-discretizations

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RESEARCH ARTICLE

Trajectory planning for quasilinear parabolic distributed parameter systems based on finite-difference semi-discretizations

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In this contribution, a flatness-based approach is considered for the solution of the trajectory planning problem for quasilinear parabolic distributed parameter systems (DPS) by making use of finite-difference semi-discretizations. It is shown that the method yields solutions which are equivalent to results known from the infinite-dimensional trajectory planning for a certain class of quasi-linear parabolic distributed parameter systems. Furthermore, the methodology being proposed can also be applied to systems with general analytic nonlinearities. As analytical convergence results are not available in this case, a numerical test criterion for the convergence behaviour is suggested.

Keywords: Trajectory planning; differential flatness; distributed parameter systems; finite-difference semi-discretization; diffusion-convection-reaction systems

1 Introduction and problem description

Consider the following partial differential equation (PDE) of parabolic type for $x \in (0,1)$, $t > 0$

$$\rho c_p(\theta(t,x))\partial_t\theta(t,x) = \partial_x\left(\lambda(\theta(t,x))\partial_x\theta(t,x)\right) - \nu\rho c_p(\theta(t,x))\partial_x\theta(t,x) + \mu\left(\theta(t,x)\right)\theta(t,x)\ , \tag{1}$$

with boundary conditions (BCs)

$$\partial_x\theta(t,0) = 0, \quad \partial_x\theta(t,1) = g(\theta(t,1), u(t)),\ t > 0\ , \tag{2}$$

the initial condition (IC)

$$\theta(0,x) = \theta_0(x),\ x \in [0,1] \tag{3}$$

and the boundary control input $u(t)$. The parameters of this infinite-dimensional system λ (possibly interpreted as a diffusion coefficient), ρ (a density), c_p (a heat capacity), ν (a flow velocity) and μ (a reaction coefficient) can be chosen constant or as functions of the state $\theta(t,x)$. In order for the PDE (1) to be parabolic, it has to be guaranteed that ρc_p and λ are always positive. With this model different types of diffusion-convection-reaction (DCR) systems, e. g. different types of heat conduction problems or chemical reactors, can be described. Thereby, the restriction of the spatial domain to the interval $x \in [0,1]$ can be obtained by a suitable scaling.

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The BC at $x = 0$ represents a zero-flow as it is encountered at a perfectly insulated wall or when considering situations where symmetry occurs, for example, a homogeneous heat conductor that is uniformly heated or cooled at both ends. The BC at $x = 1$ will be defined for the subsequently considered examples. In general it will only be required that $g(\theta(t, 1), u(t))$ is at least locally solvable for $u(t)$.

The typical control tasks for the systems modelled by (1)–(3) include stabilizing stationary profiles as well as trajectory planning, i. e. finding a control input $u(t)$ such that some output, usually obtained by evaluating a spatial profile at a certain position, i. e. $y(t) = \theta(t, x_{\mathrm{out}})$ follows a desired trajectory $y^*(t)$. In the following, the considered output is located at the boundary opposite to the controlled boundary, i. e. $x_{\mathrm{out}} = 0$. The more specific problem considered in the following is the transition between stationary profiles along a desired reference trajectory and within a finite time interval $t \in [0, \tau]$, where $0 < \tau < \infty$ is the prescribed transition time. Examples are the start-up or shut-down of chemical reactors or the reheating of metal slabs in furnaces to enable their further processing in hot rolling mills.

For the solution of finite-dimensional trajectory planning problems, the property of flatness (Fliess et al. 1995), which allows for a parametrization of the state and input by a so-called flat output, has proven to be a useful tool. More recently, flatness-based methods have been extended for use with infinite-dimensional systems. The approach taken can be motivated by a well known result (Gevrey 1913, Goursat 1927, Widder 1975) for the linear heat equation $\partial_t\theta(t, x) = \partial_{xx}\theta(t, x)$ with BCs $\theta(t, 0) = y(t)$, $\partial_x\theta(t, 0) = 0$, where it suffices for $y(t)$ to meet certain growth conditions so that the solution represented by the power series

$$\theta(t, x) = \sum_{n=0}^{\infty} a_n(t) \frac{x^n}{n!} \tag{4}$$

exists. The coefficients $a_n(t)$ of (4) can then be calculated from a differential recursion $a_{n+2}(t) = \dot{a}_n(t)$ with $a_0(t) = y(t)$ and $a_1(t) = 0$, leading to $\theta(t, x) = \sum_{n=0}^{\infty} y^{(n)}(t) \frac{x^{2n}}{(2n)!}$. Here and in the following $y^{(n)}(t)$ describes the n-th time-derivative of $y(t)$. It is clear that a control at the boundary $x = 1$ can then be parametrized by $y(t)$ and its derivatives, such that the analogy to the notion of flatness in the finite-dimensional case becomes evident. This approach has been successfully applied to the trajectory planning problem for various systems governed by linear and certain quasi-linear parabolic PDEs, see e. g., Laroche et al. (2000), Lynch and Rudolph (2002).

The core of the approach outlined above is the existence of a power series solving the given problem. However, given analytic parameters in the state other than low degree polynomials, power series are in general no longer applicable due to increasing computational complexity, which is required to determine the series coefficients, and the resulting radius of convergence, which might well be zero for any interesting parameter range. Therefore, the approach taken in this contribution consists in semi-discretizing the problem (1)–(3) by means of finite differences in the spatial coordinate and to apply flatness-based trajectory planning to the obtained finite-dimensional system. With this it is possible to tackle also systems with parameters that only have to meet differentiability conditions with respect to the state.

Weighted residual approaches as the Galerkin or the Finite Element method are frequently-used as a means of discretization for PDEs. The choice of the finite difference method is justified, besides its easy applicability, by some advantageous properties both in view of system analysis and controller design. First of all, nonlinearities can be handled quite easily and nearly all types

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of BCs can be introduced, which is in contrast to e. g. the Galerkin approach, where the inclusion of nonlinearities is a rather tedious procedure and BCs can only be considered approximately or with heavy constraints on the trial functions to be chosen, see, e. g., Zienkiewicz and Morgan (1983), Fletcher (1984). Secondly, and as is shown subsequently in this contribution, finite-difference semi-discretizations directly allow to determine a physically meaningful flat output in order to parametrize the discretized state variable and the boundary input. While this may still be the case with some Finite Element approaches with local test functions, the property is very likely to be lost when using global test functions.

The presented method of trajectory planning based on finite-difference semi-discretizations is interesting to be considered for several reasons. First of all, it yields results that are consistent with the ones obtained by power series solutions, see, e. g., Ollivier and Sedoglavic (2001) for similar results for the linear heat equation. Furthermore, the application of the presented method to more general systems with parameters that are analytic functions of the state is possible, as will be demonstrated in simulation (see also Utz et al. 2007), and computationally quite effective.

The paper is organized as follows. In Section 2, the method of flatness-based trajectory planning based on finite-difference semi-discretizations is presented. In Section 3, it is shown that for polynomial nonlinearities the solution of the trajectory planning problem based on finite differences is equivalent to a solution obtained by power series and that convergence can be proven analytically under certain conditions. Furthermore, a convergence test based on numerical considerations is motivated. In the concluding section, Section 4, the method presented in Section 2 is applied to a trajectory planning problem with general analytic nonlinearities for the heat-up process of a steel slab.

2 Flatness-based parametrization for the finite-difference semi-discretized DCR-system

In this section, the solution of the trajectory planning problem for the system (1)–(3) is outlined. The first step thereby consists in deriving the finite-difference semi-discretization and in proving that the resulting ODE system is differentially flat. The parametrization of the state and the control input can then be found in a straightforward manner.

2.1 *Finite-difference semi-discretization*

The methodology to obtain the semi-discretized system pursued in this contribution is the discretization along the spatial coordinate x using finite differences on an equidistant grid with N grid elements and the nodes $x_0 = 0, x_1 = \delta_N, \ldots, x_k = k\delta_N, \ldots, x_N = 1$ where $\delta_N = 1/N$. For any given N this results in a system of $N+1$ ODEs for the state variable $\theta(t, x)$ evaluated at the nodes $\theta_{N,k}(t) = \theta(t, x_k)$, where the first index refers to the number of grid elements used. For the sake of readability in the following, time-dependencies as in $\theta_{N,k}(t)$ are omitted whenever they are clear from the context. In general, a central difference scheme is used yielding the approximate derivatives

$$\partial_x \theta_{N,k} = \frac{1}{2\delta_N}(\theta_{N,k+1} - \theta_{N,k-1}) + \mathcal{O}(\delta_N^2) \tag{5a}$$

$$\partial_{xx} \theta_{N,k} = \frac{1}{\delta_N^2}(\theta_{N,k+1} - 2\theta_{N,k} + \theta_{N,k-1}) + \mathcal{O}(\delta_N^2)\ . \tag{5b}$$

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In order to approximate the squared first derivative $(\partial_x\theta_k(t))^2$, as it results from a state-dependent diffusion coefficient λ, it is usually more practical in view of trajectory planning to use the following approximation scheme

$$(\partial_x\theta_{N,k})^2 = \frac{1}{\delta_N^2}\left(\theta_{N,k+1}-\theta_{N,k}\right)\left(\theta_{N,k}-\theta_{N,k-1}\right)+\mathcal{O}(\delta_N^2)\ . \tag{6}$$

By applying the introduced approximations (5) and (6) to (1) and evaluating the BCs (2) to obtain $\theta_{N,-1}=\theta_{N,1}$ and $\theta_{N,N+1}=\theta_{N,N-1}+2\delta_N g(\theta_{N,N},u_N)$, a system of ODEs is obtained. The notation u_N is thereby chosen in view of trajectory planning to indicate the control input calculated based on a semi-discretization with N grid elements. For the node numbered $k=0$ the discretization (5a) is used for $(\partial_x\theta_k(t))^2$ yielding

$$\rho c_p(\theta_{N,0})\dot{\theta}_{N,0} = \frac{2\lambda(\theta_{N,0})}{\delta_N^2}\left(\theta_{N,1}-\theta_{N,0}\right)+\mu(\theta_{N,0})\theta_{N,0}\ . \tag{7}$$

In all other cases the expression $(\partial_x\theta_k(t))^2$ is discretized according to (6). Thus, the semi-discretized approximation reads as

$$\begin{aligned}\rho c_p(\theta_{N,k})\dot{\theta}_{N,k} = {}& \frac{\lambda(\theta_{N,k})}{\delta_N^2}\left(\theta_{N,k+1}-2\theta_{N,k}+\theta_{N,k-1}\right)+\frac{\partial_\theta\lambda(\theta_{N,k})}{\delta_N^2}(\theta_{N,k+1}-\theta_{N,k})(\theta_{N,k}-\theta_{N,k-1})\\ &-\frac{\nu\rho c_p(\theta_{N,k})}{2\delta_N}(\theta_{N,k+1}-\theta_{N,k-1})+\mu(\theta_{N,k})\theta_{N,k}\end{aligned} \tag{8}$$

for $k=1,\dots,N-1$ and

$$\begin{aligned}\rho c_p(\theta_{N,N})\dot{\theta}_{N,N} = {}& \frac{2\lambda(\theta_{N,N})}{\delta_N^2}\left(-\theta_{N,N}+\theta_{N,N-1}\right)-\frac{\partial_\theta\lambda(\theta_{N,N})}{\delta_N^2}\left(\theta_{N,N-1}-\theta_{N,N}\right)^2\\ &+\mu(\theta_{N,N})\theta_{N,N}+\Omega(\theta_{N,N-1},\theta_{N,N})g\left(\theta_{N,N},u_N\right)\end{aligned} \tag{9}$$

with

$$\Omega(\theta_{N,N-1},\theta_{N,N}) = \left(\frac{2\lambda(\theta_{N,N})}{\delta_N}+\frac{2\partial_\theta\lambda(\theta_{N,N})}{\delta_N}\left(\theta_{N,N}-\theta_{N,N-1}\right)-\nu\rho c_p(\theta_{N,N})\right) \tag{10}$$

for the node numbered $k=N$.

The ICs obtained by evaluating (3) at the nodes

$$\theta_{N,k}(0)=\theta_0(x_k),\ k=0,\dots,N \tag{11}$$

complete the finite-dimensional semi-discretized approximation of the DPS (1)–(3).

2.2 *Flatness-based state and input parametrization*

From the finite-difference semi-discretization, a certain structure emerges in the ODE system (7)–(11) that can be exploited for the flatness-based parametrization. Solving each ODE for the state

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variable with the highest index, the following relation is obtained from (7)

$$\theta_{N,1} = \frac{\delta_N^2}{2\lambda(\theta_{N,0})}\left(\rho c_p(\theta_{N,0})\dot{\theta}_{N,0} - \mu(\theta_{N,0})\theta_{N,0}\right) + \theta_{N,0} =: \Psi_0(\theta_{N,0}, \dot{\theta}_{N,0}) \ , \tag{12}$$

and (8) and (9) lead to the recursion

$$\begin{aligned} \theta_{N,k+1} = 2\delta_N \Bigg(& \rho c_p(\theta_{N,k})\dot{\theta}_{N,k} - \mu(\theta_{N,k})\theta_{N,k} + \left(\frac{2\lambda(\theta_{N,k})}{\delta_N^2} + \frac{\partial_\theta \lambda(\theta_{N,k})}{\delta_N^2}\theta_{N,k}\right)(\theta_{N,k} - \theta_{N,k-1}) \\ & - \frac{\nu \rho c_p(\theta_{N,k})}{2\delta_N}\theta_{N,k-1} + \frac{\lambda(\theta_{N,k})}{\delta_N^2}\theta_{N,k-1}\Bigg) \Bigg/ \Omega(\theta_{N,k-1}, \theta_{N,k}) =: \Psi_k\left(\theta_{N,k-1}, \theta_{N,k}, \dot{\theta}_{N,k}\right) \end{aligned} \tag{13}$$

$$\begin{aligned} u_N(t) = \bar{g}\Bigg(& \theta_{N,N}, \Bigg(\rho c_p(\theta_{N,N})\dot{\theta}_{N,N} - \mu(\theta_{N,N})\theta_{N,N} + \left(\frac{2\lambda(\theta_{N,N})}{\delta_N^2} + \frac{\partial_\theta \lambda(\theta_{N,N})}{\delta_N^2}(\theta_{N,N} - \theta_{N,N-1})\right) \\ & (\theta_{N,N} - \theta_{N,N-1})\Bigg) \Bigg/ \Omega(\theta_{N,N-1}, \theta_{N,N})\Bigg) =: \Psi_N\left(\theta_{N,N-1}, \theta_{N,N}, \dot{\theta}_{N,N}\right) \ . \end{aligned} \tag{14}$$

where $\bar{g}(\cdot, g(\cdot, u_N(t))) = u_N(t)$. Next it will be shown that these equations already constitute a flat parametrization of all states and of the control input. Choosing $y(t) = \theta_{N,0}$ as the flat output it follows from equation (12) that $\theta_{N,1}$ can be parametrized by $y(t)$ and $\dot{y}(t)$. Differentiating the same equation (12) with respect to time

$$\dot{\theta}_{N,1} = \frac{\partial \Psi_0}{\partial \theta_{N,0}}\dot{\theta}_{N,0} + \frac{\partial \Psi_0}{\partial \dot{\theta}_{N,0}}\ddot{\theta}_{N,0} \tag{15}$$

and inserting this result into (13) for $k = 1$ directly yields a parametrization of $\theta_{N,2}$ by $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$. Obviously, this procedure can be analogously continued for $k = 1, \dots, N-1$ with

$$\dot{\theta}_{N,k+1} = \frac{\partial \Psi_k}{\partial \theta_{N,k-1}}\dot{\theta}_{N,k-1} + \frac{\partial \Psi_k}{\partial \dot{\theta}_{N,k}}\ddot{\theta}_{N,k} + \frac{\partial \Psi_k}{\partial \theta_{N,k}}\dot{\theta}_{N,k} \ , \tag{16}$$

such that every state $\theta_{N,k}$, $k = 1, \dots, N$ as well as the control input u_N are recursively parametrized by $y(t)$ and its first $N+1$ time derivatives. Obviously, the parameter functions λ and ρc_p have to be sufficiently smooth with respect to θ. In view of the considerations to be made in the next section it has even to be ensured that the functions λ and ρc_p are analytic in θ.

Remark 1: Of course, in this general formulation it is assumed that the equations (7), (8) and (9) can actually be solved for $\theta_{N,1}$, $\theta_{N,k+1}$ and $u_N(t)$, respectively. In the given context, this means that $\Omega(\cdot,\cdot) \neq 0$ and that $\bar{g}(\cdot,\cdot)$ exists, conditions which are satisfied in the examples under consideration.

3 Convergence analysis

The crucial question to be answered when solving the trajectory planning problem for distributed parameter systems based on a (semi-)discretized model is whether the resulting controller also solves the motion planning problem of the infinite-dimensional system. For this it is useful

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to arrange the control inputs obtained for the semi-discretizations with different numbers N of grid elements as introduced in Section 2 in form of a sequence $(u_N(t))_{N=1,2,\dots}$. In Section 2.2 it was shown that the finite-difference semi-discretized system (7)–(11) constitutes a flat system. It is known in this case that applying the control input (14) parametrized by a suitably differentiable trajectory $y(t)$ and a finite number of its time-derivatives will result in $\theta_{N,0}(t) = y(t)$. By developing the corresponding trajectories of the semi-discretized state $\theta_{N,k}(t)$ in a power series and considering the case of infinitely small discretization step sizes δ_N, it will be shown subsequently that this power series is equivalent to the power series solution of the underlying DPS (1)–(3). In the case of affine parameters the convergence of this power series can be proven explicitly, which results in inequality constraints on the system parameters and the trajectory for the flat output. For more general situations a numerical convergence test is presented to generalize these results.

3.1 *Polynomial nonlinearities*

3.1.1 From finite differences to power series

In the following, the parameters $\rho c_p(\theta)$, $\lambda(\theta)$ and $\mu(\theta)$ are considered to be given in the form of finite order polynomials, i. e.

$$\rho c_p(\theta) = \sum_{j=0}^{J_1} p_j \theta^j; \quad \lambda(\theta) = \sum_{j=0}^{J_2} q_j \theta^j; \quad \mu(\theta) = \sum_{j=0}^{J_3} r_j \theta^j \ . \tag{17}$$

In order to show that the finite-difference semi-discretization (7)–(11) of the DPS (1)–(3) with parameters (17) converges to a power series formulation for infinitely small discretization step size, i. e. $N \to \infty$, it is assumed that the solutions $\theta_{N,k}$, $k = 0, \dots, N$ are analytic. On this assumption $\theta(t, x)$ can be developed in a power series at an arbitrary node $x = k\delta_N$ in the form

$$\theta_{N,k} = \theta(k\delta_N, t) = \sum_{n=0}^{\infty} a_n(t)(k\delta_N)^n \ . \tag{18}$$

Substituting (18) into the ODEs (7)–(9) and utilizing the relation

$$\begin{gathered}\left(\sum_{n=0}^{\infty}((k+m)\delta_N)^n a_n(t)\right)^j = \sum_{n=0}^{\infty}((k+m)\delta_N)^n \alpha_n^j(t), \quad m \in \{+1, 0, -1\} \\ \text{with} \quad \alpha_n^j(t) = \sum_{i=0}^{n} \alpha_i^{j-1}(t) a_{n-i}(t),\ \alpha_n^1(t) = a_n(t),\ \alpha_n^0(t) = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases} ,\end{gathered} \tag{19}$$

and the abbreviations for the binomial terms

$$(k+1)^j = \sum\nolimits_{i=0}^{j} \binom{j}{i} k^i =: \chi_{k+}^j \tag{20a}$$

$$(k-1)^j = \sum\nolimits_{i=0}^{j} \binom{j}{i} (-1)^{j-i} k^i =: \chi_{k-}^j \ , \tag{20b}$$

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the following relation for $n \geq 0$ is obtained by sorting for equal powers of δ_N

$$
\begin{aligned}
\sum_{j=0}^{J_1} p_j k^n \sum_{i=0}^{n} \alpha_i^j \dot{a}_{n-i}(t) = \sum_{j=0}^{J_2} q_j \underbrace{\sum_{i=0}^{n+2} \alpha_i^j k^i a_{n+2-i}(t) \left(\chi_{k+}^{n+2-i} - 2k^{n+2-i} + \chi_{k-}^{n+2-i}\right)}_{(*)} \\
+ \sum_{j=0}^{J_2-1} (j+1) q_{j+1} \sum_{i=0}^{n+2} \alpha_i^j k^i \underbrace{\sum_{l=0}^{n+2-i} a_l(t) a_{n+2-i-l}(t) \left(\chi_{k+}^{l} - k^l\right) \left(k^{n+2-i-l} - \chi_{k-}^{n+2-i-l}\right)}_{(**)} \\
- \frac{\nu}{2} \sum_{j=0}^{J_1} p_j \sum_{i=0}^{n+1} \alpha_i^j k^i a_{n+1-i}(t) \left(\chi_{k+}^{n+1-i} - \chi_{k-}^{n+1-i}\right) + \sum_{j=0}^{J_3} r_j \alpha_n^{j+1} k^n \ . \qquad (21)
\end{aligned}
$$

From equation (19) it is clear that $\alpha_n^j(t)$ is always affine in the coefficient with the maximal index $a_n(t)$. Keeping in mind that $(*) = (**) = 0$ for $i = n+2$ it can be deduced that equation (21) is affine in $a_{n+2}(t)$ and can therefore be easily solved for this coefficient. Note that $(**) = 0$ for $i = n+2$ as well as for $i = 0$ and $l = \{0, n+2\}$, i. e. the expression is independent of $a_{n+2}(t)$.

Rearranging equation (21) and isolating the coefficients of $a_{n+2}(t)$ yields

$$
\begin{aligned}
a_{n+2}(t) \left(\chi_{k+}^{n+2} - 2k^{n+2} + \chi_{k-}^{n+2}\right) \underbrace{\sum_{j=0}^{J_2} q_j \alpha_0^j}_{1/\beta_{J_2}(t)} = k^n \sum_{i=0}^{n} \dot{a}_{n-i}(t) \sum_{j=0}^{J_1} p_j \alpha_i^j \\
- \sum_{i=1}^{n+1} k^i a_{n+2-i}(t) \left(\chi_{k+}^{n+2-i} - 2k^{n+2-i} + \chi_{k-}^{n+2-i}\right) \sum_{j=0}^{J_2} q_j \alpha_i^j \\
- \sum_{i=0}^{n+1} k^i \sum_{l=0}^{n+2-i} a_l(t) a_{n+2-i-l}(t) \left(\chi_{k+}^{l} - k^l\right) \left(k^{n+2-i-l} - \chi_{k-}^{n+2-i-l}\right) \sum_{j=0}^{J_2-1} (j+1) q_{j+1} \alpha_i^j \\
+ \frac{\nu}{2} \sum_{i=0}^{n+1} k^i a_{n+1-i}(t) \left(\chi_{k+}^{n+1-i} - \chi_{k-}^{n+1-i}\right) \sum_{j=0}^{J_1} p_j \alpha_i^j(t) - k^n \sum_{j=0}^{J_3} r_j \alpha_n^{j+1} \ , \qquad (22)
\end{aligned}
$$

with $1/\beta_{J_2}(t) = \sum_{j=0}^{J_2} q_j \alpha_0^j = \sum_{j=0}^{J_2} q_j a_0^j$ a polynomial of order J_2 in $a_0(t)$. The whole equation is in fact a polynomial of order n in k as can be found by considering the coefficients containing χ_{k-} or χ_{k+}, i. e.

$$
\begin{aligned}
k^i (\chi_{k+}^{n+2-i} - 2k^{n+2-i} + \chi_{k-}^{n+2-i}) = k^{n+2} + \tbinom{n+2-i}{1} k^{n+1} + \tbinom{n+2-i}{2} k^n + \ldots - 2k^{n+2} \\
+ k^{n+2} - \tbinom{n+2-i}{1} k^{n+1} + \tbinom{n+2-i}{2} k^n + \ldots = 2\tbinom{n+2-i}{2} k^n + o(k^{n-1}) \\
= (n+2-i)(n+1-i) k^n + o(k^{n-1}),\ i = 0, \ldots, n+1 \ , \qquad (23a)
\end{aligned}
$$

$$
\begin{aligned}
k^i (\chi_{k+}^{l} - k^l)(k^{n+2-l-i} - \chi_{k-}^{n+2-i-l}) = k^i (l k^{l-1} + o(k^{l-2}))((n+2-i-l) k^{n+1-i-l} + o(k^{n-i-l})) \\
= l(n+2-i-l) k^n + o(k^{n-1}),\ i = 0, \ldots, n+1,\ l = 0, \ldots, n+2-i \ , \qquad (23b)
\end{aligned}
$$

$$\text{and}\quad k^i(\chi_{k+}^{n+1-i} - \chi_{k-}^{n+1-i}) = 2(n+1-i)k^n + o(k^{n-1}),\ i = 0,\dots,n+1\ , \tag{23c}$$

where $o(k^{n-1})$ refers to the remaining polynomial in k with degree less or equal $n-1$. For $k > 0$ equation (22) can then be divided by k^n and for vanishing discretization step size $\delta_N \to 0$, i. e. equivalently $k \to \infty$, only the coefficients of k^n as established in (22)–(23) have to be considered and constitute the following recursion relation for the coefficients $a_{n+2}(t)$, $n \geq 0$ of the power series (18)

$$\begin{aligned} a_{n+2}(t)\frac{(n+2)(n+1)}{\beta_{J_2}(t)} &= \sum_{i=0}^{n} \dot{a}_{n-i}(t)\sum_{j=0}^{J_1} p_j\alpha_i^j - \sum_{i=1}^{n+1} a_{n+2-i}(t)(n+2-i)(n+1-i)\sum_{j=0}^{J_2} q_j\alpha_i^j \\ &\quad - \sum_{i=0}^{n+1}\sum_{l=0}^{n+2-i} a_l(t)a_{n+2-i-l}(t)l(n+2-i-l)\sum_{j=0}^{J_2-1}(j+1)q_{j+1}\alpha_i^j \\ &\quad + \nu\sum_{i=0}^{n+1} a_{n+1-i}(t)(n+1-i)\sum_{j=0}^{J_1} p_j\alpha_i^j - \sum_{j=0}^{J_3} r_j\alpha_n^{j+1}\ . \end{aligned} \tag{24}$$

The recursion can be started using the flat output and the BC at $x = 0$

$$a_0(t) = y(t), \quad a_1(t) = 0. \tag{25}$$

Remark 1: For parameters given in the form of (17), this result can also be obtained by directly solving the trajectory planning problem with a power-series ansatz to the solution, see, e. g., Lynch and Rudolph (2002), Rudolph (2003), Dunbar et al. (2003) for some examples with various parameters that are affine or quadratic in the state.

3.1.2 Convergence result

In a second step, it has to be proven that the power series (18) with coefficients defined by (24) and (25) exists. This will be shown by means of the trajectory planning problem for the DPS (1)–(3) with parameters affine in the state, i. e.

$$\rho c_p(\theta) = p_0 + p_1\theta, \quad \lambda(\theta) = q_0 + q_1\theta, \quad \mu(\theta) = r_0 + r_1\theta\ . \tag{26}$$

Clearly, according to (24) the desired trajectory $y^*(t)$ has to be chosen from a class of infinitely differentiable functions. However, it cannot be an analytic function because this would not allow for a transition from one stationary point to another. In order to overcome this problem the class of Gevrey functions is considered, see, e. g., Hua and Rodino (1996), which are defined as follows.

Definition 3.1: A function $f(t)$ is called Gevrey of order α if for given $M_f, R_f > 0$

$$\sup_{t\in\ ^{+}} |f^{(n)}(t)| \leq M_f\frac{n!^{\alpha}}{R_f^n},\ \forall n \in\ ^{+}\ . \tag{27}$$

To shorten the notation $f(t)$ is also said to belong to $C_{M_f,R_f,\alpha}$. Note that if $f(t) \in C_{M_f,R_f,\alpha}$, then polynomials in $f(t)$ as well as $1/f(t)$ also belong to the same Gevrey class α, only the constants M_f and R_f in (27) differ.

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The convergence result can then be stated as follows:

Theorem 3.2 : *The power series* (18) *with coefficients determined by* (24) *and* (25) *converges with radius of convergence* $\varrho \geq 1$*, if* $a_0(t) = y^*(t)$ *is Gevrey of class* α *with* $1 \leq \alpha \leq 2$*, and if the parameters of the problem satisfy the set of conditions*

$$\frac{9}{2}|q_1|M^2 < 1\ , \tag{28a}$$

$$\left(\frac{9}{2}|q_1| + \frac{55}{36}\nu|p_1|\right) M^2 + \frac{\nu}{2}|p_0|M \leq 1\ , \tag{28b}$$

$$\left(\frac{9}{2}|q_1| + \frac{55}{18}\nu|p_1| + \frac{1445}{384}\frac{|p_1|}{\tilde{R}} + 6|r_1|\right) M^2 + \left(\nu|p_0| + 3\frac{|p_0|}{\tilde{R}} + 3|r_0|\right) M \leq 1\ . \tag{28c}$$

Thereby $\tilde{R} = R/2^{2\alpha}$*,* $M = \max\{M_{y^*}, M_\beta\}$ *and* $R = \min\{R_{y^*}, R_\beta\}$*.*

The proof of the theorem relies on the idea to bound all power series coefficients and their derivatives with respect to time, which allows the calculation of a radius of convergence by the Cauchy-Hadamard theorem. Stipulating a radius of convergence $\varrho \geq 1$ then determines the conditions given in Theorem 3.2. Since the proof is rather technical and contains some lengthy calculations it will be summarized in Appendix A.

Remark 2 : As can be clearly seen from conditions (28), the set of admissible physical parameters $\mathcal{P} := \{p_0, p_1, q_0, q_1, \nu, r_0, r_1\}$ and trajectories represented by M and R is rather limited and no guarantee can be given a priori, whether a trajectory for a given system can be found or what the conditions on the physical parameters would be such that a given trajectory satisfies conditions (28). However, this can be checked rather comfortably by solving the inequalities (28) for a certain set of parameters by means of modern computer algebra systems.

3.1.3 Simulation example

To illustrate the implications of this theorem, some simulation results for a DCR-system (1)–(3) with $g = 2u(t) - \theta$ and parameters according to (26) are considered. With the flat output $y(t) = \theta(t, 0)$, the control task under consideration consists in trajectory planning for the setpoint transition from $y(0) = y_0 = 7$ to $y(T) = y_T = 8$. In order to be able to solve the trajectory planning problem, a suitable reference trajectory $y^*(t)$ has to be found. Basically, this trajectory has to comply with the desired stationary output values at the beginning ($t = 0$) and at the end ($t = T$) of the setpoint transition, i. e.

$$y^*(0) = y_0,\ y^*(T) = y_T \tag{29}$$

$$y^{*(k)}(0) = y^{*(k)}(T) = 0,\ \text{for } k = 1, \ldots, N + 1. \tag{30}$$

As a suitable reference trajectory for this type of problems the following easily parameterizable time functions $y^*(t)$ have been proposed in Fliess et al. (1997)

$$\Phi_{\gamma,\tau}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq T \\ \dfrac{\int_0^t \phi_{\gamma,T}(\tilde{t})\mathrm{d}\tilde{t}}{\int_0^T \phi_{\gamma,T}(\tilde{t})\mathrm{d}\tilde{t}} & \text{if } t \in (0, T) \end{cases} \tag{31}$$

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with

$$\phi_{\gamma,T}(t) = \begin{cases} 0 & \text{if } t \notin (0,T) \\ \exp\left(\dfrac{-1}{[(1-\frac{t}{T})\frac{t}{T}]^{\gamma}}\right) & \text{if } t \in (0,T) \end{cases}, \tag{32}$$

where the Gevrey class α is determined by the parameter $\gamma = 1/(\alpha - 1)$.

The trajectory planning problem is solved with the described method based on finite-difference semi-discretization with various numbers N of grid elements, desired output trajectories belonging to different Gevrey classes, and two sets of parameters $\mathcal{P}$. The first parameter set $\mathcal{P}_1 = \{0.00001, 0.000001, 0.1, 0.002, 0.5, 0.0002, -0.0001\}$ together with M_{y^*}, R_{y^*} resulting from the choice of $y^*(t) = y_0 + (y_T - y_0)\Phi_{\gamma,1}(t)$ fulfills the conditions of Theorem 3.2 for $\alpha = 1.8$. The second parameter set $\mathcal{P}_2 = \{0.1, 0.01, 0.1, 0.002, 0.5, 0.02, -0.01\}$ does not. Additionally, the trajectory planning problem is solved for a trajectory belonging to $C_{M_{y^*},R_{y^*},2.4}$, where the requirements of Theorem 3.2 are not satisfied, too. The resulting control input trajectories $u_N(t)$ are shown in Figure 1 for $N = 2$, 10 and 18. In the same figure, the corresponding simulation results for the output $y_N(t)$ are displayed. The simulations of the feedforward-controlled DPS (1)–(3) with $u(t)$ replaced by $u_N(t)$ are performed using the Matlab solver `pdepe` on a sufficiently fine grid.

It can be seen that the motion planning problem is solved in a satisfactory way not only in the case where the conditions of Theorem 3.2 are satisfied, but also in cases, where the trajectory or the system parameters do not comply with these conditions. In the example scenarios shown in Figure 1, only for the parameter set $\mathcal{P}_2$ and $\alpha = 2.4$, oscillations in the control input $u_N(t)$, indicating a divergent behaviour, can be observed.

3.2 *Numerical convergence considerations*

It is evident from the previous section that an analytical convergence analysis is of limited use for applications mainly due to two major reasons. Firstly, the convergence can only be shown for very special cases and yields rather restrictive conditions, which do not seem to be very sharp as far as it can be presumed from the simulation results in Figure 1. Furthermore, it is known from research results in summability methods for power series solutions, see, e. g., Meurer (2005) that diverging sums may still be used to determine a control input solving a given motion planning problem. Therefore, a numerical test is presented that allows us to evaluate from a limited number of calculated trajectories whether the method will converge or not. The test is motivated by using the previously considered DCR-system as an example.

From Figure 1 it can be deduced that non-convergence of the sequence of control inputs $(u_N(t))_{N=1,2,\ldots}$ is related to increasing oscillations, especially at the beginning and at the end of the transition. Due to the diffusive character of the considered equations, these oscillations are significantly attenuated in the simulated output. Therefore, a criterion is sought after that detects the onset of oscillations in the trajectory of the control input. It is furthermore preferable to work with the control input $u_N(t)$ because this quantity is independent of any solver used to simulate the system equations. However, for $u_N(t)$ the limit for $N \to \infty$ is usually not known, which means that differences between control inputs calculated for different numbers N_m and N_n of grid elements have to be considered, i. e. $u_{N_m}(t) - u_{N_n}(t)$, and convergence has to be verified in the sense of Cauchy, i. e. for a sequence $(a_n)_{n=1,2,\ldots}$ if $\forall \epsilon > 0\ \exists n_0 = n_0(\epsilon)$, s.t. $|a_m - a_n| < \epsilon\ \forall m, n > n_0$. As the elements of the sequence $(u_N(t))_{N=1,2,\ldots}$ are in fact

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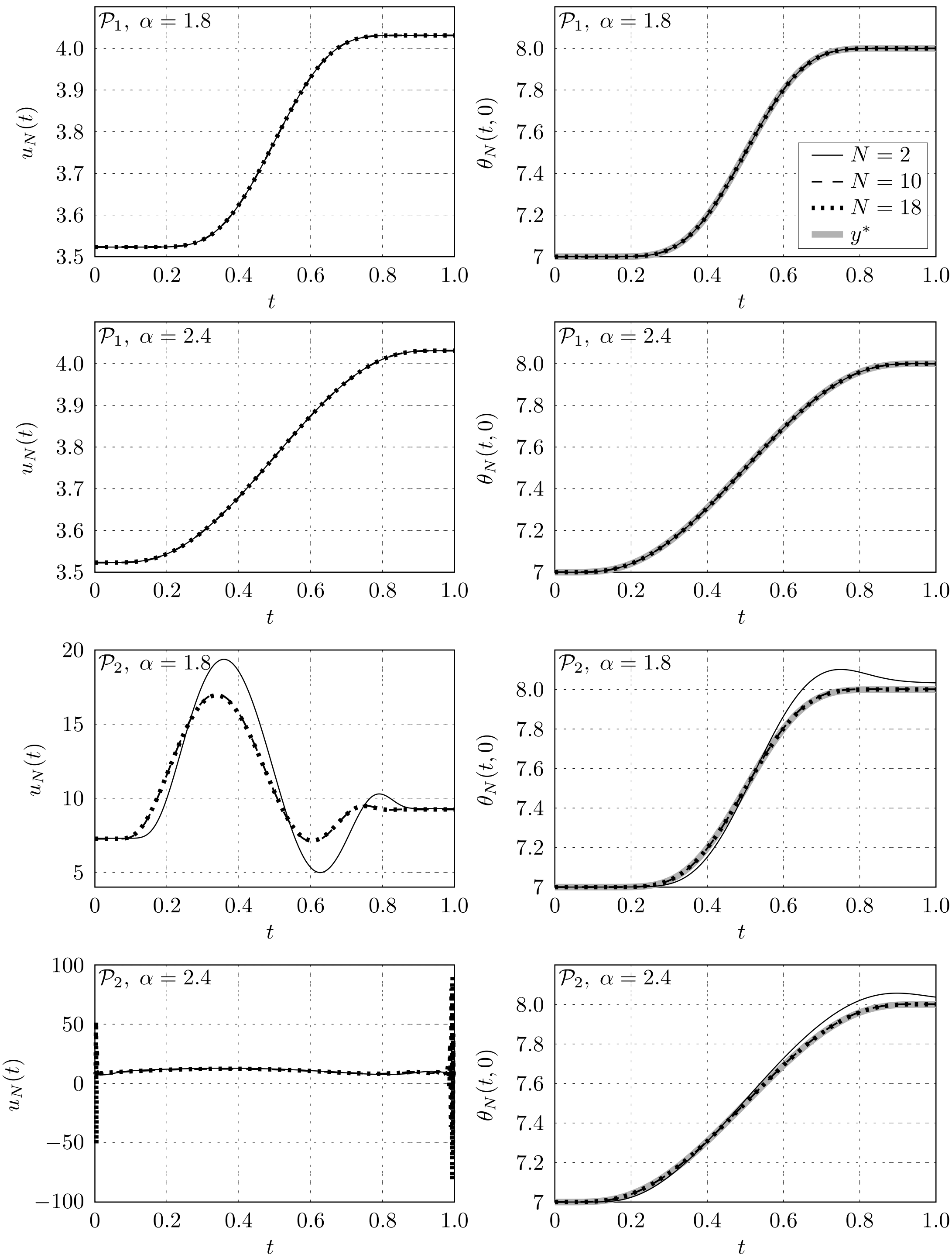

Figure 1. Control input $u_N(t)$ and output trajectories $\theta_N(t,0)$ for the trajectory planning of the DCR-system for the parameter sets $\mathcal{P}_1$ and $\mathcal{P}_2$, desired trajectories of different Gevrey classes α and different numbers N of grid elements.

sampled functions of time the following is considered

$$\sup_{t\in[0,T]} \left|\left(u_{N_n}(t) - u_{N_m}(t)\right)/\Delta y\right|, \quad N_n, N_m \in [1, N]\,. \tag{33}$$

The scaling factor $\Delta y = y_T - y_0$ is merely used to ensure comparability for different setpoint

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transitions (in the previous example of Figure 1, $\Delta y = 1$).

In Figure 2 the logarithms of (33) are shown up to $N = 18$ grid elements. For $\mathcal{P}_1$ with $\alpha = 1.8$ it can be seen that $u_N(t)$ does not change any more for $N \geq 4$. The same is true for $\mathcal{P}_1$ with $\alpha = 2.4$ and $\mathcal{P}_2$ with $\alpha = 1.8$, although in the latter case there is a larger difference between $u_2(t)$ and $u_N(t)$ for $N \geq 4$, which confirms the results from Figure 1. In the case $\mathcal{P}_2$ with $\alpha = 2.4$ it is evident that the difference for any pair of (N_m, N_n) is increasing, which indicates the oscillations that can be observed in Figure 1. Although the criterion cannot give definite assertions on the behaviour for even larger N, its feasibility to estimate the quality of the control inputs is justified.

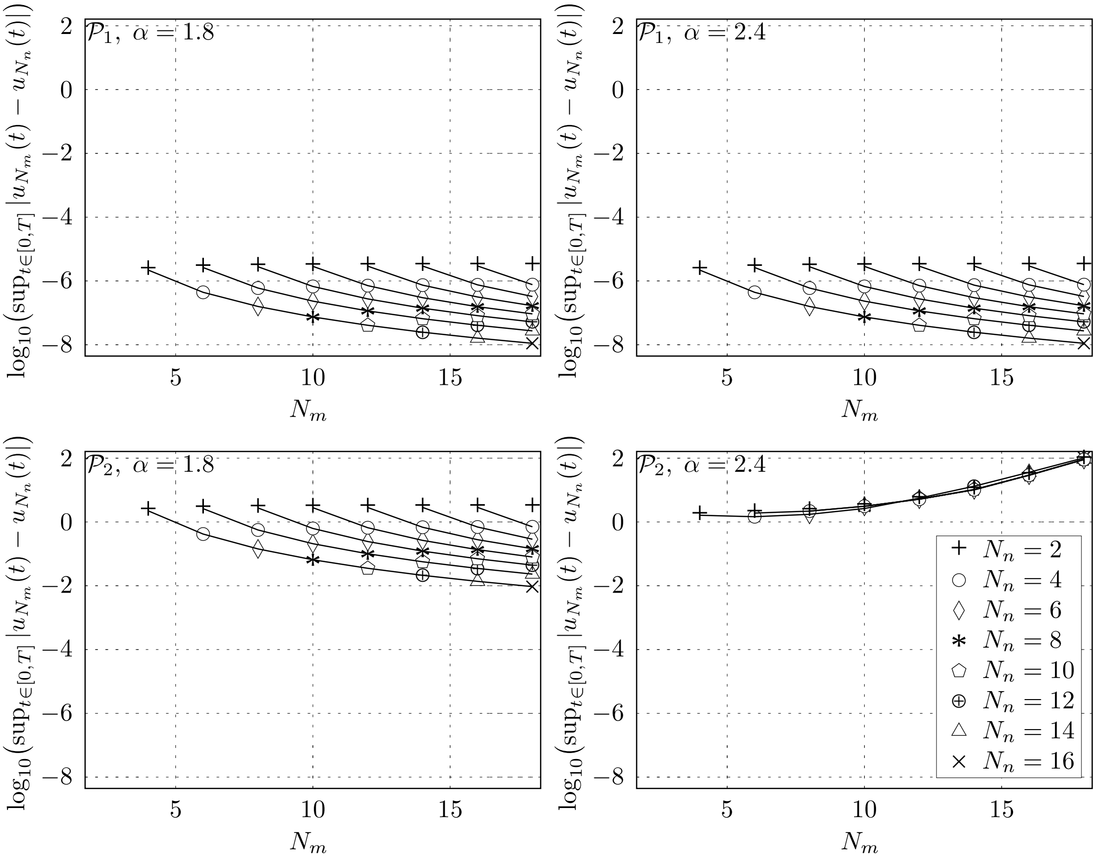

Figure 2. Numerical convergence criterion for the trajectory planning of the DCR-system.

In the following section, this numerical criterion is applied to a trajectory planning problem, where no analytical results are available for the verification of the convergence of the determined parametrization.

4 Example: Heat-up of a steel slab

Trajectory planning based on finite-difference semi-discretization is demonstrated for the heat-up process of steel slabs in reheating furnaces as it is done for example in hot rolling mills to reach a certain final temperature distribution within the slab required for the hot rolling process, see, e. g., Wild et al. (2009). In this context, radiation represents the primary mode of heat transfer within the furnace such that energy is exchanged between the furnace and the slab along the slab surface. Modeling of the temperature distribution within a slab may be reduced to the heat equation in a spatially 1-dimensional domain (McGuinness and Taylor 2004, Wild et al. 2009), which is defined by half the height of the slab, $L = 0.145\,\mathrm{m}$. Then, the BCs can be written according to (2) with $g(\theta(t,1),u(t)) = \frac{\epsilon\sigma}{\lambda(\theta(t,1))}\left(u^4(t) - \theta^4(t,1)\right)$ where $\epsilon = 0.7$ denotes the emissivity coefficient and σ is the Stefan-Boltzmann constant. Due to the large temperature range the temperature-dependence of the thermal conductivity λ and of the heat capacity c_p, which undergoes large variations at around $1000 - 1100\,\mathrm{K}$ due to the re-crystallization of steel, has to be taken into account. The density $\rho = 7880.8\,\frac{\mathrm{kg}}{\mathrm{m}^3}$ can be assumed constant.

The trajectory planning problem for the slab then concerns the calculation of a wall temperature $u(t)$ such that the core temperature of the slab $\theta(t,0)$ follows a desired trajectory $y^*(t) = y_0 + \Delta y \Phi_{\gamma,T}$ with $\Delta y = y_T - y_0$, $y_0 = 300\,\mathrm{K}$ and $y_T = 1500\,\mathrm{K}$, and a transition time of $T = 10\,\mathrm{h}$.

In order to solve this trajectory planning problem with the method outlined in Section 2, the material parameters, which are usually identified for certain temperature values and stored in tables, see, e. g., BISRA (1953), Harste (1989), have to be approximated by sufficiently smooth functions. Henceforth, functions of the form

$$\lambda(\theta) = \frac{\eta_1}{2}\left(\tanh\left(\frac{\theta - \eta_2}{\eta_3}\right) + 1\right) + \frac{(\eta_4\theta + \eta_5)}{2}\left(\tanh\left(\frac{\theta - \eta_6}{\eta_7}\right) + 1\right) + \eta_8 \tag{34}$$

$$c_p(\theta) = \left(\frac{(\kappa_1\theta + \kappa_2)}{2}\left(\tanh\left(\frac{\theta - \kappa_3}{\kappa_4}\right) + 1\right) + \frac{(\kappa_5\theta + \kappa_6)}{2}\left(\tanh\left(\frac{\theta - \kappa_7}{\kappa_8}\right) + 1\right)\right)^{-1} \tag{35}$$

are used, with the parameters $\eta_{1,\ldots,8}$, $\kappa_{1,\ldots,8}$ chosen so as to best fit the experimental values, see Figure 3. In Figure 4, the control inputs resulting from the trajectory planning for a desired

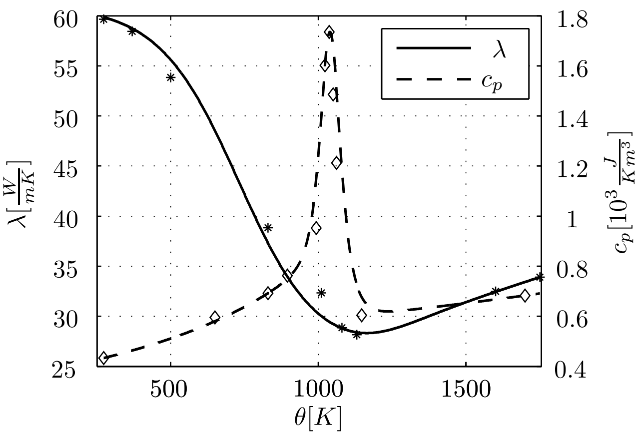

Figure 3. Tabulated values marked with $*$ and $\diamond$ and approximating functions of $\lambda(\theta)$ and $\rho c_p(\theta)$, respectively.

trajectory of Gevrey class $\alpha = 1.8$ and 4.5 are depicted for different numbers N of grid elements. Note that the maximum number N of grid elements is chosen smaller compared to the previous example of the DCR-system in Section 3.1.3 due to the fact that evaluating the differential recursion (14) becomes a tedious calculation because of the derivatives of the functions (34) and (35). It has to be noted, however, that evaluating power series solutions for the trajectory planning problem would certainly be even more difficult, since multiple Cauchy products would have to be calculated. The simulation results of the core temperature of the slab $\theta_{N,0} = \theta_N(t,0)$ are also shown in Figure 4. Thereby the simulation of the feedforward controlled system (1)–(3)

with $u(t)$ replaced by $u_N(t)$ is performed by using the Matlab solver `pdepe` on a sufficiently fine grid and by using the tabulated values of the parameters according to Figure 3. It can be seen in

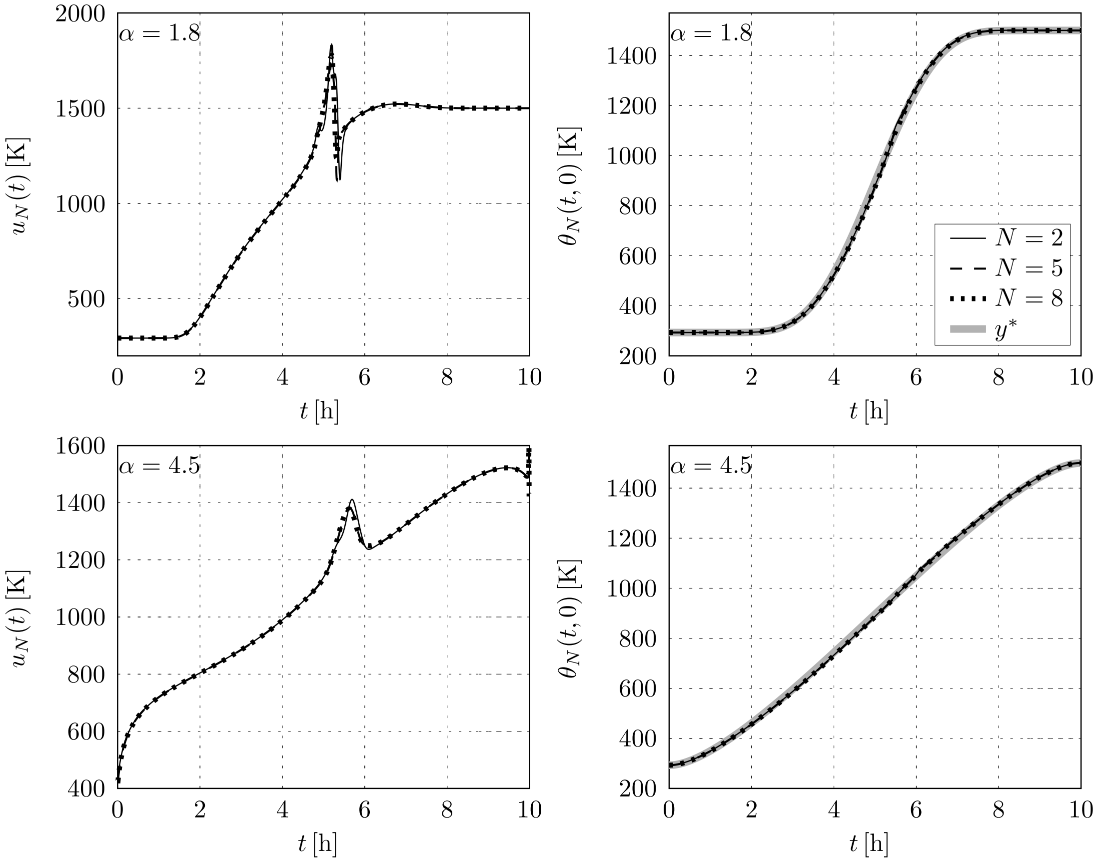

Figure 4. Control inputs $u_N(t)$ and output trajectories $\theta_N(t, 0)$ for the trajectory planning of the heat-up process for desired output trajectories of different Gevrey classes α and different numbers N of grid elements.

Figure 4 that the control input $u_N(t)$ at the boundary has to changed in a fast manner in order to maintain the desired heat-up during the re-crystallization phase. Prescribing the flat output to slow the heat-up during the re-crystallization can significantly reduce these steep gradients, see, e. g., Utz et al. (2007). Apart from this it can be observed for $\alpha = 1.8$ that the control input changes significantly around the re-crystallization temperature from $N = 2$ to 5, but no more for $N = 8$. At any rate, there is no oscillation visible at the beginning or end of the trajectory. This is in contrast to the control trajectory obtained for a desired output trajectory of Gevrey class $\alpha = 4.5$.

The numerical criterion evaluated for the considered trajectory planning problem is given for different values of α in Figure **??**. For $\alpha = 1.8$ the criterion confirms the simulation result, and it also indicates a convergent behaviour for $\alpha = 2$ and $\alpha = 2.5$. In the case $\alpha = 4.5$, the onset of oscillations can be determined. However, this becomes evident only for $N \geq 6$, illustrating the already mentioned problem that the presented test cannot be seen as a formal proof of convergence as divergent behaviour may only appear if the control input $u_N(t)$ is calculated for larger numbers N of grid elements.

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5 Conclusions

This contribution presents an approach to the solution of the trajectory planning problem for infinite-dimensional systems governed by parabolic PDEs based on finite-difference semi-discretizations. In the case of quasi-linear systems with parameters being polynomials in the state the resulting control input is shown to be equivalent to a power series solution of the trajectory planning problem for vanishing discretization step size. In some special cases it can be shown that the power series converges under certain conditions on the system parameters and the trajectory.

The presented method can easily be applied to quasi-linear problems involving parameters depending on the state and nonlinear boundary conditions, where analytical convergence results are not available. Therefore, a numerical test criterion for the convergence behaviour is suggested by exploiting the analogy to examples where analytical results are known. The feasibility of the presented method is demonstrated by means of the practical example of a heat-up trajectory planning problem for steel slabs.

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Appendix A: Proof of Theorem 3.2

The proof of Theorem 3.2 is based on induction to show that the coefficients $a_n(t)$ of the power series (18) with coefficients defined by (24) and (25) satisfy

$$\sup_{t\in {}^{+}} |a_n^{(l)}(t)| \leq \begin{cases} \dfrac{M\epsilon^n}{\tilde{R}^l}\dfrac{(n+l-2)!^\alpha}{n!^\alpha}, & n+l\geq 2 \\ \dfrac{M\epsilon^n}{\tilde{R}^l}, & \text{otherwise} \end{cases} \tag{A1}$$

with ϵ a positive constant depending on the parameters M and $R = 2^{2\alpha}\tilde{R}$ and the system parameters. Using the theorem of Cauchy-Hadamard a relation between ϵ and the radius of convergence $\varrho = (\limsup_{n\to\infty} \sqrt[n]{a_n})^{-1}$ of the power series $\sum_{i=0}^{\infty} a_n x^n$ can be established according to

$$\frac{1}{\varrho} = \limsup_{n\to\infty} \sqrt[n+2]{|a_{n+2}|} \leq \limsup_{n\to\infty} \sqrt[n+2]{\frac{M\epsilon^{n+2}}{(n+2)^\alpha (n+1)^\alpha}} = \epsilon\ . \tag{A2}$$

Hence, the condition $\epsilon \leq 1$ yields a radius of convergence of at least 1, which is sufficient in the considered set-up.

First, let us recall the equation defining the power series coefficients (24) with the paramet-

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ers (26)

$$
\begin{aligned}
a_0(t) &= y(t), \quad a_1(t) = 0, \\
a_{n+2}(t) &= \frac{\beta_1(t)}{(n+2)(n+1)} \Bigg(p_0 \dot{a}_n(t) + p_1 \sum_{i=0}^{n} \dot{a}_{n-i}(t) a_i(t) \\
&\quad - q_1(n+1) \sum_{i=1}^{n-1} (n+1-i) a_{n+1-i}(t) a_{i+1}(t) + \nu \big[p_0(n+1) a_{n+1}(t) \\
&\quad + p_1 \sum_{i=0}^{n} (n+1-i) a_{n+1-i}(t) a_i(t) \big] - r_0 a_n(t) - r_1 \sum_{i=0}^{n} a_{n-i}(t) a_i(t) \Bigg) .
\end{aligned} \tag{A3}
$$

Since $a_0(t) = y(t)$, Lemma A.1, given at the end of this appendix, implies that (A1) is true for $a_0^{(l)}(t)$. For $a_1(t) \equiv 0$ it is generally true, and for $a_n(t)$, $n \geq 0$, the inequality condition (A1) will be shown by induction. The time derivative of $a_{n+2}(t)$ can be easily deduced using the Leibniz formula and can be bounded as follows

$$
\begin{aligned}
|a_{n+2}^{(l)}| &\leq |p_0| \sum_{r=0}^{l} \binom{l}{r} \frac{|\beta_1^{(l-r)}||a_n^{(r+1)}|}{(n+2)(n+1)} + |p_1| \sum_{i=0}^{n} \sum_{r=0}^{l} \binom{l}{r} \frac{|\beta_1^{(l-r)}|}{(n+2)(n+1)} \sum_{s=0}^{r} \binom{r}{s} |a_{n-i}^{(r-s+1)}||a_i^{(s)}| \\
&\quad + |q_1| \sum_{i=1}^{n-1} \frac{(n+1-i)}{n+2} \sum_{r=0}^{l} \binom{l}{r} |\beta_1^{(l-r)}| \sum_{s=0}^{r} \binom{r}{s} |a_{n+1-i}^{(r-s)}||a_{i+1}^{(s)}| \\
&\quad + \nu |p_0| \sum_{r=0}^{l} \binom{l}{r} \frac{|\beta_1^{(l-r)}|}{(n+2)} |a_{n+1}^{(r)}| + \nu |p_1| \sum_{i=0}^{n} \frac{(n+1-i)}{(n+2)(n+1)} \sum_{r=0}^{l} \binom{l}{r} |\beta_1^{(l-r)}| \sum_{s=0}^{r} \binom{r}{s} |a_{n+1-i}^{(r-s)}||a_i^{(s)}| \\
&\quad + |r_0| \sum_{r=0}^{l} \binom{l}{r} \frac{|\beta_1^{(l-r)}|}{(n+2)(n+1)} |a_n^{(r)}| + |r_1| \sum_{i=0}^{n} \sum_{r=0}^{l} \binom{l}{r} \frac{|\beta_1^{(l-r)}|}{(n+2)(n+1)} \sum_{s=0}^{r} \binom{r}{s} |a_{n-i}^{(r-s)}||a_i^{(s)}| .
\end{aligned} \tag{A4}
$$

In order to be able to express (A4) in the form of (A1) several cases have to be distinguished. For this, observe that (A4) simplifies significantly for $n = 0$ and $n = 1$. Hence, using (A1) and $a_1(t) \equiv 0$ it follows that

$$
\begin{aligned}
|a_2^{(l)}| &\leq |p_0| \frac{M^2}{2\tilde{R}^{l+1}} \left[\sum_{r=0}^{l} \binom{l}{r} \frac{(l-r)!^\alpha (r+1)!^\alpha}{2^{2\alpha(l+1)}} \right] \\
&\quad + |p_1| \frac{M^3}{2\tilde{R}^{l+1}} \left[\sum_{r=0}^{l} \binom{l}{r} \frac{(l-r)!^\alpha}{2^{2\alpha(l+1)}} \sum_{s=0}^{r} \binom{r}{s} (r-s+1)!^\alpha s!^\alpha \right] \\
&\quad + |r_0| \frac{M^2}{2\tilde{R}^{l}} \left[\sum_{r=0}^{l} \binom{l}{r} \frac{(l-r)!^\alpha}{2^{2\alpha l}} \right] + |r_1| \frac{M^3}{2\tilde{R}^{l}} \left[\sum_{r=0}^{l} \binom{l}{r} \frac{(l-r)!^\alpha}{2^{2\alpha l}} \sum_{s=0}^{r} \binom{r}{s} (r-s)!^\alpha s!^\alpha \right] ,
\end{aligned} \tag{A5}
$$

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$$
\begin{aligned}
|a_3^{(l)}| \leq \nu|p_0|\frac{M^2\epsilon^2}{3\cdot 2^\alpha \tilde{R}^l}&\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha r!^\alpha}{2^{2\alpha(l-r)}}\right]\\
&+\nu|p_1|\frac{M^3\epsilon^2}{3\cdot 2^\alpha \tilde{R}^l}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{2^{2\alpha(l-r)}}\sum_{s=0}^{r}\binom{r}{s}\frac{(r-s)!^\alpha s!^\alpha}{2^{2\alpha s}}\right],
\end{aligned}
\tag{A6}
$$

while

$$
\begin{aligned}
|a_{n+2}^{(l)}| \leq |p_0|\frac{M^2\epsilon^n}{\tilde{R}^{l+1}}&\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha(n+r-1)!^\alpha}{(n+2)(n+1)2^{2\alpha(l-r)}n!^\alpha}\right]\\
&+|p_1|\frac{M^3\epsilon^n}{\tilde{R}^{l+1}}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{(n+2)(n+1)2^{2\alpha(l-r)}}\times\right.\\
&\sum_{s=0}^{r}\binom{r}{s}\left(\frac{(n+r-s-1)!^\alpha s!^\alpha}{2^{2\alpha s}n!^\alpha}+\frac{(r-s+1)!^\alpha(n+s-2)!^\alpha}{2^{2\alpha(r-s+1)}n!^\alpha}\right)\\
&\left.+\sum_{i=2}^{n-2}\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{(n+2)(n+1)2^{2\alpha(l-r)}}\underbrace{\sum_{s=0}^{r}\binom{r}{s}\frac{(n-i+r-s-1)!^\alpha(i+s-2)!^\alpha}{(n-i)!^\alpha i!^\alpha}}_{\overset{(A14),\,(A15)}{\leq}\frac{(n-i-1)!^\alpha(i-2)!^\alpha(n+r-2)!^\alpha}{(n-2)!^\alpha(n-i)!^\alpha i!^\alpha}}\right]\\
&+|q_1|\frac{M^3\epsilon^{n+2}}{\tilde{R}^l}\sum_{i=1}^{n-1}\frac{n+1-i}{n+2}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{2^{2\alpha(l-r)}}\underbrace{\sum_{s=0}^{r}\binom{r}{s}\frac{(n-i+r-s-1)!^\alpha(i+s-1)!^\alpha}{(n+1-i)!^\alpha(i+1)!^\alpha}}_{\overset{(A14),\,(A15)}{\leq}\frac{(n-i-1)!^\alpha(i-1)!^\alpha(n+r-1)!^\alpha}{(n-1)!^\alpha(n+1-i)!^\alpha(i+1)!^\alpha}}\right]\\
&+\nu|p_0|\frac{M^2\epsilon^{n+1}}{\tilde{R}^l}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha(n+r-1)!^\alpha}{(n+2)2^{2\alpha(l-r)}(n+1)!^\alpha}\right]\\
&+\nu|p_1|\frac{M^3\epsilon^{n+1}}{\tilde{R}^l}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{(n+2)2^{2\alpha(l-r)}}\sum_{s=0}^{r}\binom{r}{s}\frac{(n+r-s-1)!^\alpha s!^\alpha}{2^{2\alpha s}(n+1)!^\alpha}\right.\\
&\left.+\sum_{i=2}^{n-1}\frac{n+1-i}{(n+2)(n+1)}\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{2^{2\alpha(l-r)}}\underbrace{\sum_{s=0}^{r}\binom{r}{s}\frac{(n-i+r-s-1)^\alpha(i+s-2)!^\alpha}{(n+1-i)!^\alpha i!^\alpha}}_{\overset{(A14),\,(A15)}{\leq}\frac{(n-i-1)!^\alpha(i-2)!^\alpha(n+r-2)!^\alpha}{(n-2)!^\alpha(n+1-i)!^\alpha i!^\alpha}}\right]\\
&+|r_0|\frac{M^2\epsilon^n}{\tilde{R}^l}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha(n+r-2)!^\alpha}{(n+2)(n+1)2^{2\alpha(l-r)}n!^\alpha}\right]+|r_1|\frac{M^3\epsilon^n}{\tilde{R}^l}\left[\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{(n+2)(n+1)2^{2\alpha(l-r)}}\times\right.\\
&\sum_{s=0}^{r}\binom{r}{s}\left(\frac{(n+r-s-2)!^\alpha s!^\alpha}{2^{2\alpha s}n!^\alpha}+\frac{(r-s)!^\alpha(n+s-2)!^\alpha}{2^{2\alpha(r-s)}n!^\alpha}\right)\\
&\left.+\sum_{i=2}^{n-2}\sum_{r=0}^{l}\binom{l}{r}\frac{(l-r)!^\alpha}{(n+2)(n+1)2^{2\alpha(l-r)}}\underbrace{\sum_{s=0}^{r}\binom{r}{s}\frac{(n-i+r-s-2)!^\alpha(i+s-2)!^\alpha}{(n-i)!^\alpha i!^\alpha}}_{\overset{(A14),\,(A15)}{\leq}\frac{(n-i-2)!^\alpha(i-2)!^\alpha(n+r-3)!^\alpha}{(n-3)!^\alpha(n-i)!^\alpha i!^\alpha}}\right]
\end{aligned}
\tag{A7}
$$

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for $n \geq 2$. Keeping in mind the bound (A1), it is essential for the proof that all expressions in square brackets in (A5)–(A7) multiplied by $(n+2)!^\alpha/(n+l)!^\alpha$ with n that applies to equation (A5), (A6) and (A7), respectively, can be bounded for $1 \leq \alpha \leq 2$ by an expression independent of l and n. It can be shown that all these expressions are monotonically decreasing in l for $\alpha \geq 1$ and monotonically decreasing in n for $\alpha \leq 2$. Thus, all expressions in Equations (A5) and (A6) can be bounded by evaluating them at $l = 0$ and $\alpha = 1$, and the expressions in Equation (A7) consisting of a single sum can be bounded by evaluating them at $l = 0$, $\alpha = 2$ and $n = 2$. In the cases where the expression in square brackets consists of multiple sums, it has to be considered that some of these sums only yield non-zero values for $n \geq 3$ or $n \geq 4$ and are monotonically decreasing thereafter. Consequently, these expressions can be bounded by the maximum value resulting from an evaluation at $l = 0$, $\alpha = 2$ and $n \in [2, 3, 4]$. Applying these bounds and rearranging the inequalities yields

$$|a_2^{(l)}| \leq \frac{M\epsilon^2}{\tilde{R}^l}\frac{l!^\alpha}{2^\alpha}\underbrace{\left[\frac{2^{\alpha-1}}{\epsilon^2}\left(\frac{M}{4\tilde{R}}|p_0| + \frac{M^2}{4\tilde{R}}|p_1| + M|r_0| + M^2|r_1|\right)\right]}_{=T_1(\epsilon)} \tag{A8}$$

$$|a_3^{(l)}| \leq \frac{M\epsilon^3}{\tilde{R}^l}\frac{(l+1)!^\alpha}{3!^\alpha}\underbrace{\left[\frac{3^{\alpha-1}}{\epsilon}\left(M\nu|p_0| + M^2\nu|p_1|\right)\right]}_{=T_2(\epsilon)} \tag{A9}$$

$$|a_{n+2}^{(l)}| \leq \frac{M\epsilon^{n+2}}{\tilde{R}^l}\frac{(n+l)!^\alpha}{(n+2)!^\alpha}$$
$$\underbrace{\frac{1}{\epsilon^2}\left[\frac{9}{2}|q_1|M^2\epsilon^2 + \left(\nu|p_0|M + \frac{55}{18}\nu|p_1|M^2\right)\epsilon + 3|p_0|\frac{M}{\tilde{R}} + \frac{1445}{384}|p_1|\frac{M^2}{\tilde{R}} + 3|r_0|M + 6|r_1|M^2\right]}_{=T_3(\epsilon)}\,. \tag{A10}$$

Obviously, the relations

$$T_1(\epsilon) = 1,\; T_2(\epsilon) = 1 \text{ and } T_3(\epsilon) = 1 \tag{A11}$$

determine conditions under which (A1) is satisfied for $n = 0$, $n = 1$ and $n \geq 2$, respectively. These conditions can be obtained by solving the equations (A11) and stipulating $0 \leq \epsilon \leq 1$ as mentioned at the beginning of the proof.

It turns out that the conditions determined by $T_3(\epsilon)$ are the most restrictive ones in the sense that the admissible ranges of the coefficients are subsets of those determined by $T_1(\epsilon)$ and $T_2(\epsilon)$. By reformulating $T_3 = 1$ as a quadratic equation $a\epsilon^2 + b\epsilon + c = 0$ with

$$\begin{aligned} a &= \frac{9}{2}|q_1|M^2 - 1, \quad 0 < b = \nu|p_0|M + \frac{55}{18}\nu|p_1|M^2 \\ \text{and} \quad 0 < c &= 3|p_0|\frac{M}{\tilde{R}} + \frac{1445}{384}|p_1|\frac{M^2}{\tilde{R}} + 3|r_0|M + 6|r_1|M^2\,, \end{aligned} \tag{A12}$$

this equation only has a solution $\epsilon \in {}^{+}$ if $a < 0$ leading to condition (28a). The only possible solution takes the form

$$0 \leq \epsilon = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \overset{!}{\leq} 1\,.$$

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This inequality is satisfied if $-2a - b \geq 0$, i. e. condition (28b), and $a + b + c \leq 0$, i. e. condition (28c), which concludes the proof.

The following lemmas are utilized at several occasions in the proof of Theorem 3.2 and are given here without proof.

Lemma A.1: *(Lynch and Rudolph 2002) If y : $\;^{+} \to$ is Gevrey of order α then for* $0 \leq k \leq l$

$$\sup_{t \in \;^{+}} |y^{(l)}(t)| \leq \frac{M}{(R/2^{k\alpha})^l} (l - k)!^{\alpha} \tag{A13}$$

Lemma A.2: *(Gevrey 1918)*

$$\sum_k L_k^{\alpha} \leq \left(\sum_k L_k \right)^{\alpha} \ , \quad \alpha \geq 1, \ L_k \geq 0 \tag{A14}$$

Lemma A.3: *(Petkovsek et al. 1996)*

$$\frac{i!j!(i+j+n+1)!}{(i+j+1)!} = \sum_{k=0}^{n} \binom{n}{k} (j+k)!(i+n-k)! \ , \quad i, j, n \geq 0 \tag{A15}$$

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