TECHNISCHE UNIVERSITÄT WIEN

AUTOMATION & CONTROL INSTITUTE
INSTITUT FÜR AUTOMATISIERUNGS- & REGELUNGSTECHNIK

MATHEMATICAL MODELING

Lecture and Exercises
2025S

Univ.-Prof. Dr.-Ing. Wolfgang Kemmetmüller
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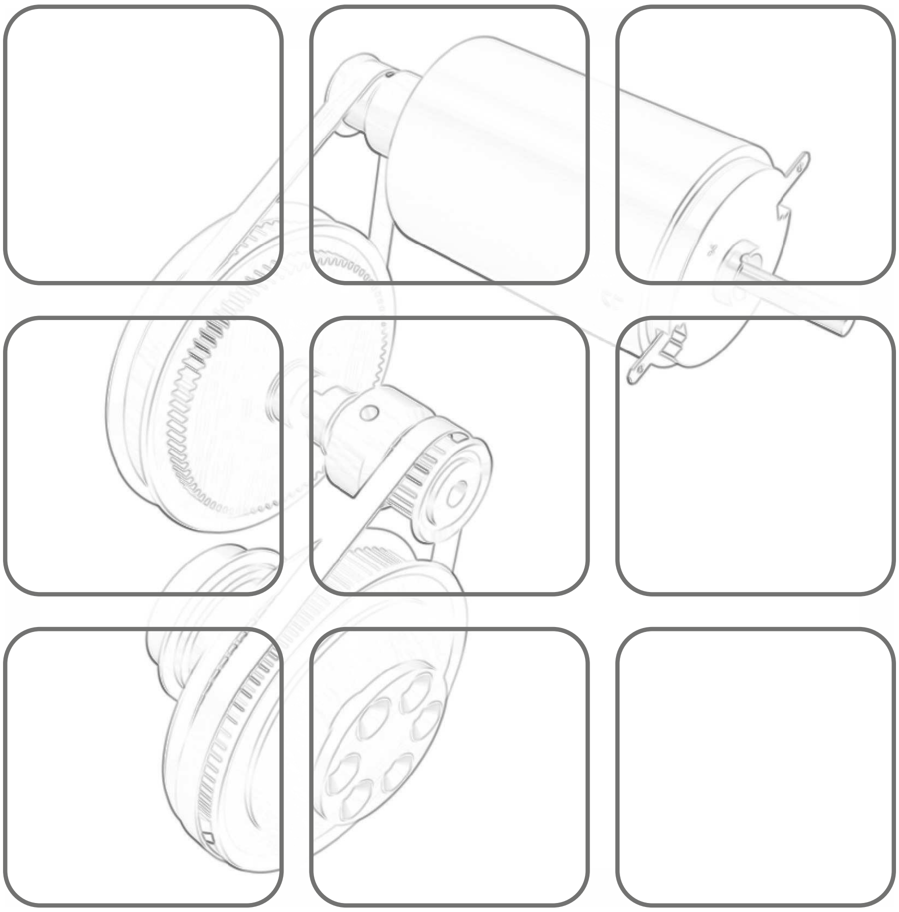

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TU Wien
Automation and Control Institute
Complex Dynamical Systems Group

Gußhausstraße 27–29
1040 Wien
Phone: +43 1 58801 – 37615
Internet: https://www.acin.tuwien.ac.at

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1 Introduction

This lecture deals with the modeling of technical systems. As a first step, it is therefore necessary to clarify what is meant by a *system*. Simply put, *a system is the connection of different components that are interconnected to form a whole for the purpose of performing certain tasks*. The interaction of a system with the system environment takes place via the so-called *input* or *output variables*, see Figure 1.1.

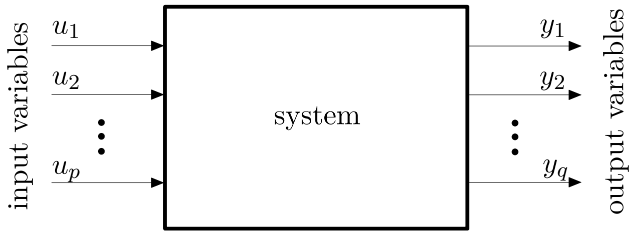


Figure 1.1: On the concept of systems.

The input variables u_1, $u_2, \ldots, u_p$ are variables that act on the system from the system environment and are not influenced by the behavior of the system itself. A distinction is made between input variables with which the system can be influenced in an control engineering sense (*manipulated variables*) and input variables that are not under our control (*disturbance variables*). The output variables y_1, $y_2, \ldots, y_q$ are variables that are generated by the system and in turn influence the system environment. Output variables that can be measured are also called *measured variables*.

A *model* is essentially a *limited representation of reality* that takes into account the essential properties of the system for the task at hand. In a *mathematical model*, the behavior of the real system is represented in an abstract form, for example by algebraic equations, ordinary or partial differential equations. At this point it is important to emphasize that *no mathematical model can represent a system exactly*. Rather, a mathematical model is always a *compromise* between *model complexity* and *model accuracy* with respect to the desired properties. In order to develop a mathematical model that is suitable for the respective question, various steps of *decomposition* (breaking down the system into individual subsystems and components), *reduction* and *abstraction* (omitting details that are irrelevant for the task and transferring to a simpler substitute system) and *aggregation* (combining components and subsystems into a whole) must be carried out, sometimes in recurring loops. These steps can only be systematized to a limited extent, which is why the creation of a suitable mathematical model is at least partly an art and always will be. The mathematical model forms the basis not only for *system analysis*, in which the static and dynamic behavior of the system is investigated as a function of the input variables and system parameters, but also for *system synthesis*, i.e. the design of the overall system. The latter point includes in particular the design of suitable sensors and actuators up to the control design, which will be dealt with in detail in the Automation lecture next semester.

Basically, a distinction is made between theoretical and experimental modeling. In *experimental modeling*, the mathematical model is created on the basis of the measured input and output variables in such a way that the input-output behavior is reproduced as well as possible. This type of modeling is also called *system identification* and models that are based exclusively on experimental information are called *black box models*. Since black box models are based solely on experimental results and use no (or very little) a priori knowledge of the system, the model obtained in this way is only valid for the data set covered by the identification. The main advantage is that relatively little knowledge about the system is required. In contrast, in *theoretical modeling*, the mathematical models are derived on the basis of fundamental physical laws. In this context, one also speaks of *white box models* or *first-principles models*. Between the black box and white box models, there are different degrees of *gray box models*, depending on the ratio of experimental to physically based model information. It should be mentioned here that it is generally not possible to derive a mathematical model exclusively from physical laws and to parameterize it completely. Some so-called constitutive parameters (friction parameters, leakage inductances, leakage oil flow coefficients) have to be determined from experiments, even if the model approach is physically motivated. The advantages of these latter models (white box models with few experimentally determined constitutive parameters) are the very good *extrapolatability* of the model beyond the data obtained by experiments, a high reliability, a good insight into the model, as well as the fact that the model is *scalable* and also applicable to systems not yet realized (prototyping). The disadvantage is that this type of modeling is generally relatively time-consuming and requires a thorough understanding of the system. In this lecture, we will focus exclusively on the latter type of mathematical models.

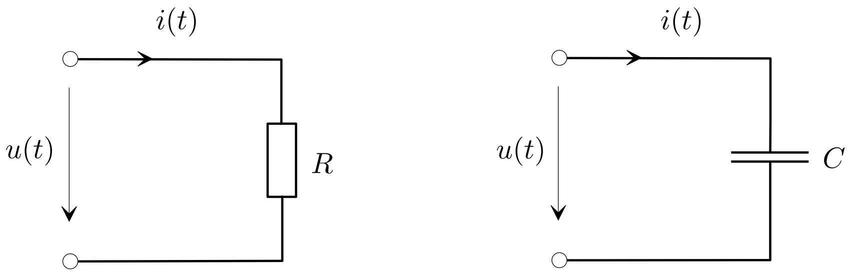


Figure 1.2: On static and dynamic systems.

In the following, consider the two simple electrical systems of Figure 1.2, namely a resistor and an ideal capacitor, with the input variable $i(t)$ (current), the output variable $u(t)$ (voltage) and time t. For the resistor R, the output variable at any time t is uniquely determined by the input variable at time t, namely

$$u(t) = Ri(t) \ . \tag{1.1}$$

Systems of this type, whose output variables depend only on the instantaneous value of the input variables, are called *static systems*. In contrast, to calculate the voltage $u(t)$ of the capacitor C at time t, the input current $i(\tau)$ for the entire past $\tau \leq t$ must be known, since

$$u(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau = \underbrace{\frac{1}{C}\int_{-\infty}^{t_0} i(\tau)\,\mathrm{d}\tau}_{u(t_0)=u_0} + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,\mathrm{d}\tau\ . \tag{1.2}$$

If the input variable $i(\tau)$ is known only for the time interval $t_0 \leq \tau \leq t$, then the voltage of the capacitor at time t_0 must also be known as the initial condition $u(t_0) = u_0$. As can be seen from (1.2), the initial condition contains all the information about the past $\tau < t_0$. One also says that $u(t_0)$ describes the internal *state* of the capacitor system at time t_0. Systems of this type, whose output variables depend not only on the instantaneous value of the input variables but also on their past, are called *dynamic systems*.

If, for a system according to Figure 1.1, as in the case of the resistor and the capacitor, the values of the output variables $y_1, y_2, \dots, y_q$ at time t depend exclusively on the course of the input variables $u_1(\tau), u_2(\tau), \dots, u_p(\tau)$ for $\tau \leq t$, then the system is called *causal*. Since all technically realizable systems are causal, we will restrict ourselves to this case in the following.

The previous considerations now allow us to give the general definition of the state variables of a dynamic system:

Definition 1.1 (State)**.** If for a dynamic system there exist variables $x_1, \dots, x_n$ with the property that the output variables $y_1, y_2, \dots, y_q$ at any time t are uniquely determined by the course of the input variables $u_1(\tau), u_2(\tau), \dots, u_p(\tau)$ on the interval $t_0 \leq \tau \leq t$ and the values of $x_1(t_0), \dots, x_n(t_0)$ for any t_0, then the variables $x_1, \dots, x_n$ are called *state variables* of the system.

Exercise 1.1. Which variable would you choose as the state variable for an inductor? Justify your answer.

Solution of exercise 1.1. The current or the flux linkage of the inductor.

Dynamic systems that can be characterized by a finite number n of state variables are also called *finite-dimensional systems* of *order* n. These finite-state systems, often also called *lumped-parameter systems*, are described by mathematical models in the form of ordinary differential equations and algebraic equations. Within the scope of this lecture, we restrict ourselves to the class of finite-state systems that can be described by an *explicit mathematical model* of the following form:

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= f_1(x_1,\dots,x_n,u_1,\dots u_p,t), & x_1(t_0) &= x_{1,0}\\
\frac{\mathrm{d}}{\mathrm{d}t}x_2 &= f_2(x_1,\dots,x_n,u_1,\dots u_p,t), & x_2(t_0) &= x_{2,0}\\
&\vdots\\
\frac{\mathrm{d}}{\mathrm{d}t}x_n &= f_n(x_1,\dots,x_n,u_1,\dots u_p,t), & x_n(t_0) &= x_{n,0}
\end{aligned}\right\}\ \text{State differential equations with initial conditions} \tag{1.3a}$$

$$\left.\begin{aligned} y_1 &= h_1(x_1, \dots, x_n, u_1, \dots\ u_p, t) \\ y_2 &= h_2(x_1, \dots, x_n, u_1, \dots\ u_p, t) \\ &\vdots \\ y_q &= h_q(x_1, \dots, x_n, u_1, \dots\ u_p, t) \end{aligned}\right\} \quad \text{Output equations} \tag{1.3b}$$

If the input, output, and state variables are combined into column vectors

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_p \end{bmatrix}^{\mathrm{T}} \tag{1.4a}$$

$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_q \end{bmatrix}^{\mathrm{T}} \tag{1.4b}$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^{\mathrm{T}} \tag{1.4c}$$

and, to simplify the notation, a dot is written above the variable to be derived instead of $\frac{\mathrm{d}}{\mathrm{d}t}$, then (1.3) can be written in compact vector notation in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.5a}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, t) \tag{1.5b}$$

The variables $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{x}$ are called the *input*, *output* and *state* of the dynamic mathematical model.

If the state $\mathbf{x}$ is considered as an element of an n-dimensional vector space, then this vector space is also called the *state space*. The state of a system at time t can then be represented as a point in the n-dimensional state space. The curve of all these points in state space for variable time t in a time interval is also called a *trajectory*, see Figure 1.3 for an illustration of a trajectory in 3-dimensional state space.

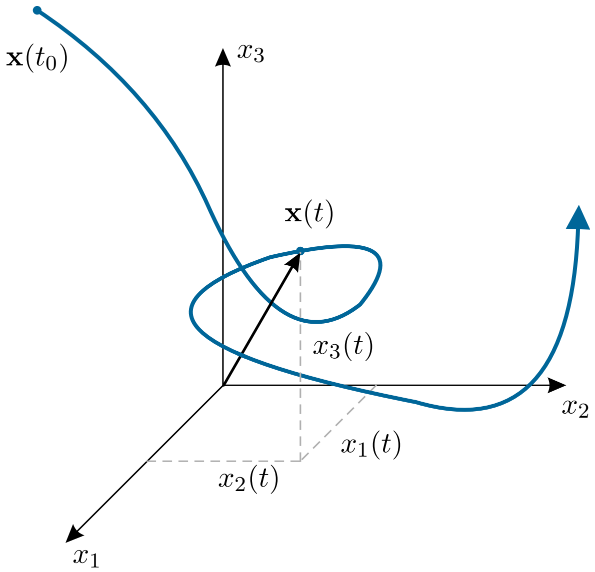


Figure 1.3: On the concept of trajectory.

For the sake of completeness, it should be mentioned that *systems with infinite-dimensional state*, also called *distributed-parameter systems*, are described by partial differential equations. Examples are beams, plates, flow fields, and electromagnetic fields.

Example 1.1. As a simple example of modeling, consider the electrical series resonant circuit from Fig. 1.4.

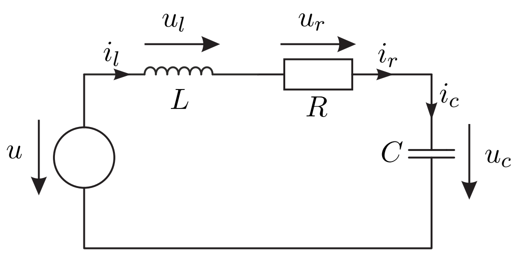


Figure 1.4: Series resonant circuit.

In the first step, the component equations are formulated. The (linear) electrical resistance R is described by

$$u_r(t) = Ri_r(t) \tag{1.6}$$

cf. (1.1). According to (1.2), the capacitor C can be modeled by

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t) = \frac{\mathrm{d}}{\mathrm{d}t}(Cu_c(t)) = C\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) = i_c(t), \quad u_c(0) = u_{c0} \tag{1.7}$$

and the inductor L by

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t}(Li_l(t)) = L\frac{\mathrm{d}}{\mathrm{d}t}i_l(t) = u_l(t), \quad i_l(0) = i_{l0} \tag{1.8}$$

Furthermore, the balance equations, i.e., the node and mesh equations, must be satisfied in the electrical network. These are

$$i_r(t) = i_l(t) \tag{1.9a}$$
$$i_c(t) = i_l(t) \tag{1.9b}$$
$$u_l(t) = -u_r(t) - u_c(t) + u(t) \tag{1.9c}$$

and inserted into (1.6)-(1.8) follows

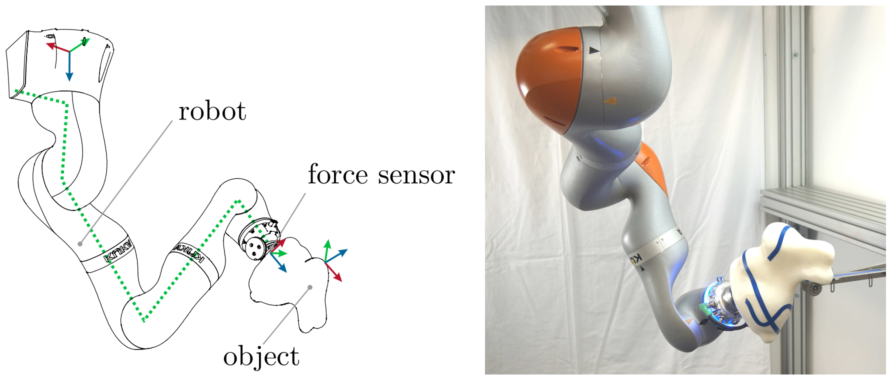
$$\dot{\mathbf{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_l(t) \\ u_c(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{L}(-u_c(t) - Ri_l(t) + u(t)) \\ \frac{1}{C}i_l(t) \end{bmatrix} = \mathbf{f}(\mathbf{x}, u), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{1.10}$$

As output of the system one can choose e.g. the voltage u_c at the capacitor, i.e. $y = u_c$.

Based on this model, the behavior of the system can be analyzed. For example, the influence of the parameters of the system (R, L, C), the initial values $\mathbf{x}(0)^{\mathrm{T}} = \begin{bmatrix} i_l(0) & u_c(0) \end{bmatrix}$ and the input variable $u(t)$ on the dynamic system behavior can be calculated. Today, very advanced computer programs, such as MAPLE or MATLAB, are available for this purpose. The creation and analysis of the model using MAPLE can be found in the file `Serienschwingkreis.mw`, which can be downloaded from the institute's homepage https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

This lecture focuses on the systematic modeling of mechanical rigid body systems, which occur in many real systems (at least as subsystems). Two typical applications with rigid body systems are shown below as examples.

Example 1.2 (Robot). One of the most important technical applications of rigid body systems are (industrial) robots. Fig. 1.5 shows a lightweight robot from KUKA. This robot has 7 degrees of freedom and is designed for direct interaction with humans. For this purpose, it has a sensor for the torque in each joint, which enables, for example, the detection of collisions with obstacles. Typical applications of this robot are manipulation tasks or assembly tasks that are performed in cooperation with humans.



(a) (b)

Figure 1.5: KUKA lightweight robot: (a) Sketch of the system, (b) Photo of the robot.

The (optimal) planning and control of the position and orientation of the end-effector are challenging tasks that are based on a mathematical model of the robot. One goal of this lecture is therefore to model the kinematics and dynamics of such robots.

Videos of applications of the robot can be viewed at `https://www.acin.tuwien.ac.at/bachelor/modellbildung/`. The video `Kollaborative Synchronisation eines 7 Achsroboters` shows the interaction of the robot with a target moved by a human. The video `Pfadfolgeregelung mit Konzepten für den Pfadfortschritt` shows possible ways of interaction between humans and robots. The application of adhesive tape on a complex 3D surface is shown in the video `Oberflächen-basierte Pfadfolgeregelung für das Ablegen von Klebestreifen`. Finally, the use of the robot for the swing-up and stabilization of a spherical pendulum is shown in the video `Aufschwingen eines sphärischen Pendels`.

Example 1.3 (Gyroscope). Another application with mechanical rigid body systems as an essential part are micro-mechanical gyroscopes. Gyroscopes are required in many current applications, such as mobile phones, game consoles or navigation systems. In the automotive sector, gyroscopes are used to measure the rotation of the vehicle around the vertical and lateral axis. These measured values are used (in combination with other sensors), for example, for the electronic stability control (ESP) of vehicles, see Fig. 1.6.

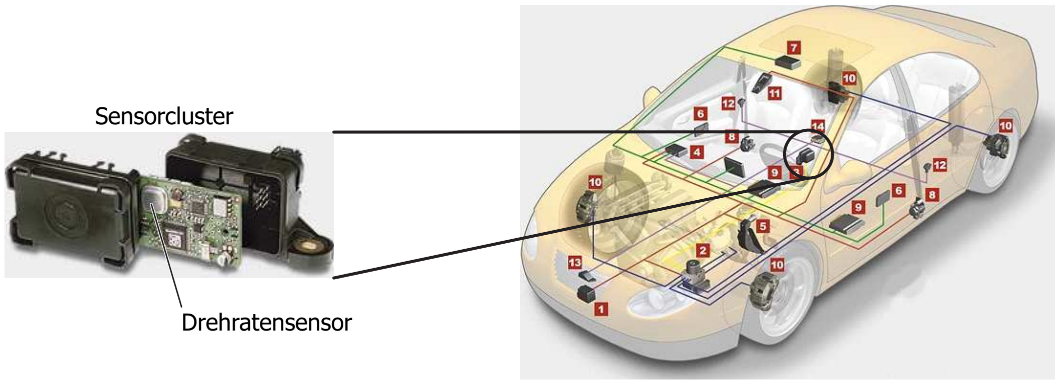


Figure 1.6: Application of a gyroscope in the automotive sector.

A possible design of a micro-mechanical gyroscope is shown in Fig. 1.7. It consists of a number of rigid bodies coupled to each other by elastic elements (springs). These rigid bodies can be set into vibration in the x-direction by means of capacitive actuators. The working principle of this gyroscope is essentially based on the fact that when a rotational speed occurs, additional vibrations in the y- or z-direction are excited due to the Coriolis effect. These are detected with capacitive sensors, whereby the amplitude is a measure of the angular velocity. The entire sensor has an extension of $1000\,\mu\text{m} \times 2000\,\mu\text{m} \times 100\,\mu\text{m}$ and is manufactured in silicon by etching techniques.

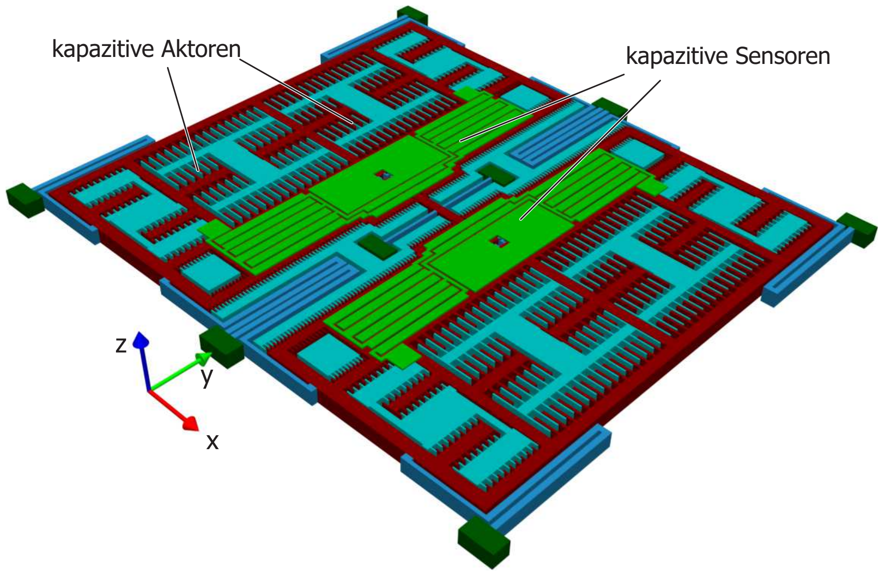


Figure 1.7: Mechanical structure of the micro-mechanical gyroscope.

This example shows that mechanical rigid body systems also occur in very small applications.

1.1 References

[1.0] R. Cannon, *Dynamics of Physical Systems*. New York: McGraw-Hill, 1967.

[1.0] F. E. Cellier, *Continuous System Modeling*. New York: Springer, 1991.

[1.0] S. W. Director and R. A. Rohrer, *Introduction to Systems Theory*. New York: McGraw-Hill, 1972.

[1.0] O. Föllinger and D. Franke, *Einführung in die Zustandsbeschreibung dynamischer Systeme*. München, Wien: Oldenbourg, 1982.

[1.0] R. Isermann, *Mechatronische Systeme: Grundlagen*. Berlin Heidelberg: Springer, 2008.

[1.0] D. Karnopp, D. Margolis, and R. Rosenberg, *System Dynamics*. New Jersey: John Wiley & Sons, 2006.

[1.0] G. Ludyk, *Theoretische Regelungstechnik 1*. Berlin, Heidelberg: Springer, 1995.

[1.0] D. G. Luenberger, *Introduction to Dynamic Systems*. New York: John Wiley & Sons, 1979.

[1.0] L. Padulo and M. A. Arbib, *System Theory*. Philadelphia: W.B. Saunders Company, 1974.

[1.0] J. L. Shearer and B. T. Kulakowski, *Dynamic Modeling and Control of Engineering Systems*. New York: Macmillan Publishing Company, 1990.

2 Point Kinematics

Kinematics describes the motion of bodies or individual material points in space with respect to a reference frame. This chapter describes the basics of point kinematics, i.e., the description of the motion of point masses in space. If one considers the *inertial* Cartesian coordinate system $(0xyz)$ with the origin 0 and the orthonormal basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ as the reference frame, i.e.,

$$\begin{bmatrix} \mathbf{e}_x^{\mathrm{T}}\mathbf{e}_x & \mathbf{e}_x^{\mathrm{T}}\mathbf{e}_y & \mathbf{e}_x^{\mathrm{T}}\mathbf{e}_z \\ \mathbf{e}_y^{\mathrm{T}}\mathbf{e}_x & \mathbf{e}_y^{\mathrm{T}}\mathbf{e}_y & \mathbf{e}_y^{\mathrm{T}}\mathbf{e}_z \\ \mathbf{e}_z^{\mathrm{T}}\mathbf{e}_x & \mathbf{e}_z^{\mathrm{T}}\mathbf{e}_y & \mathbf{e}_z^{\mathrm{T}}\mathbf{e}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{2.1}$$

then the position vector $\mathbf{r}$ from the origin 0 to a material point P can be described in the form

$$\mathbf{r}(t) = r_x(t)\mathbf{e}_x + r_y(t)\mathbf{e}_y + r_z(t)\mathbf{e}_z \tag{2.2}$$

with the time-parameterized components $r_x(t)$, $r_y(t)$, and $r_z(t)$, see Figure 2.1. The

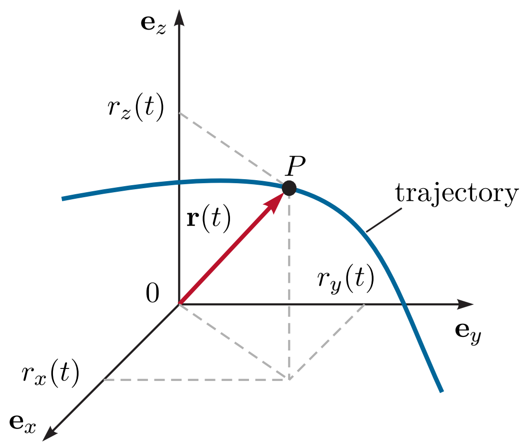


Figure 2.1: Trajectory in a Cartesian coordinate system.

velocity $\mathbf{v}(t)$ and the acceleration $\mathbf{a}(t)$ of the material point P are obtained by time differentiation in the form

$$\mathbf{v}(t) = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z = \dot{r}_x\mathbf{e}_x + \dot{r}_y\mathbf{e}_y + \dot{r}_z\mathbf{e}_z \tag{2.3}$$

and

$$\mathbf{a}(t) = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z = \ddot{r}_x\mathbf{e}_x + \ddot{r}_y\mathbf{e}_y + \ddot{r}_z\mathbf{e}_z , \tag{2.4}$$

respectively, where v_x, v_y, v_z and a_x, a_y, a_z describe the respective components with respect to the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. It should be noted that in the following the total time derivative of a function $x(t)$ is denoted by $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ and $\ddot{x}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$, respectively. In the simplest case, if the coordinate system can be chosen such that the position vector $\mathbf{r}(t)$ coincides with a coordinate axis for all times t, one speaks of a *rectilinear motion*.

Example 2.1. A mass is accelerated linearly by a motor according to the acceleration profile shown in Figure 2.2.

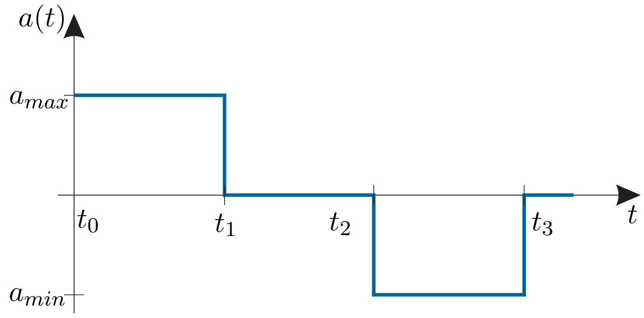


Figure 2.2: Acceleration $a(t)$ as a function of time.

How large must the time t_3 and the minimum acceleration a_{min} be chosen so that at time $t = t_3$ the velocity is zero and the position assumes a given value x_{soll}? It is assumed that at time $t = t_0$ we have $v(t_0) = v_0 = 0$, $x(t_0) = x_0 = 0$.

For the time interval $t_0 \leq t \leq t_1$ the velocity and position profiles are calculated as

$$v_1(t) = v(t_0) + \int_{t_0}^{t} a_{max}\,\mathrm{d}\tau = \underbrace{v_0}_{=0} + a_{max}(t - t_0) \tag{2.5a}$$

$$x_1(t) = x(t_0) + \int_{t_0}^{t} a_{max}(\tau - t_0)\,\mathrm{d}\tau = \underbrace{x_0}_{=0} + \frac{1}{2}a_{max}(t - t_0)^2 \ , \tag{2.5b}$$

for $t_1 \leq t \leq t_2$ it follows

$$v_2(t) = v_1(t_1) + \int_{t_1}^{t} 0 \ \mathrm{d}\tau = a_{max}(t_1 - t_0) \tag{2.6a}$$

$$x_2(t) = x_1(t_1) + \int_{t_1}^{t} a_{max}(t_1 - t_0)\,\mathrm{d}\tau = \frac{1}{2}a_{max}(t_1 - t_0)^2 + a_{max}(t_1 - t_0)(t - t_1) \tag{2.6b}$$

and for $t_2 \leq t \leq t_3$ we obtain

$$v_3(t) = v_2(t_2) + \int_{t_2}^{t} a_{min}\,\mathrm{d}\tau = a_{max}(t_1 - t_0) + a_{min}(t - t_2) \tag{2.7a}$$

$$x_3(t) = x_2(t_2) + \int_{t_2}^{t} v_3(\tau)\,\mathrm{d}\tau = \frac{1}{2}a_{max}\left(t_0^2 - t_1^2\right) + a_{max}(t_1 - t_0)t + \frac{1}{2}a_{min}(t - t_2)^2 \ . \tag{2.7b}$$

With the velocity at time $t = t_3$

$$v_3(t_3) = a_{max}(t_1 - t_0) + a_{min}(t_3 - t_2) \tag{2.8}$$

the desired time t_3 is calculated from the condition $v_3(t_3) = 0$ to be

$$t_3 = t_2 - \frac{a_{max}}{a_{min}}(t_1 - t_0) \tag{2.9}$$

and the desired position $x_3(t_3) = x_{soll}$, with

$$x_3(t_3) = \frac{1}{2a_{min}} a_{max}(t_1 - t_0)(a_{min}(2t_2 - t_0 - t_1) + a_{max}(t_0 - t_1)), \tag{2.10}$$

is reached by the acceleration

$$a_{min} = \frac{-a_{max}^2(t_1 - t_0)^2}{a_{max}(t_1 - t_0)(t_1 + t_0 - 2t_2) + 2x_{soll}} \tag{2.11}$$

The position and velocity profiles are shown in Figure 2.3.

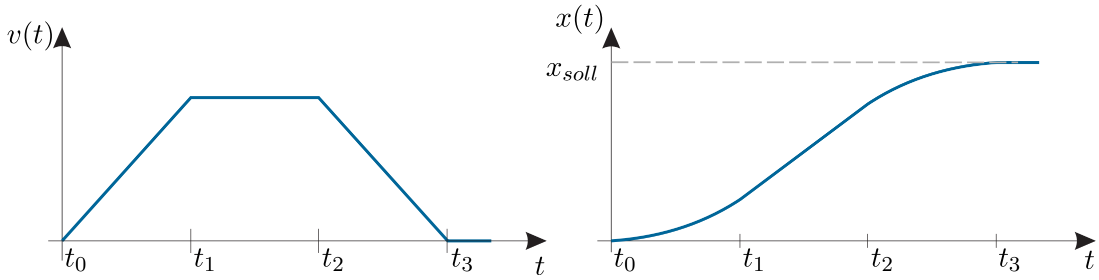


Figure 2.3: Velocity profile $v(t)$ and position profile $x(t)$.

The solution of this example using MAPLE is shown in the file `Beispiel_2_1.mw`, which can be downloaded from the institute's homepage https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

In the following, the motion of a material point P in the xy-plane with respect to the inertial coordinate system $(0xy)$ is considered and described with the aid of *polar coordinates*

$$r_x(t) = r(t)\cos(\varphi(t)) \quad \text{and} \quad r_y(t) = r(t)\sin(\varphi(t)) \tag{2.12}$$

see Figure 2.4. Thus, the position vector from the origin 0 to a material point P is

$$\mathbf{r}(t) = r(t)\cos(\varphi(t))\mathbf{e}_x + r(t)\sin(\varphi(t))\mathbf{e}_y \ . \tag{2.13}$$

The velocity $\mathbf{v}(t)$ according to (2.3) is obtained by applying the chain rule of differentiation in the form

$$\mathbf{v}(t) = \left(\frac{\partial}{\partial r}\mathbf{r}\right)\dot{r} + \left(\frac{\partial}{\partial \varphi}\mathbf{r}\right)\dot{\varphi} \ , \tag{2.14}$$

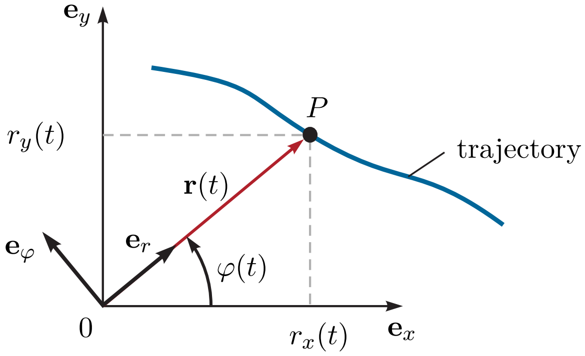


Figure 2.4: Trajectory in a polar coordinate system.

where the basis vectors of the polar coordinates are

$$\tilde{\mathbf{e}}_r = \frac{\partial}{\partial r}\mathbf{r} = \cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y \tag{2.15a}$$

$$\tilde{\mathbf{e}}_\varphi = \frac{\partial}{\partial \varphi}\mathbf{r} = -r\sin(\varphi)\mathbf{e}_x + r\cos(\varphi)\mathbf{e}_y \ . \tag{2.15b}$$

The vectors $\tilde{\mathbf{e}}_r$ and $\tilde{\mathbf{e}}_\varphi$ form a valid basis of a coordinate system if and only if the matrix

$$\mathbf{J} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r\sin(\varphi) & r\cos(\varphi) \end{bmatrix} \tag{2.16}$$

is non-singular, i.e., $\det(\mathbf{J}) = r \neq 0$. This is the case everywhere except at the point $r = 0$. If one normalizes the basis vectors to a length of 1

$$\mathbf{e}_r = \frac{\tilde{\mathbf{e}}_r}{\|\tilde{\mathbf{e}}_r\|_2} \quad \text{and} \quad \mathbf{e}_\varphi = \frac{\tilde{\mathbf{e}}_\varphi}{\|\tilde{\mathbf{e}}_\varphi\|_2} \tag{2.17}$$

with

$$\|\tilde{\mathbf{e}}_r\|_2 = \sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1 \quad \text{and} \quad \|\tilde{\mathbf{e}}_\varphi\|_2 = r \ , \tag{2.18}$$

then (2.14) can be written in the form

$$\mathbf{v}(t) = v_r\mathbf{e}_r + v_\varphi\mathbf{e}_\varphi = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi \tag{2.19}$$

with the components $v_r = \dot{r}$ (*radial component*) and $v_\varphi = r\dot{\varphi}$ (*circular component*) of the velocity $\mathbf{v}(t)$ with respect to the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$. In the time $\mathrm{d}t$ the position vector $\mathbf{r}(t)$ sweeps over an angle $\mathrm{d}\varphi$ and the time rate of change of the angle $\omega = \dot{\varphi}$ is called *angular velocity*. For a pure circular motion (see Figure 2.5) the radial velocity component is $v_r = 0$ and for the circular velocity component we have $v_\varphi = r\omega$.

Exercise 2.1. Show that the velocity components of a material point P in space with respect to the normalized basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ in spherical coordinates

$$r_x = r\sin(\theta)\cos(\varphi), \quad r_y = r\sin(\theta)\sin(\varphi), \quad r_z = r\cos(\theta) \tag{2.20}$$

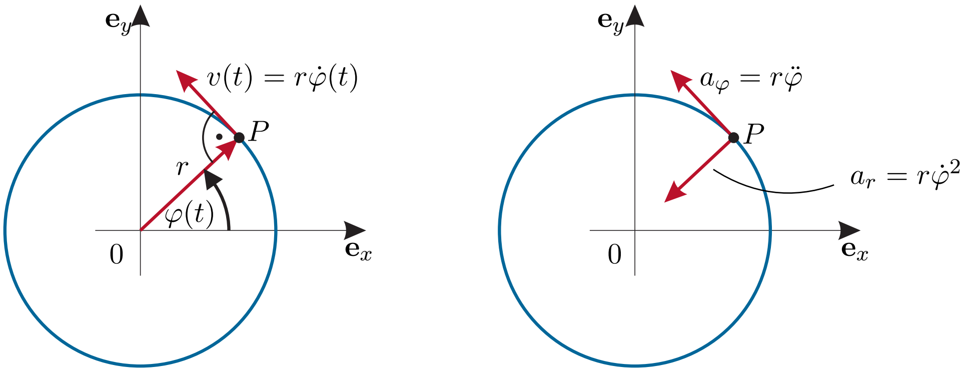


Figure 2.5: Circular path in a polar coordinate system.

can be calculated as

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_\varphi = r\sin(\theta)\dot{\varphi} \ . \tag{2.21}$$

The solution to this exercise using Maple is presented in the file `Aufgabe_2_1_und_2_2.mw`, which can be downloaded at `https://www.acin.tuwien.ac.at/bachelor/modellbildung/`.

The components of the acceleration $\mathbf{a}(t)$ in polar coordinates with respect to the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are obtained by total time differentiation of $\mathbf{v}(t)$ according to (2.19)

$$\mathbf{a}(t) = a_r\mathbf{e}_r + a_\varphi\mathbf{e}_\varphi = \dot{v}_r\mathbf{e}_r + v_r\dot{\mathbf{e}}_r + \dot{v}_\varphi\mathbf{e}_\varphi + v_\varphi\dot{\mathbf{e}}_\varphi \ , \tag{2.22}$$

where it must be noted that the basis vectors (see (2.15) and (2.17))

$$\mathbf{e}_r = \cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y \tag{2.23a}$$
$$\mathbf{e}_\varphi = -\sin(\varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y \tag{2.23b}$$

also change over time. One now tries to express $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\varphi$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\varphi$. To do this, (2.23) is inverted

$$\mathbf{e}_x = \cos(\varphi)\mathbf{e}_r - \sin(\varphi)\mathbf{e}_\varphi \tag{2.24a}$$
$$\mathbf{e}_y = \sin(\varphi)\mathbf{e}_r + \cos(\varphi)\mathbf{e}_\varphi \tag{2.24b}$$

and substituted into $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\varphi$, i.e.

$$\begin{aligned}\dot{\mathbf{e}}_r &= -\sin(\varphi)\dot{\varphi}\mathbf{e}_x + \cos(\varphi)\dot{\varphi}\mathbf{e}_y \\ &= -\sin(\varphi)\dot{\varphi}(\cos(\varphi)\mathbf{e}_r - \sin(\varphi)\mathbf{e}_\varphi) + \cos(\varphi)\dot{\varphi}(\sin(\varphi)\mathbf{e}_r + \cos(\varphi)\mathbf{e}_\varphi) \\ &= \dot{\varphi}\mathbf{e}_\varphi\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\dot{\mathbf{e}}_\varphi &= -\cos(\varphi)\dot{\varphi}\mathbf{e}_x - \sin(\varphi)\dot{\varphi}\mathbf{e}_y \\ &= -\cos(\varphi)\dot{\varphi}(\cos(\varphi)\mathbf{e}_r - \sin(\varphi)\mathbf{e}_\varphi) - \sin(\varphi)\dot{\varphi}(\sin(\varphi)\mathbf{e}_r + \cos(\varphi)\mathbf{e}_\varphi) \\ &= -\dot{\varphi}\mathbf{e}_r \ .\end{aligned} \tag{2.26}$$

Substituting (2.25) and (2.26) into (2.22)

$$\begin{aligned}\mathbf{a}(t) &= \dot{v}_r\mathbf{e}_r + v_r\dot{\mathbf{e}}_r + \dot{v}_\varphi\mathbf{e}_\varphi + v_\varphi\dot{\mathbf{e}}_\varphi \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\varphi}\mathbf{e}_\varphi + (\dot{r}\dot{\varphi} + r\ddot{\varphi})\mathbf{e}_\varphi + r\dot{\varphi}(-\dot{\varphi}\mathbf{e}_r) \\ &= \left(\ddot{r} - r\dot{\varphi}^2\right)\mathbf{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\mathbf{e}_\varphi \ ,\end{aligned} \tag{2.27}$$

then the *radial acceleration* is $a_r = \ddot{r} - r\dot{\varphi}^2$ and the *circular acceleration* is $a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi}$. For a pure circular motion, the tangential component simplifies to $a_\varphi = r\ddot{\varphi}$ and the radial component $a_r = -r\dot{\varphi}^2$ is also called *centripetal acceleration*, see Figure 2.5.

It should be mentioned at this point that in the general case of a coordinate transformation, the time derivatives of the basis vectors can be expressed very elegantly using the so-called *Christoffel symbols* with the help of the basis vectors themselves. Efficient ways to calculate these Christoffel symbols can be found e.g. in [2.0].

Exercise 2.2. Show that the acceleration components of a material point P in space with respect to the normalized basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ in spherical coordinates

$$r_x = r\sin(\theta)\cos(\varphi), \quad r_y = r\sin(\theta)\sin(\varphi), \quad r_z = r\cos(\theta)$$

can be calculated as

$$\begin{aligned}a_r &= \ddot{r} - r\dot{\theta}^2 - r\sin^2(\theta)\dot{\varphi}^2 \\ a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin(\theta)\cos(\theta)\dot{\varphi}^2 \\ a_\varphi &= (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\sin(\theta) + 2r\dot{\varphi}\dot{\theta}\cos(\theta) \ .\end{aligned}$$

Remark: Use a computer algebra system to solve this exercise!

The solution to this exercise using MAPLE is presented in the file `Aufgabe_2_1_und_2_2.mw`, which can be downloaded at https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

2.1 References

[2.0] J. Simmonds, *A Brief on Tensor Analysis*, S. Axler, F. Gehring, and K. Ribet, Eds. New York: Springer, 1994.

[2.0] O. Bruhns and T. Lehmann, *Elemente der Mechanik I: Einführung, Statik*. Braunschweig/Wiesbaden: Vieweg, 1993.

[2.0] O. Bruhns and T. Lehmann, *Elemente der Mechanik II: Elastostatik*. Braunschweig/Wiesbaden: Vieweg, 1994.

[2.0] O. Bruhns and T. Lehmann, *Elemente der Mechanik III: Kinetik*. Braunschweig/Wiesbaden: Vieweg, 1994.

[2.0] M. Calkin, *Lagrangian and Hamiltonian Mechanics*. Singapore: World Scientific, 1996.

[2.0] D. Gross, W. Hauger, J. Schröder, and W. A. Wall, *Technische Mechanik 1: Statik*. Berlin Heidelberg: Springer, 2006.

[2.0] D. Gross, W. Hauger, J. Schröder, and W. A. Wall, *Technische Mechanik 2: Elastostatik*. Berlin Heidelberg: Springer, 2007.

[2.0] D. Gross, W. Hauger, J. Schröder, and W. A. Wall, *Technische Mechanik 3: Kinetik*. Berlin Heidelberg: Springer, 2008.

[2.0] D. Gross, W. Hauger, J. Schröder, and E. A. Werner, *Technische Mechanik 4: Hydromechanik, Elemente der Höheren Mechanik, Numerische Methoden*. Berlin Heidelberg: Springer, 2008.

[2.0] D. Gross, W. Ehlers, and P. Wriggers, *Formeln und Aufgaben zur Technischen Mechanik 1: Statik*. Berlin Heidelberg: Springer, 2008.

[2.0] D. Gross, W. Ehlers, and P. Wriggers, *Formeln und Aufgaben zur Technischen Mechanik 2: Elastostatik, Hydrostatik*. Berlin Heidelberg: Springer, 2007.

[2.0] D. Gross, W. Ehlers, and P. Wriggers, *Formeln und Aufgaben zur Technischen Mechanik 3: Kinetik, Hydromechanik*. Berlin Heidelberg: Springer, 2007.

[2.0] H. Goldstein, *Klassische Mechanik*. Wiesbaden: AULA-Verlag, 1991.

[2.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 1: Theoretical Basis*. Berlin: Springer, 2002.

[2.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 2: Applications*. Berlin: Springer, 2002.

[2.0] B. Heimann, W. Gerth, and K. Popp, *Mechatronik: Komponenten - Methoden - Beispiele*. Leipzig: Fachbuchverlag, 2003.

[2.0] R. Isermann, *Mechatronische Systeme: Grundlagen*. Berlin Heidelberg: Springer, 2008.

[2.0] L. Meirovitch, *Dynamics and Control of Structures*. New York: John Wiley & Sons, 1990.

[2.0] A. Shabana, *Dynamics of Multibody Systems*. Cambridge: Cambridge University Press, 1998.

[2.0] W. Schiehlen and P. Eberhard, *Technische Dynamik: Modelle für Regelung und Simulation*. Wiesbaden: Teubner, 2004.

[2.0] M. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: John Wiley & Sons, 1989.

[2.0] P. Wellstead, *Physical System Modelling*. London: Academic Press, 1979.

[2.0] F. Ziegler, *Mechanics of Solids and Fluids*. New York: Springer, 1995.

3 Newton's Laws

3.1 Force Systems

In the context of this lecture, only point forces acting at discrete points (*points of application*) of a *rigid body* are considered. A rigid body has the property that under the action of forces the distance between any two points of the body always remains the same. The direction of the force is described by its *line of action* and by the direction of the force vector. The SI unit of force is Newton ($\mathrm{N} = \mathrm{kg\,m/s^2}$).

In a so-called *central force system*, all individual forces $\mathbf{f}_i$, $i = 1, \dots, n$, act at the same point of application and the resulting force $\mathbf{f}_R$ is given by (see Figure 3.1)

$$\mathbf{f}_R = \sum_{i=1}^{n} \mathbf{f}_i \ . \tag{3.1}$$

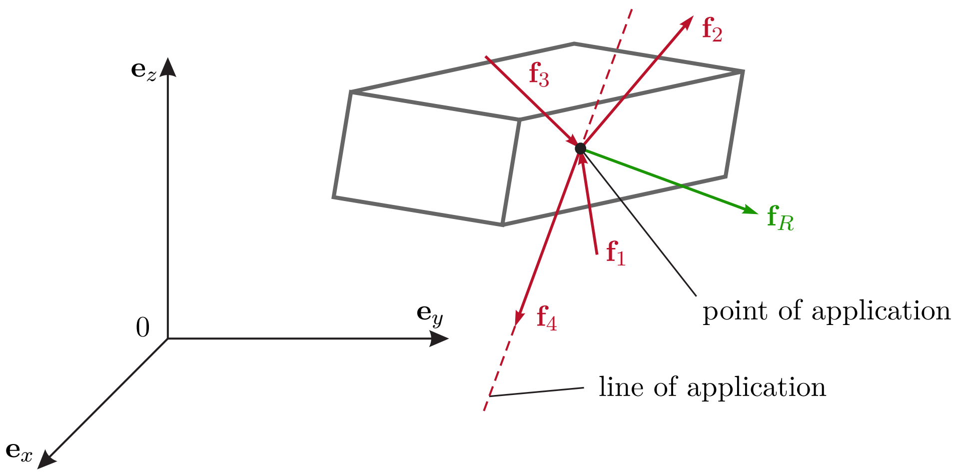


Figure 3.1: Central force system.

If the forces $\mathbf{f}_i$ are expressed in terms of their components in the coordinate system $(0xyz)$ with the orthonormal basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, i.e. $\mathbf{f}_i = f_{i,x}\mathbf{e}_x + f_{i,y}\mathbf{e}_y + f_{i,z}\mathbf{e}_z$, $i = 1, \dots, n$, then (3.1) becomes

$$\mathbf{f}_R = \sum_{i=1}^{n}(f_{i,x}\mathbf{e}_x + f_{i,y}\mathbf{e}_y + f_{i,z}\mathbf{e}_z) = \underbrace{\sum_{i=1}^{n} f_{i,x}}_{f_{R,x}}\mathbf{e}_x + \underbrace{\sum_{i=1}^{n} f_{i,y}}_{f_{R,y}}\mathbf{e}_y + \underbrace{\sum_{i=1}^{n} f_{i,z}}_{f_{R,z}}\mathbf{e}_z \ . \tag{3.2}$$

A central force system is now in *equilibrium* if the resulting force vanishes

$$\mathbf{f}_R = \mathbf{0} \quad \text{or} \quad f_{R,x} = 0,\ f_{R,y} = 0,\ f_{R,z} = 0. \tag{3.3}$$

The *third Newton's law (law of interaction)* states that for every force there is always an equal and opposite reaction force (*actio equals reactio*). For example, if you press your finger on the table top, then an equal and opposite force acts from the table top on your finger. This can be illustrated by cutting the two bodies apart at the point of contact finger/table top and drawing the corresponding forces (*cutting principle*), see Figure 3.2.

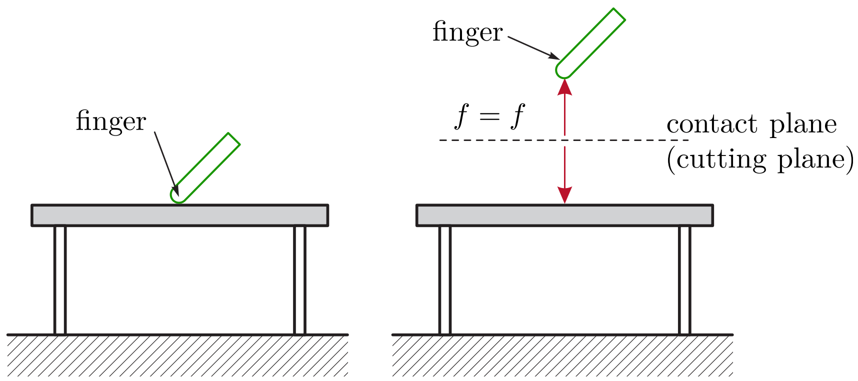


Figure 3.2: Forces between the table top and the hand pressing on it.

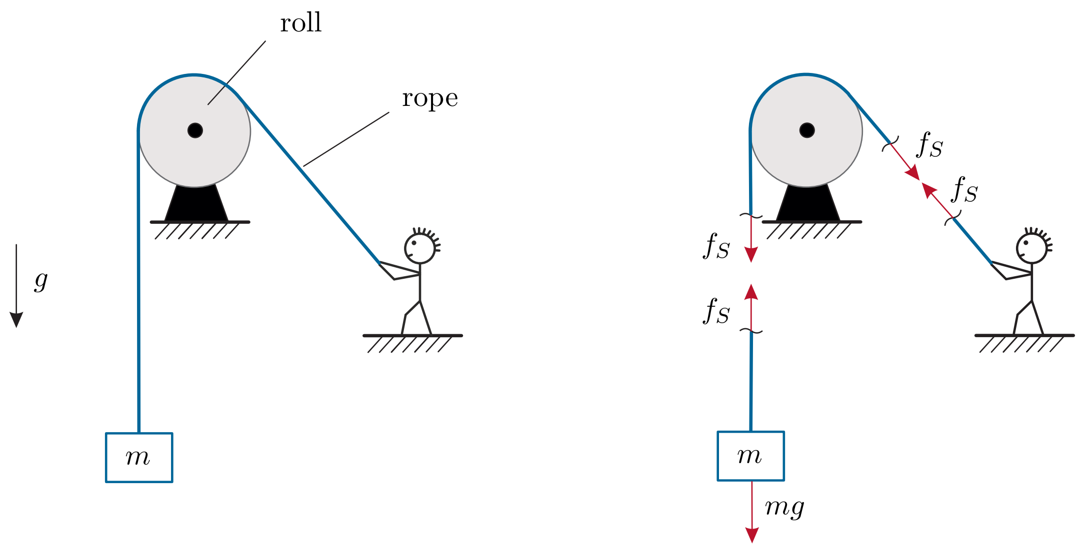


Figure 3.3: Force in a rope.

Another example is shown in Figure 3.3. Assuming that the weight of the rope is negligible and the pulley is frictionless, then the rope force f_S acts on the mass m and the person must also apply the force f_S to hold the load.

> *Example* 3.1. A cylinder of mass m with radius r is held on a smooth plane by a rope of length l attached to its center, see Figure 3.4(a). The forces acting on the isolated cylinder are shown in Figure 3.4(b).

Since the central force system is in equilibrium, according to (3.2) it must hold that

$$\mathbf{e}_x : f_N - f_S \sin(\alpha) = 0 \tag{3.4a}$$

$$\mathbf{e}_z : f_S \cos(\alpha) - mg = 0 \tag{3.4b}$$

with the acceleration due to gravity $g \approx 9.81 \text{ m/s}^2$ and the angle $\alpha = \arcsin(r/l)$. From (3.4) the forces f_S and f_N can now be calculated in the form

$$f_S = \frac{mg}{\cos(\alpha)} \quad \text{and} \quad f_N = mg\tan(\alpha) \tag{3.5}$$

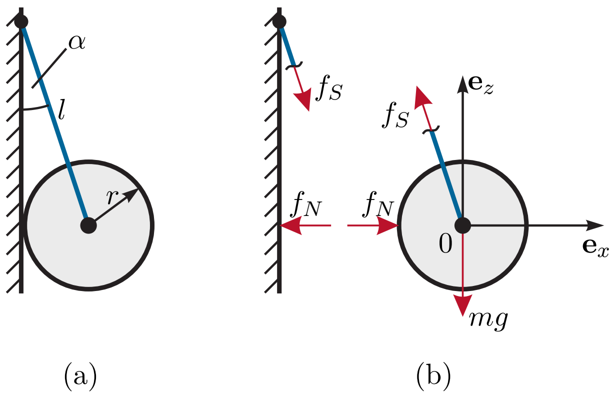


Figure 3.4: Cylinder on a rope.

Exercise 3.1. A vertical mast M is braced by ropes according to Figure 3.5. What are the magnitudes of the forces f_{S1} and f_{S2} in ropes 1 and 2 and the force f_M in the mast when the tensile force f_{S3} is applied to rope 3?

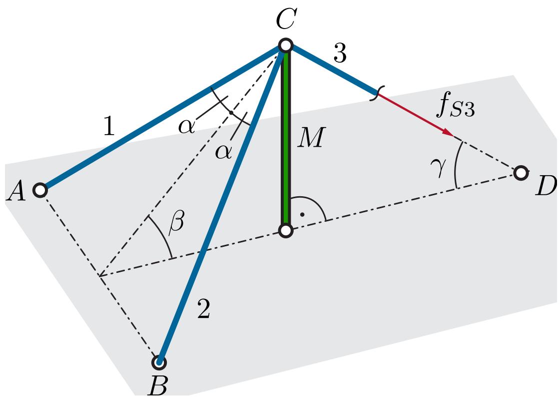


Figure 3.5: Vertical mast with three ropes.

Solution of exercise 3.1.

$$f_{S1} = f_{S2} = f_{S3}\frac{\cos(\gamma)}{2\cos(\alpha)\cos(\beta)} \quad \text{and} \quad f_M = -f_{S3}\frac{\sin(\beta+\gamma)}{\cos(\beta)}$$

The solution to this Exercise using MAPLE is shown in the file `Aufgabe_3_1.mw`, which can be downloaded at https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

In a *general force system*, the individual forces do not act at a single point of application and therefore can no longer be combined into a single resultant force, see Figure 3.6. In this case, the forces – if they are not in equilibrium – cause not only a translational

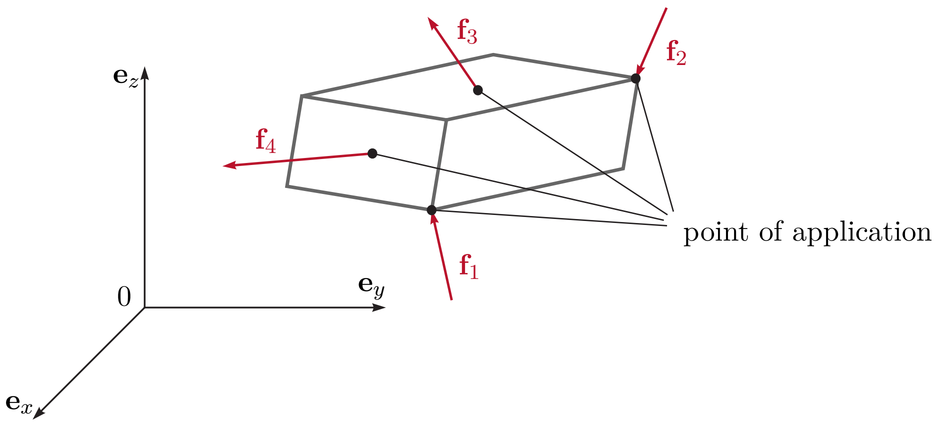


Figure 3.6: General force system.

displacement of the rigid body but also a rotation. In the simplest case, consider the rigid body of Figure 3.7, where the two forces $f_{z,1}\mathbf{e}_z$ and $f_{z,2}\mathbf{e}_z$ produce a resulting *torque* about the axis of rotation $\mathbf{e}_y$ and thus rotate the rigid body about this axis, if the *lever rule* $f_{z,1}l_1 = f_{z,2}l_2$ (force times lever arm equals load times lever arm) is not satisfied. The torque about the axis of rotation is counted positive if the effect of the torque is in

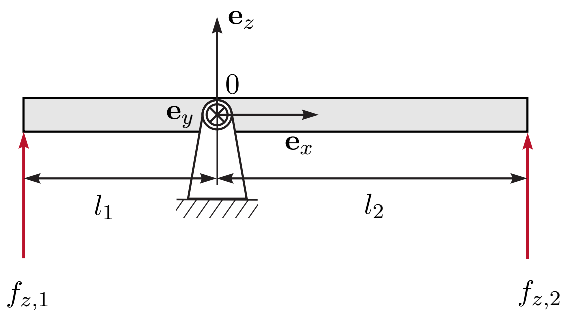


Figure 3.7: Beam mounted on a pivot.

the direction of the directional vector belonging to the axis, according to the right-hand rule. For positive force components $f_{z,1}$ and $f_{z,2}$, the torque $\tau_{y,1}^{(0)} = f_{z,1}l_1$ about the axis of rotation $\mathbf{e}_y$ is positive and the torque $\tau_{y,2}^{(0)} = -f_{z,2}l_2$ is negative for Figure 3.7. The SI unit of torque is Newton-meter ($\mathrm{Nm} = \mathrm{kg\,m^2/s^2}$).

The torque

$$\boldsymbol{\tau}^{(0)} = \tau_x^{(0)}\mathbf{e}_x + \tau_y^{(0)}\mathbf{e}_y + \tau_z^{(0)}\mathbf{e}_z \tag{3.6}$$

of the force

$$\mathbf{f} = f_x\mathbf{e}_x + f_y\mathbf{e}_y + f_z\mathbf{e}_z \tag{3.7}$$

with respect to point 0 in the Cartesian coordinate system $(0xyz)$ with the position vector $\mathbf{r}$ from point 0 to the point of application of the force P, see Figure 3.8,

$$\mathbf{r} = r_x\mathbf{e}_x + r_y\mathbf{e}_y + r_z\mathbf{e}_z \tag{3.8}$$

is given by

$$\tau_x^{(0)} = (r_y f_z - r_z f_y), \quad \tau_y^{(0)} = (r_z f_x - r_x f_z), \quad \tau_z^{(0)} = (r_x f_y - r_y f_x)\ . \tag{3.9}$$

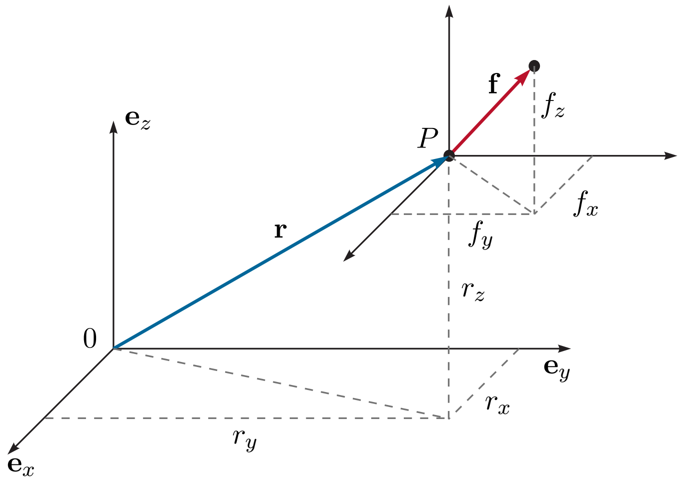


Figure 3.8: On the torque of the force $\mathbf{f}$ with respect to the point 0.

It can be seen immediately that the torque can be written in the form[1].

$$\boldsymbol{\tau}^{(0)} = \mathbf{r} \times \mathbf{f} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} r_y f_z - r_z f_y \\ r_z f_x - r_x f_z \\ r_x f_y - r_y f_x \end{bmatrix} \tag{3.10}$$

[1]For a simplified and more compact notation, the components of the vector quantities are often combined in a single vector, i.e. with $\mathbf{f}^{\mathrm{T}} = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}$ or $\mathbf{r}^{\mathrm{T}} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix}$, $\mathbf{f} = f_x\mathbf{e}_x + f_y\mathbf{e}_y + f_z\mathbf{e}_z$ or $\mathbf{r} = r_x\mathbf{e}_x + r_y\mathbf{e}_y + r_z\mathbf{e}_z$ is meant.

If several torques $\boldsymbol{\tau}_i^{(A)}$, $i = 1, \ldots, n$ with respect to the same point A act on a rigid body, then the resulting torque $\boldsymbol{\tau}_R^{(A)}$ is calculated as

$$\boldsymbol{\tau}_R^{(A)} = \sum_{i=1}^{n} \boldsymbol{\tau}_i^{(A)} = \underbrace{\sum_{i=1}^{n} \tau_{i,x}^{(A)}}_{\boldsymbol{\tau}_{R,x}^{(A)}} \mathbf{e}_x + \underbrace{\sum_{i=1}^{n} \tau_{i,y}^{(A)}}_{\boldsymbol{\tau}_{R,y}^{(A)}} \mathbf{e}_y + \underbrace{\sum_{i=1}^{n} \tau_{i,z}^{(A)}}_{\boldsymbol{\tau}_{R,z}^{(A)}} \mathbf{e}_z \ . \tag{3.11}$$

A general force system according to Figure 3.6 can always be reduced with respect to an *arbitrarily chosen* reference point A by a resulting force $\mathbf{f}_R$ at the point of application A and a resulting torque $\boldsymbol{\tau}_R^{(A)}$ with respect to this point A. A general force system is now in *equilibrium* if both the resulting force $\mathbf{f}_R$ and the resulting torque $\boldsymbol{\tau}_R^{(A)}$ vanish, i.e.

$$\text{Balance of forces:} \quad \mathbf{f}_R = \mathbf{0} \qquad \text{or} \quad f_{R,x} = 0,\ f_{R,y} = 0,\ f_{R,z} = 0 \tag{3.12a}$$

$$\text{Balance of torques:} \quad \boldsymbol{\tau}_R^{(A)} = \mathbf{0} \qquad \text{or} \quad \tau_{R,x}^{(A)} = 0,\ \tau_{R,y}^{(A)} = 0,\ \tau_{R,z}^{(A)} = 0 \ . \tag{3.12b}$$

Example 3.2. Consider the rigid body of Figure 3.9 with the forces

$$\mathbf{f}_A = \begin{bmatrix} f_{A,x} \\ f_{A,y} \\ f_{A,z} \end{bmatrix}, \quad \mathbf{f}_B = \begin{bmatrix} f_{B,x} \\ f_{B,y} \\ f_{B,z} \end{bmatrix}, \quad \mathbf{f}_C = \begin{bmatrix} f_{C,x} \\ f_{C,y} \\ f_{C,z} \end{bmatrix} \tag{3.13}$$

at the points of application A, B and C as well as the reference point D with the corresponding position vectors

$$\mathbf{r}_{0A} = \begin{bmatrix} a_x/2 \\ 0 \\ a_z/2 \end{bmatrix}, \quad \mathbf{r}_{0B} = \begin{bmatrix} a_x/2 \\ a_y/2 \\ a_z \end{bmatrix}, \quad \mathbf{r}_{0C} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_{0D} = \begin{bmatrix} a_x \\ 0 \\ 0 \end{bmatrix} . \tag{3.14}$$

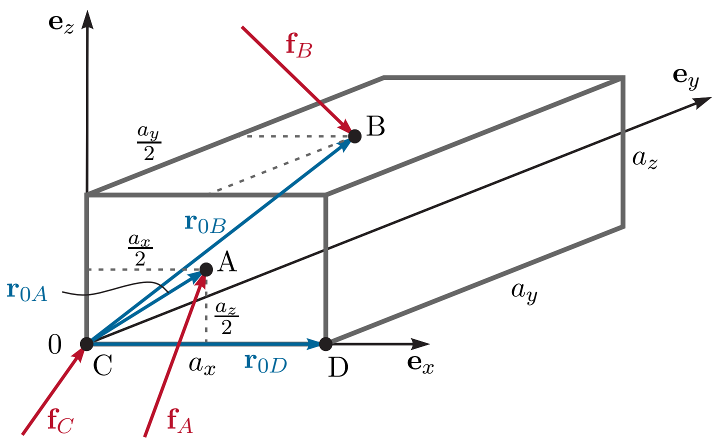


Figure 3.9: Reduction of a general force system.

For the reference point D, the torques follow as

$$\boldsymbol{\tau}_A^{(D)} = \underbrace{(\mathbf{r}_{0A} - \mathbf{r}_{0D})}_{\mathbf{r}_{DA}} \times \mathbf{f}_A = \begin{bmatrix} -a_x/2 \\ 0 \\ a_z/2 \end{bmatrix} \times \begin{bmatrix} f_{A,x} \\ f_{A,y} \\ f_{A,z} \end{bmatrix} = \begin{bmatrix} -f_{A,y}a_z/2 \\ f_{A,z}a_x/2 + f_{A,x}a_z/2 \\ -f_{A,y}a_x/2 \end{bmatrix} \tag{3.15a}$$

$$\boldsymbol{\tau}_B^{(D)} = \underbrace{(\mathbf{r}_{0B} - \mathbf{r}_{0D})}_{\mathbf{r}_{DB}} \times \mathbf{f}_B = \begin{bmatrix} -a_x/2 \\ a_y/2 \\ a_z \end{bmatrix} \times \begin{bmatrix} f_{B,x} \\ f_{B,y} \\ f_{B,z} \end{bmatrix} = \begin{bmatrix} f_{B,z}a_y/2 - f_{B,y}a_z \\ f_{B,z}a_x/2 + f_{B,x}a_z \\ -f_{B,y}a_x/2 - f_{B,x}a_y/2 \end{bmatrix} \tag{3.15b}$$

$$\boldsymbol{\tau}_C^{(D)} = \underbrace{(\mathbf{r}_{0C} - \mathbf{r}_{0D})}_{\mathbf{r}_{DC}} \times \mathbf{f}_C = \begin{bmatrix} -a_x \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} f_{C,x} \\ f_{C,y} \\ f_{C,z} \end{bmatrix} = \begin{bmatrix} 0 \\ a_x f_{C,z} \\ -a_x f_{C,y} \end{bmatrix} \tag{3.15c}$$

and the general force system $\mathbf{f}_A$, $\mathbf{f}_B$ and $\mathbf{f}_C$ can be replaced by the resulting force $\mathbf{f}_R = \mathbf{f}_A + \mathbf{f}_B + \mathbf{f}_C$ and by the resulting torque $\boldsymbol{\tau}_R^{(D)} = \boldsymbol{\tau}_A^{(D)} + \boldsymbol{\tau}_B^{(D)} + \boldsymbol{\tau}_C^{(D)}$. Thus, the equilibrium conditions result from the balance of forces ($\mathbf{f}_R = \mathbf{0}$)

$$\mathbf{e}_x: \quad f_{A,x} + f_{B,x} + f_{C,x} = 0 \tag{3.16a}$$
$$\mathbf{e}_y: \quad f_{A,y} + f_{B,y} + f_{C,y} = 0 \tag{3.16b}$$
$$\mathbf{e}_z: \quad f_{A,z} + f_{B,z} + f_{C,z} = 0 \tag{3.16c}$$

and the balance of torques ($\boldsymbol{\tau}_R^{(D)} = \mathbf{0}$)

$$\mathbf{e}_x: \quad -f_{A,y}a_z/2 + f_{B,z}a_y/2 - f_{B,y}a_z = 0 \tag{3.17a}$$
$$\mathbf{e}_y: \quad f_{A,z}a_x/2 + f_{A,x}a_z/2 + f_{B,z}a_x/2 + f_{B,x}a_z + f_{C,z}a_x = 0 \tag{3.17b}$$
$$\mathbf{e}_z: \quad -f_{A,y}a_x/2 - f_{B,y}a_x/2 - f_{B,x}a_y/2 - f_{C,y}a_x = 0\ . \tag{3.17c}$$

Exercise 3.2. Give the balance of torques for the example in Figure 3.9 about the reference point C.

Solution of exercise 3.2.

$$\boldsymbol{\tau}_A^{(C)} = \begin{bmatrix} -f_{A,y}a_z/2 \\ -f_{A,z}a_x/2 + f_{A,x}a_z/2 \\ f_{A,y}a_x/2 \end{bmatrix}, \boldsymbol{\tau}_B^{(C)} = \begin{bmatrix} f_{B,z}a_y/2 - f_{B,y}a_z \\ -f_{B,z}a_x/2 + f_{B,x}a_z \\ f_{B,y}a_x/2 - f_{B,x}a_y/2 \end{bmatrix}, \boldsymbol{\tau}_C^{(C)} = \mathbf{0}$$

Solution in MAPLE: `Beispiel_Aufgabe_3_2.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Example 3.3. The mechanism shown in Figure 3.10(a) is pivotally mounted at point A and held at points B and C by a rope. It is assumed that the rope pulleys are mounted frictionless and that the rope mass as well as the thickness of the individual beams can be neglected. If the mechanism is cut free, one obtains the forces shown in Figure 3.10(b).

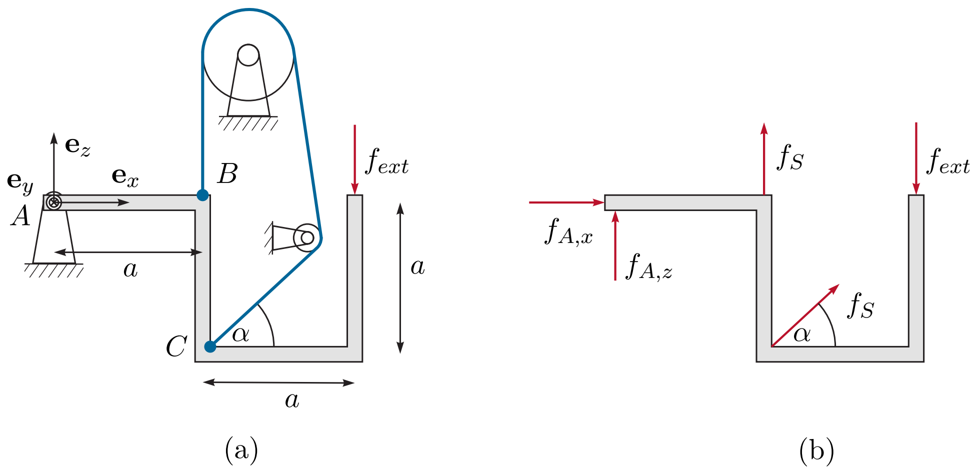


Figure 3.10: Simple mechanism.

In equilibrium, the balance of forces

$$\mathbf{e}_x: \quad f_{A,x} + f_S\cos(\alpha) = 0 \tag{3.18a}$$

$$\mathbf{e}_z: \quad f_{A,z} + f_S + f_S\sin(\alpha) - f_{ext} = 0 \tag{3.18b}$$

and the balance of torques (chosen reference point A)

$$\mathbf{e}_y: \quad -af_S + 2af_{ext} - af_S(\sin(\alpha) + \cos(\alpha)) = 0 \tag{3.19}$$

must be fulfilled.

Solution in MAPLE: `Beispiel_3_3.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

3.2 Center of Gravity

The previous considerations now allow the definition of the so-called *center of gravity* of a rigid body. For this purpose, consider in a first step a massless rigid rod which connects the point masses m_i, $i = 1, \dots, n$ according to Figure 3.11. Due to the acceleration of gravity

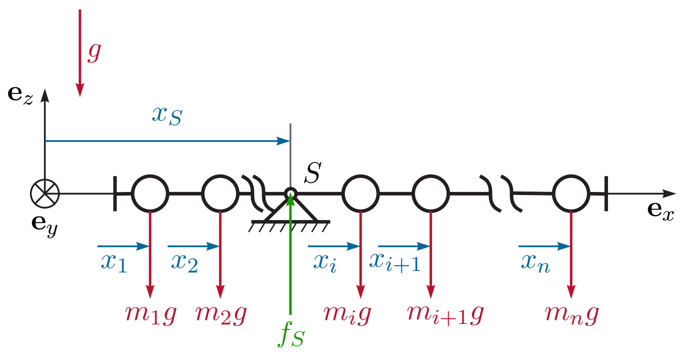


Figure 3.11: Definition of the center of gravity: Massless rod with point masses.

g in the negative $\mathbf{e}_z$ direction, the gravitational forces $\mathbf{f}_i = -m_i g \mathbf{e}_z$, $i = 1, \dots, n$ act on the rod. It is now known that the forces $\mathbf{f}_i$, $i = 1, \dots, n$ with respect to an *arbitrarily chosen* reference point A can be replaced by a resultant force $\mathbf{f}_R$ at the point of application A and a resultant torque $\boldsymbol{\tau}_R^{(A)}$ with respect to this point A. The center of gravity now describes that point of application S at which the resultant torque $\boldsymbol{\tau}_R^{(S)}$ vanishes and thus the rod can be held in equilibrium solely by suspending it at point S with the holding force f_S. From the balance of torques

$$\mathbf{e}_y: \quad \sum_{i=1}^{n} m_i g x_i - f_S x_S = 0 \tag{3.20}$$

and the balance of forces

$$\mathbf{e}_z: \quad -\sum_{i=1}^{n} m_i g + f_S = 0 \tag{3.21}$$

one can calculate x_S in the form

$$x_S = \frac{\sum\limits_{i=1}^{n} m_i x_i}{\sum\limits_{i=1}^{n} m_i} . \tag{3.22}$$

Exercise 3.3. Show that for a general system consisting of n rigidly coupled point masses with masses m_i and position vectors $\mathbf{r}_i$ from the origin 0 of the coordinate system $(0xyz)$ to the point masses, the position vector $\mathbf{r}_S$ to the center of gravity is calculated as follows

$$\mathbf{r}_S = \frac{\sum\limits_{i=1}^{n} m_i \mathbf{r}_i}{\sum\limits_{i=1}^{n} m_i} . \tag{3.23}$$

This can now be directly transferred to a general rigid body. Assume that the rigid body has the volume $\mathcal{V}$ and the (position-dependent) density $\rho(x, y, z)$. The mass m of the rigid body is then given by

$$m = \int_{\mathcal{V}} \rho(x, y, z) \,\mathrm{d}\mathcal{V} . \tag{3.24}$$

The center of gravity S with the position vector $\mathbf{r}_S$ measured in the coordinate system $(0xyz)$ is now that point at which the body would have to be suspended (holding force f_S) so that the body is in equilibrium independent of the direction of the acceleration due to gravity g. Assuming that the acceleration due to gravity acts in the direction of $\mathbf{e}_g$, then the volume element $\mathrm{d}\mathcal{V}$ is acted upon by the force $g\rho(x, y, z)\,\mathrm{d}\mathcal{V}\mathbf{e}_g$ due to the acceleration of gravity and by the torque $\mathbf{r} \times g\rho(x, y, z)\,\mathrm{d}\mathcal{V}\mathbf{e}_g = \mathbf{r}g\rho(x, y, z)\,\mathrm{d}\mathcal{V} \times \mathbf{e}_g$ with respect to the coordinate origin 0, see Figure 3.12. The equilibrium conditions are

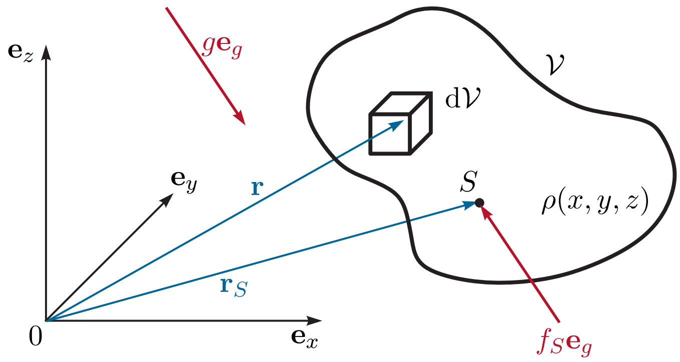


Figure 3.12: Definition of the center of gravity of a rigid body.

obtained by integration over the rigid body volume $\mathcal{V}$ again from the balance of forces

$$-f_S\mathbf{e}_g + g\underbrace{\int_{\mathcal{V}} \rho(x, y, z)\,\mathrm{d}\mathcal{V}}_{m}\mathbf{e}_g = \mathbf{0} \tag{3.25}$$

and the balance of torques

$$-(\mathbf{r}_S \times f_S\mathbf{e}_g) + g\int_{\mathcal{V}} \mathbf{r}\rho(x, y, z)\,\mathrm{d}\mathcal{V} \times \mathbf{e}_g = \mathbf{0} . \tag{3.26}$$

Substituting $f_S = mg$ from (3.25) into (3.26) yields

$$\left(-mg\mathbf{r}_S + g\int_{\mathcal{V}} \mathbf{r}\rho(x, y, z)\,\mathrm{d}\mathcal{V}\right) \times \mathbf{e}_g = \mathbf{0} \tag{3.27}$$

and since $\mathbf{e}_g$ is arbitrary, the expression in parentheses must vanish identically, i.e.

$$\mathbf{r}_S = \frac{\int\limits_{\mathcal{V}} \mathbf{r}\rho(x,y,z)\,\mathrm{d}\mathcal{V}}{m} \tag{3.28}$$

or in component notation $\mathbf{r}_S = r_{S,x}\mathbf{e}_x + r_{S,y}\mathbf{e}_y + r_{S,z}\mathbf{e}_z$

$$r_{S,x} = \frac{\int\limits_{\mathcal{V}} x\rho(x,y,z)\,\mathrm{d}\mathcal{V}}{m}, \quad r_{S,y} = \frac{\int\limits_{\mathcal{V}} y\rho(x,y,z)\,\mathrm{d}\mathcal{V}}{m}, \quad r_{S,z} = \frac{\int\limits_{\mathcal{V}} z\rho(x,y,z)\,\mathrm{d}\mathcal{V}}{m} \; . \tag{3.29}$$

If a body is composed of several sub-bodies $j = 1, \ldots, N$ with volumes $\mathcal{V}_j$ and density $\rho_j(x,y,z)$, then the position vectors $\mathbf{r}_{Sj}$ to the centers of gravity of the sub-bodies measured in the same coordinate system $(0xyz)$ are calculated as

$$\mathbf{r}_{Sj} = \frac{\int\limits_{\mathcal{V}_j} \mathbf{r}\rho_j(x,y,z)\,\mathrm{d}\mathcal{V}_j}{m_j} \quad \text{with} \quad m_j = \int_{\mathcal{V}_j} \rho_j(x,y,z)\,\mathrm{d}\mathcal{V}_j \; . \tag{3.30}$$

From this it can be seen immediately that the center of gravity of the entire body according to (3.28) can be calculated in the form

$$\begin{aligned} \mathbf{r}_S &= \frac{\int\limits_{\mathcal{V}_1} \mathbf{r}\rho_1(x,y,z)\,\mathrm{d}\mathcal{V}_1 + \ldots + \int\limits_{\mathcal{V}_j} \mathbf{r}\rho_j(x,y,z)\,\mathrm{d}\mathcal{V}_j + \ldots + \int\limits_{\mathcal{V}_N} \mathbf{r}\rho_N(x,y,z)\,\mathrm{d}\mathcal{V}_N}{m} \\ &= \frac{\sum\limits_{j=1}^{N} \mathbf{r}_{Sj} m_j}{\sum\limits_{j=1}^{N} m_j} . \end{aligned} \tag{3.31}$$

Example 3.4. For the homogeneous rigid body (density ρ is constant) of Figure 3.13, the position vector to the center of gravity is sought.

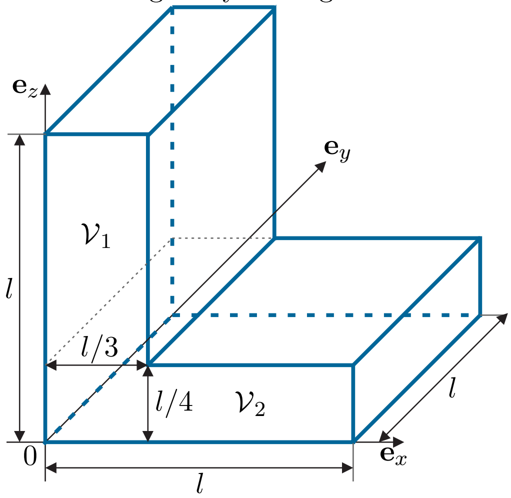


Figure 3.13: Calculation of the center of gravity of composite bodies.

For this purpose, the centers of gravity are first calculated separately for the two volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ according to (3.30). For the first part of the body with volume $\mathcal{V}_1$ follows

$$
\begin{aligned}
r_{S1,x} &= \frac{\rho}{m_1}\int_{l/4}^{l}\int_0^l\int_0^{l/3} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho}{\rho\frac{l}{3}\frac{3l}{4}l}\frac{l^4}{24} = \frac{l}{6}\\
r_{S1,y} &= \frac{\rho}{m_1}\int_{l/4}^{l}\int_0^l\int_0^{l/3} y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4}{l^3}\frac{l^4}{8} = \frac{l}{2}\\
r_{S1,z} &= \frac{\rho}{m_1}\int_{l/4}^{l}\int_0^l\int_0^{l/3} z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4}{l^3}\frac{5l^4}{32} = \frac{5l}{8}
\end{aligned}
$$

and for the second sub-body $\mathcal{V}_2$ follows

$$
\begin{aligned}
r_{S2,x} &= \frac{\rho}{m_2}\int_0^{l/4}\int_0^l\int_0^l x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\rho}{\rho\frac{l}{4}l^2}\frac{l^4}{8} = \frac{l}{2}\\
r_{S2,y} &= \frac{\rho}{m_2}\int_0^{l/4}\int_0^l\int_0^l y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4}{l^3}\frac{l^4}{8} = \frac{l}{2}\\
r_{S2,z} &= \frac{\rho}{m_2}\int_0^{l/4}\int_0^l\int_0^l z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4}{l^3}\frac{l^4}{32} = \frac{l}{8}\ .
\end{aligned}
$$

Hence, according to (3.31), the position vector of the center of gravity of the entire body is

$$\mathbf{r}_S = \frac{1}{m_1 + m_2}(m_1 \mathbf{r}_{S1} + m_2 \mathbf{r}_{S2}) = \frac{1}{\frac{\rho}{4} l^3 + \frac{\rho}{4} l^3} \left(\frac{\rho}{4} l^3 \begin{bmatrix} \frac{l}{6} \\ \frac{l}{2} \\ \frac{5l}{8} \end{bmatrix} + \frac{\rho}{4} l^3 \begin{bmatrix} \frac{l}{2} \\ \frac{l}{2} \\ \frac{l}{8} \end{bmatrix} \right) = \begin{bmatrix} \frac{l}{3} \\ \frac{l}{2} \\ \frac{3l}{8} \end{bmatrix} . \quad (3.32)$$

Solution in MAPLE: `Beispiel_3_4.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Exercise 3.4. Calculate the center of gravity of a homogeneous hemisphere according to Figure 3.14.

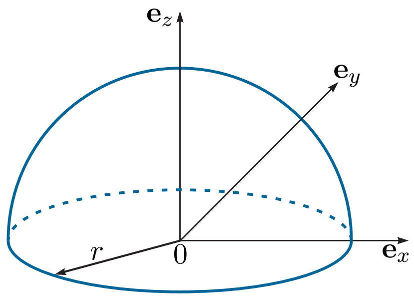


Figure 3.14: Center of gravity of a homogeneous hemisphere.

Solution of exercise 3.4.

$$\mathbf{r}_S = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{8} r \end{bmatrix}$$

Solution in MAPLE: `Aufgabe_3_4.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

3.3 Conservation of Momentum

The *second Newton's law (law of conservation of momentum)* formulated for a point mass states that the temporal change of momentum $\mathbf{p} = m\mathbf{v}$ is equal to the force $\mathbf{f}$ acting on the point mass, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{f} \quad (3.33)$$

with the mass m and the velocity $\mathbf{v}$. Note that the formulation (3.33) is only valid with respect to a resting reference coordinate system (*inertial system*). For the systems considered in this lecture, the Earth can be considered as an inertial system.

Example 3.5. A ball of mass m is launched from a height h above the ground with the velocity $v(0) = v_0 > 0$, see Figure 3.15.

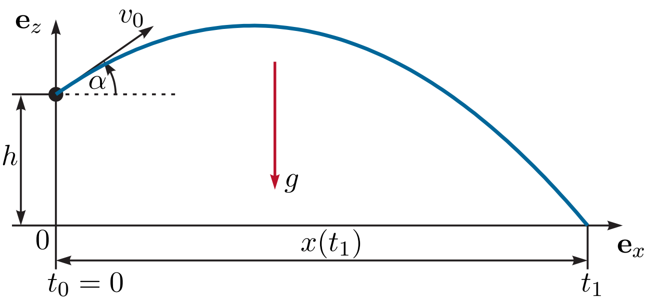


Figure 3.15: Projectile motion.

In the following, we want to calculate at which angle α the ball has to be launched in order to maximize the throwing distance under the assumption of vanishing air friction. Since the mass m is constant, the law of conservation of momentum in the inertial system $(0xyz)$ reads as

$$m\ddot{x} = 0 \quad \text{and} \quad m\ddot{z} = -mg \tag{3.34}$$

with the initial conditions $x(0) = 0$, $\dot{x}(0) = v_0 \cos(\alpha)$, $z(0) = h$, $\dot{z}(0) = v_0 \sin(\alpha)$. From (3.34) with $\dot{x}(t) = v_x(t)$ and $\dot{z}(t) = v_z(t)$ one obtains

$$v_x(t) = \dot{x}(0) = v_0 \cos(\alpha) \tag{3.35a}$$
$$x(t) = v_0 \cos(\alpha) t \tag{3.35b}$$
$$v_z(t) = -gt + v_0 \sin(\alpha) \tag{3.35c}$$
$$z(t) = -g\frac{t^2}{2} + v_0 \sin(\alpha) t + h\ . \tag{3.35d}$$

The time t can now be eliminated in the second equation and substituted into the last equation, which results in the well-known projectile parabola

$$z = -g\frac{x^2}{2v_0^2\cos^2(\alpha)} + \tan(\alpha)x + h. \tag{3.36}$$

The time t_1 at which the ball hits the ground is obtained from the condition $z(t_1) = 0$ as

$$t_1 = \frac{v_0 \sin(\alpha) + \sqrt{v_0^2 \sin^2(\alpha) + 2gh}}{g} \tag{3.37}$$

and thus the throwing distance is

$$x(t_1) = v_0 \cos(\alpha) \frac{v_0 \sin(\alpha) + \sqrt{v_0^2 \sin^2(\alpha) + 2gh}}{g} \ . \tag{3.38}$$

To maximize the throwing distance, one differentiates $x(t_1)$ with respect to α and sets the expression equal to zero. As a result one obtains

$$\alpha_{max} = \arctan\left(\frac{v_0}{\sqrt{v_0^2 + 2gh}}\right). \tag{3.39}$$

One can easily convince oneself that for $h = 0$ the angle is $\alpha_{\max} = 45°$ and the maximum distance is $x_{\max}(t_1) = v_0^2/g$.

Solution in MAPLE: `Beispiel_3_5.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Exercise 3.5. Show the validity of (3.39).

Figure 3.16 shows two point masses m_i and m_j which are rigidly connected by a massless rod. If one cuts the rod, it follows from the cutting principle that $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$. The law of

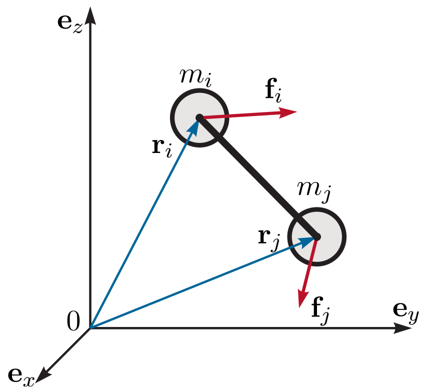


Figure 3.16: Two point masses connected by a massless rod.

conservation of momentum written separately for each point mass is

$$m_i \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i = \mathbf{f}_i + \mathbf{f}_{ij} \quad \text{and} \quad m_j \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_j = \mathbf{f}_j + \mathbf{f}_{ji} \tag{3.40}$$

or by summation and using the cutting principle $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ one obtains

$$m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_i + m_j \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_j = \underbrace{\mathbf{f}_i + \mathbf{f}_j}_{\mathbf{f}_R} + \underbrace{\mathbf{f}_{ij} + \mathbf{f}_{ji}}_{=\mathbf{0}} \ . \tag{3.41}$$

Substituting the relation for the center of gravity according to (3.23)

$$\mathbf{r}_S = \frac{m_i \mathbf{r}_i + m_j \mathbf{r}_j}{m_i + m_j} \tag{3.42}$$

into (3.41), (3.41) simplifies to

$$\underbrace{(m_i + m_j)}_{m} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathbf{r}_S = \underbrace{\mathbf{f}_i + \mathbf{f}_j}_{\mathbf{f}_R} \ . \tag{3.43}$$

It can be seen immediately that this also holds for a rigid body with volume $\mathcal{V}$, mass m according to (3.24) and position vector to the center of gravity $\mathbf{r}_S$ according to (3.28), see Figure 3.12. Namely, if one writes the law of conservation of momentum (3.33) for a mass element $\mathrm{d}m = \rho(x, y, z)\,\mathrm{d}\mathcal{V}$ with the corresponding position vector $\mathbf{r}$ and integrates over the volume $\mathcal{V}$, it follows that

$$\int_{\mathcal{V}} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathbf{r} \rho(x, y, z)\,\mathrm{d}\mathcal{V} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \underbrace{\int_{\mathcal{V}} \mathbf{r} \rho(x, y, z)\,\mathrm{d}\mathcal{V}}_{m\mathbf{r}_S} = m \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathbf{r}_S = \mathbf{f}_R \ . \tag{3.44}$$

Equation (3.44) is known in the literature as the *center of mass theorem* and states that the center of gravity with the position vector $\mathbf{r}_S$ of a system of bodies behaves like a point mass whose mass m is the sum of the masses of all the individual bodies, and on which the vector sum $\mathbf{f}_R$ of all the external forces acting on the individual bodies acts.

Example 3.6. Figure 3.17 shows a simple pulley system with two masses m_1 and m_2 connected by a massless rope over frictionless, massless pulleys.

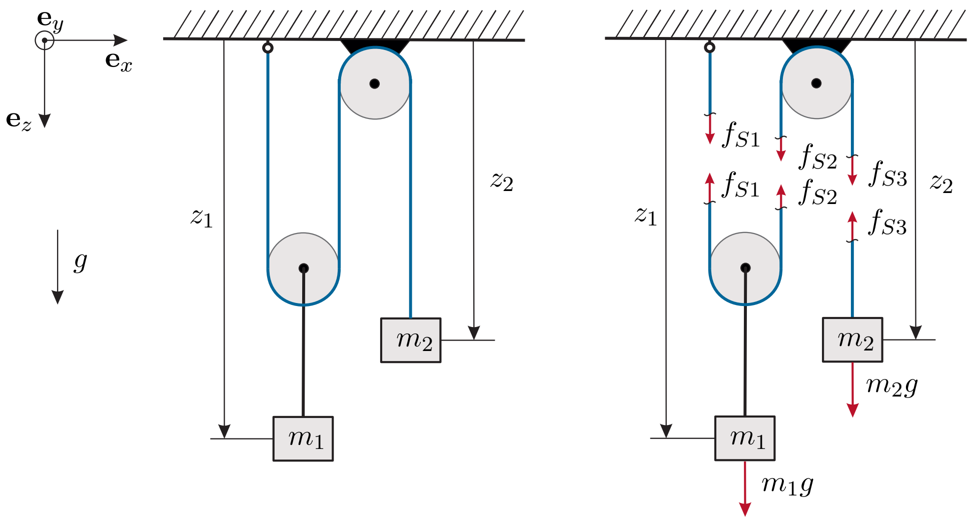


Figure 3.17: Pulley system with two masses.

The corresponding equations of motion are

$$m_1\ddot{z}_1 = m_1 g - f_{S1} - f_{S2} \tag{3.45a}$$
$$m_2\ddot{z}_2 = m_2 g - f_{S3}\ . \tag{3.45b}$$

Due to the above assumptions, the force in the entire rope is the same, i.e.

$$f_{S1} = f_{S2} = f_{S3} = f_S\ . \tag{3.46}$$

Denoting by z_{10} and z_{20} the position of the masses m_1 and m_2 at time $t = 0$, then a change of z_2 by Δz_2 causes a displacement of mass m_1 by $-\Delta z_2/2$ (pulley system), i.e.

$$z_2(t) = z_{20} + \Delta z_2(t), \quad z_1(t) = z_{10} - \frac{\Delta z_2(t)}{2}\ . \tag{3.47}$$

Substituting (3.46) and (3.47) into (3.45), one obtains

$$-\frac{m_1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta z_2 = m_1 g - 2f_S \tag{3.48a}$$
$$m_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta z_2 = m_2 g - f_S\ , \tag{3.48b}$$

from which the equation of motion of the coupled system and the rope force f_S can be calculated directly in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta z_2 = 2g\frac{2m_2 - m_1}{m_1 + 4m_2} \tag{3.49a}$$
$$f_S = \frac{3m_1 m_2 g}{4m_2 + m_1}. \tag{3.49b}$$

Solution in MAPLE: `Beispiel_3_6.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

3.3.1 Bodies with Variable Mass

The law of conservation of momentum (3.33) also applies to bodies with variable mass $m(t)$. Assuming that the body has the mass $m(t)$, the velocity $\mathbf{v}(t)$ at time t and is acted upon by the force $\mathbf{f}$. If the body now ejects the mass $\mathrm{d}\bar{m}$ with the ejection velocity $\mathbf{w}$ during the time interval $\mathrm{d}t$, then the body has the mass $m(t+\mathrm{d}t) = m(t) - \mathrm{d}\bar{m}$ and the velocity $\mathbf{v}(t+\mathrm{d}t)$ at time $t+\mathrm{d}t$. The momentum at time t is $\mathbf{p}(t) = m(t)\mathbf{v}(t)$ and the total momentum at time $t+\mathrm{d}t$ is calculated as

$$\mathbf{p}(t+\mathrm{d}t) = \underbrace{(m(t) - \mathrm{d}\bar{m})}_{m(t+\mathrm{d}t)}\underbrace{(\mathbf{v}(t) + \mathrm{d}\mathbf{v})}_{\mathbf{v}(t+\mathrm{d}t)} + \mathrm{d}\bar{m}(\mathbf{v}(t) + \mathrm{d}\mathbf{v} + \mathbf{w}(t)) = \mathbf{p}(t) + \mathrm{d}\mathbf{p} \tag{3.50}$$

or

$$\mathrm{d}\mathbf{p} = m(t)\,\mathrm{d}\mathbf{v} + \mathrm{d}\bar{m}\mathbf{w}(t)\ . \tag{3.51}$$

Thus, the law of conservation of momentum (3.33) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = m(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} + \mathbf{w}(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{m} = \mathbf{f}\ . \tag{3.52}$$

Here, $\frac{\mathrm{d}}{\mathrm{d}t}\bar{m} = \gamma > 0$ describes the *ejection rate*. With $m(t + \mathrm{d}t) = m(t) + \mathrm{d}m$ we have the mass decrease of the body due to the ejected mass

$$\frac{\mathrm{d}}{\mathrm{d}t}m = -\frac{\mathrm{d}}{\mathrm{d}t}\bar{m} = -\gamma \tag{3.53}$$

and the expression

$$\mathbf{f}_\mathrm{s} = -\gamma\mathbf{w}(t) \tag{3.54}$$

is called the *thrust*. The differential equations of a body with variable mass $m(t)$ and ejection rate $\gamma > 0$ can therefore be summarized as follows

$$m(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \mathbf{f} - \gamma\mathbf{w}(t) \tag{3.55a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m = -\gamma\ . \tag{3.55b}$$

Exercise 3.6. Calculate the mathematical model of a single-stage rocket with time-varying mass $m(t) = m_0 - m_f(t)$, where m_0 denotes the mass of the rocket before launch (dead weight + payload + fuel mass) and $m_f(t)$ the burned fuel mass. Assume that the burned fuel mass $m_f(t)$ is ejected from the rocket with the fuel ejection rate $\dot{m}_f(t) = u(t)$ at the relative velocity $\mathbf{w}(t) = -w\mathbf{e}_h$, $w > 0$, and that the rocket moves exactly against the Earth's gravitational field with the gravitational constant g.

Solution of exercise 3.6. The mathematical model is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}h &= v\\ \frac{\mathrm{d}}{\mathrm{d}t}v &= -g + \frac{w}{m}u(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}m &= -u(t)\end{aligned}$$

with the height $h(t)$ of the rocket measured from the Earth's surface, the rocket velocity $v(t)$, and the rocket mass $m(t)$.

Solution in MAPLE: `Aufgabe_3_6.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

3.4 Translational Kinetic Energy and Potential Energy

The starting point of the further considerations is again a point mass with mass m, the position vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ from the origin 0 of the inertial system $(0xyz)$, the velocity $\mathbf{v} = \dot{\mathbf{r}} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$ and the sum of the forces $\mathbf{f}_R = f_{R,x}\mathbf{e}_x + f_{R,y}\mathbf{e}_y + f_{R,z}\mathbf{e}_z$ acting on the point mass. Then, according to (3.33), the law of conservation of momentum

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \mathbf{f}_R \tag{3.56}$$

applies. The work done by the force $\mathbf{f}_R$ at time t per unit of time is called *power* (SI unit Watt $\mathrm{W} = \mathrm{N\,m/s}$)[2].

$$P = \mathbf{f}_R \cdot \mathbf{v} \tag{3.57}$$

The corresponding *energy* E transferred in the time interval $[t_0, t]$ (SI unit Joule $\mathrm{J} = \mathrm{N\,m}$) is

$$E(t) - E(t_0) = \int_{t_0}^{t} P(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} \mathbf{f}_R \cdot \mathbf{v}\,\mathrm{d}\tau \ . \tag{3.58}$$

Substituting the left-hand side of (3.56) into (3.58), one obtains the *kinetic energy* stored in the mass m at time t as

$$\begin{aligned}
T(t) &= T(t_0) + \int_{t_0}^{t}\left(m\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{v}\right)\cdot\mathbf{v}\,\mathrm{d}\tau = T(t_0) + m\int_{\mathbf{v}_0}^{\mathbf{v}}\tilde{\mathbf{v}}\cdot\mathrm{d}\tilde{\mathbf{v}}\\
&= T(t_0) + m\left(\int_{v_{0x}}^{v_x}\tilde{v}_x\,\mathrm{d}\tilde{v}_x + \int_{v_{0y}}^{v_y}\tilde{v}_y\,\mathrm{d}\tilde{v}_y + \int_{v_{0z}}^{v_z}\tilde{v}_z\,\mathrm{d}\tilde{v}_z\right)\\
&= \underbrace{T(t_0) - \frac{m}{2}\left(v_{0x}^2 + v_{0y}^2 + v_{0z}^2\right)}_{=0} + \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) = \frac{1}{2}m\mathbf{v}^{\mathrm{T}}\mathbf{v}\ ,
\end{aligned} \tag{3.59}$$

where all integrals are evaluated along a solution trajectory of the system in the time interval $[t_0, t]$ with the corresponding velocity $\mathbf{v}(t_0) = \mathbf{v}_0 = [v_{0x}, v_{0y}, v_{0z}]^{\mathrm{T}}$ and $\mathbf{v}(t) = [v_x, v_y, v_z]^{\mathrm{T}}$.

The *translational part of the kinetic energy of a rigid body* is calculated as (center of mass theorem)

$$T = \frac{1}{2}m\dot{\mathbf{r}}_S^{\mathrm{T}}\dot{\mathbf{r}}_S \tag{3.60}$$

with the total mass m and the position vector $\mathbf{r}_S$ to the center of gravity measured in the inertial system $(0xyz)$.

[2]Here and in the following, $\mathbf{f}_R \cdot \mathbf{v}$ denotes the inner product $\mathbf{f}_R \cdot \mathbf{v} = \mathbf{f}_R^{\mathrm{T}}\mathbf{v} = f_{R,x}v_x + f_{R,y}v_y + f_{R,z}v_z$.

In the next step, the corresponding *potential energy* V shall be calculated for the class of *potential forces* $\mathbf{f}_p$. For this purpose, (3.58) with $\mathbf{v} = \dot{\mathbf{r}}$ is reformulated as

$$\begin{aligned} V(t) &= V(t_0) + \int_{t_0}^{t} \mathbf{f}_p \cdot \mathbf{v} \,\mathrm{d}\tau = V(t_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{f}_p(\tilde{\mathbf{r}}) \cdot \mathrm{d}\tilde{\mathbf{r}} \\ &= V(t_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} (f_{p,x}(\tilde{x},\tilde{y},\tilde{z})\,\mathrm{d}\tilde{x} + f_{p,y}(\tilde{x},\tilde{y},\tilde{z})\,\mathrm{d}\tilde{y} + f_{p,z}(\tilde{x},\tilde{y},\tilde{z})\,\mathrm{d}\tilde{z}), \end{aligned} \tag{3.61}$$

where the integrals are again to be understood along a solution trajectory of the system in the time interval $[t_0, t]$ with the corresponding position $\mathbf{r}(t_0) = \mathbf{r}_0 = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{r}(t) = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$. The integral in (3.61) is *path-independent* if and only if[3] the *integrability conditions*

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}} f_{p,x}(\tilde{x},\tilde{y},\tilde{z}) &= \frac{\partial}{\partial \tilde{x}} f_{p,y}(\tilde{x},\tilde{y},\tilde{z}) \ , \\ \frac{\partial}{\partial \tilde{z}} f_{p,x}(\tilde{x},\tilde{y},\tilde{z}) &= \frac{\partial}{\partial \tilde{x}} f_{p,z}(\tilde{x},\tilde{y},\tilde{z}) \ , \\ \frac{\partial}{\partial \tilde{z}} f_{p,y}(\tilde{x},\tilde{y},\tilde{z}) &= \frac{\partial}{\partial \tilde{y}} f_{p,z}(\tilde{x},\tilde{y},\tilde{z}) \end{aligned} \tag{3.62}$$

are fulfilled or the Jacobian matrix of $\mathbf{f}_p = \begin{bmatrix} f_{p,x}(\tilde{x},\tilde{y},\tilde{z}) & f_{p,y}(\tilde{x},\tilde{y},\tilde{z}) & f_{p,z}(\tilde{x},\tilde{y},\tilde{z}) \end{bmatrix}^{\mathrm{T}}$ with respect to $\tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{bmatrix}^{\mathrm{T}}$ is *symmetric*, i.e.

$$\frac{\partial}{\partial \tilde{\mathbf{r}}} \mathbf{f}_p = \begin{bmatrix} \frac{\partial}{\partial \tilde{x}} f_{p,x} & \frac{\partial}{\partial \tilde{y}} f_{p,x} & \frac{\partial}{\partial \tilde{z}} f_{p,x} \\ \frac{\partial}{\partial \tilde{x}} f_{p,y} & \frac{\partial}{\partial \tilde{y}} f_{p,y} & \frac{\partial}{\partial \tilde{z}} f_{p,y} \\ \frac{\partial}{\partial \tilde{x}} f_{p,z} & \frac{\partial}{\partial \tilde{y}} f_{p,z} & \frac{\partial}{\partial \tilde{z}} f_{p,z} \end{bmatrix} = \left(\frac{\partial}{\partial \tilde{\mathbf{r}}} \mathbf{f}_p \right)^{\mathrm{T}} . \tag{3.63}$$

In this case, the force $\mathbf{f}_p$ is also called *conservative* and has a *potential* (potential energy) according to (3.61). If one now assumes that $\mathbf{r}_I$ denotes the position at which $V(\mathbf{r}_I) = 0$ (reference point), then

$$V(\mathbf{r}) = \underbrace{V(\mathbf{r}_I) + \int_{\mathbf{r}_I}^{\mathbf{r}_0} \mathbf{f}_p(\tilde{\mathbf{r}}) \cdot \mathrm{d}\tilde{\mathbf{r}}}_{=V(\mathbf{r}_0)} + \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{f}_p(\tilde{\mathbf{r}}) \cdot \mathrm{d}\tilde{\mathbf{r}} = \int_{\mathbf{r}_I}^{\mathbf{r}} \mathbf{f}_p(\tilde{\mathbf{r}}) \cdot \mathrm{d}\tilde{\mathbf{r}} \ . \tag{3.64}$$

Thus, the potential energy V depends exclusively on the final value $\mathbf{r}$ of the solution trajectory and on the reference point $\mathbf{r}_I$ and is independent of how one arrives at this final value[4]. If $\mathbf{f}_p = \begin{bmatrix} f_{p,x}(x,y,z) & f_{p,y}(x,y,z) & f_{p,z}(x,y,z) \end{bmatrix}^{\mathrm{T}}$ is conservative and thus the integrability conditions (3.62) are fulfilled, the integration path can be chosen freely and the corresponding potential can be calculated e.g. as follows

$$V(\mathbf{r}) = \int_{x_I}^{x} f_{p,x}(\tilde{x}, y_I, z_I)\,\mathrm{d}\tilde{x} + \int_{y_I}^{y} f_{p,y}(x, \tilde{y}, z_I)\,\mathrm{d}\tilde{y} + \int_{z_I}^{z} f_{p,z}(x, y, \tilde{z})\,\mathrm{d}\tilde{z} \tag{3.65}$$

with $\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{r}_I = \begin{bmatrix} x_I & y_I & z_I \end{bmatrix}^{\mathrm{T}}$.

[3]Strictly speaking, this only holds in a star-shaped set (Poincaré lemma for differential forms).
[4]Note that a change of the reference point $\mathbf{r}_I$ only causes a constant shift in V.

Example 3.7. An example is the potential energy due to gravity. If a person of mass m climbs a mountain of height h, then this person has the potential energy[a] $V = mgh$ at the top of the mountain, regardless of where they started the mountain tour and which path they took to reach the top.

[a]Here it is assumed that the sea level was chosen as the reference point.

If $V(x, y, z)$ denotes the potential energy, then V can also be written in the form

$$V = \int_{\mathbf{r}_I}^{\mathbf{r}} \frac{\mathrm{d}V}{\mathrm{d}\mathbf{r}} \,\mathrm{d}\mathbf{r} = \int_{\mathbf{r}_I}^{\mathbf{r}} \left[\frac{\partial}{\partial \tilde{x}} V(\tilde{x}, \tilde{y}, \tilde{z}) \,\mathrm{d}\tilde{x} + \frac{\partial}{\partial \tilde{y}} V(\tilde{x}, \tilde{y}, \tilde{z}) \,\mathrm{d}\tilde{y} + \frac{\partial}{\partial \tilde{z}} V(\tilde{x}, \tilde{y}, \tilde{z}) \,\mathrm{d}\tilde{z} \right] \tag{3.66}$$

and by comparison with (3.65) the following relations follow from the independence of the spatial variables x, y and z

$$f_{p,x}(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{\partial}{\partial \tilde{x}} V(\tilde{x}, \tilde{y}, \tilde{z}),\ f_{p,y}(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{\partial}{\partial \tilde{y}} V(\tilde{x}, \tilde{y}, \tilde{z}),\ f_{p,z}(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{\partial}{\partial \tilde{z}} V(\tilde{x}, \tilde{y}, \tilde{z}) \tag{3.67}$$

or

$$\mathbf{f}_p = \mathrm{grad}(V) = \nabla V \ . \tag{3.68}$$

Exercise 3.7. Show that the force that can be calculated from a potential is always *irrotational*, i.e. $\mathrm{rot}(\mathbf{f}_p) = \nabla \times \mathbf{f}_p = \mathbf{0}$.

An essential element for the lossless storage of mechanical energy is a mechanical spring. Figure 3.18 shows a mechanical spring and its nonlinear force-displacement characteristic. In the unloaded state (spring force $f_F = 0$), the spring element has the length s_0, which is also called the *relaxed length of the spring*.

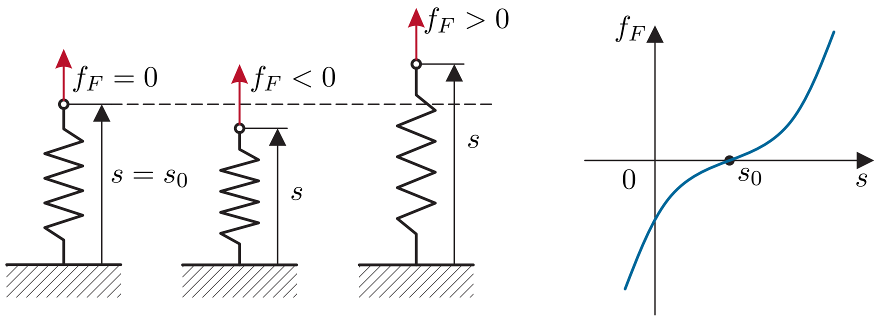


Figure 3.18: Spring element.

The potential energy of the spring with the spring force $f_F(s)$, $f_F(s_0) = 0$, is calculated according to (3.64) as

$$V(s) = \int_{s_0}^{s} f_F(\tilde{s}) \,\mathrm{d}\tilde{s} \ . \tag{3.69}$$

In the linear case, i.e. $f_F(s) = c(s - s_0)$ with the *spring constant* $c > 0$, the potential energy simplifies to

$$V(s) = \frac{1}{2} c (s - s_0)^2 \ . \tag{3.70}$$

Exercise 3.8. Figure 3.19 shows the series and parallel connection of two linear spring elements with the spring constants c_1 and c_2 and the corresponding relaxed lengths s_{01}, s_{02}.

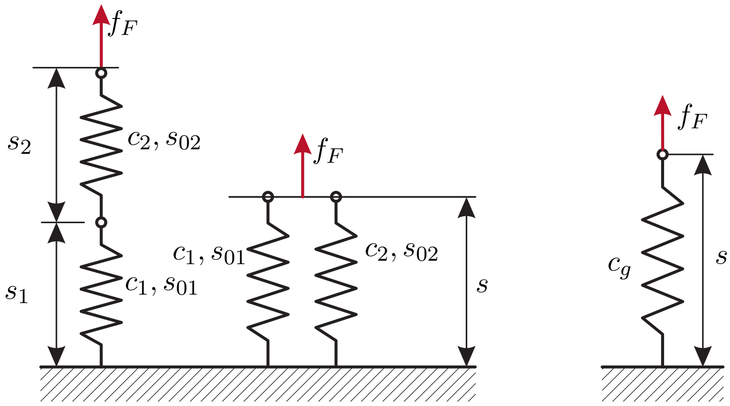


Figure 3.19: Series and parallel connection of linear spring elements.

Calculate the overall stiffness c_g and the corresponding relaxed length s_{0g} of the equivalent circuit according to Figure 3.19.

Solution of exercise 3.8.

$$\text{Series connection: } s_{0g} = s_{01} + s_{02}, \quad c_g = \frac{c_1 c_2}{c_1 + c_2}$$

$$\text{Parallel connection: } s_{0g} = \frac{c_1 s_{01} + c_2 s_{02}}{c_1 + c_2}, \quad c_g = c_1 + c_2$$

Solution in Maple: `Aufgabe_3_8.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Example 3.8. Consider the system in Figure 3.20 consisting of the two masses m_1 and m_2 as well as the two linear spring elements with spring stiffnesses $c_1 > 0$ and $c_2 > 0$ and the corresponding relaxed lengths s_{01} and s_{02}. In the following, z_1 and z_2 denote the displacement of mass m_1 or m_2 from the equilibrium position, i.e. $z_1 = s_1 - s_{01}$ and $z_2 = s_2 - s_{02}$. As shown in Figure 3.20, the two masses are connected by a leaf spring. This causes a spring force $f_{12} = c_{12}(z_1 - z_2)$, $c_{12} > 0$, due to a relative displacement of m_1 and m_2.

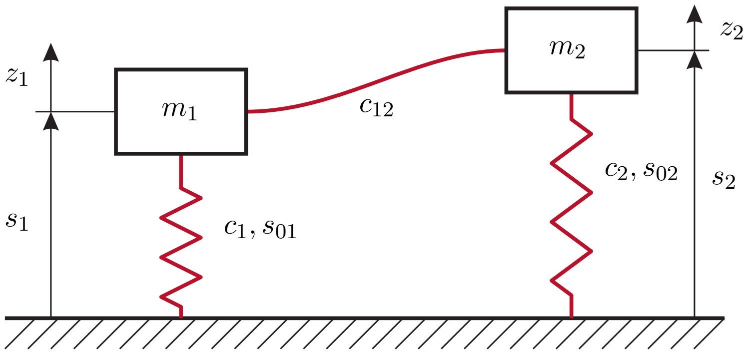


Figure 3.20: Masses with leaf spring.

Due to the cutting principle, this force must occur with different signs at the two ends of the spring, i.e. $f_{21} = -f_{12}$. Assuming that the potential energy V stored in the springs is equal to zero for $z_1 = z_2 = 0$ and combining the forces of the springs according to the displacements into a vector $\mathbf{f}_F$

$$\mathbf{f}_F = \begin{bmatrix} f_{F1}(z_1, z_2) \\ f_{F2}(z_1, z_2) \end{bmatrix} = \begin{bmatrix} c_1 z_1 + c_{12}(z_1 - z_2) \\ c_2 z_2 + c_{12}(z_2 - z_1) \end{bmatrix} , \tag{3.71}$$

then V is calculated with $\mathbf{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^{\mathrm{T}}$ in the form

$$V = \int_{\mathbf{0}}^{\mathbf{z}} \mathbf{f}_F \cdot \mathrm{d}\tilde{\mathbf{z}} = \int_{\mathbf{0}}^{\mathbf{z}} \underbrace{[c_1 \tilde{z}_1 + c_{12}(\tilde{z}_1 - \tilde{z}_2)]}_{f_{F1}(\tilde{z}_1, \tilde{z}_2)} \mathrm{d}\tilde{z}_1 + \underbrace{[c_2 \tilde{z}_2 + c_{12}(\tilde{z}_2 - \tilde{z}_1)]}_{f_{F2}(\tilde{z}_1, \tilde{z}_2)} \mathrm{d}\tilde{z}_2 \ . \tag{3.72}$$

Analogous to (3.61), the path independence of the integration of (3.72) is given, since the integrability condition

$$c_{12} = \frac{\partial f_{F1}(\tilde{z}_1, \tilde{z}_2)}{\partial \tilde{z}_2} = \frac{\partial f_{F2}(\tilde{z}_1, \tilde{z}_2)}{\partial \tilde{z}_1} = c_{12} \tag{3.73}$$

is fulfilled. The potential energy V of the force $\mathbf{f}_F = \mathbf{K}\mathbf{z}$ is then given by

$$\begin{aligned} V &= \int_0^{z_1} f_{F1}(\tilde{z}_1, 0) \, \mathrm{d}\tilde{z}_1 + \int_0^{z_2} f_{F2}(z_1, \tilde{z}_2) \, \mathrm{d}\tilde{z}_2 \\ &= \int_0^{z_1} [c_1 \tilde{z}_1 + c_{12} \tilde{z}_1] \, \mathrm{d}\tilde{z}_1 + \int_0^{z_2} [c_2 \tilde{z}_2 + c_{12} \tilde{z}_2 - c_{12} z_1] \, \mathrm{d}\tilde{z}_2 \\ &= (c_1 + c_{12}) \frac{z_1^2}{2} + (c_2 + c_{12}) \frac{z_2^2}{2} - c_{12} z_1 z_2 \\ &= \frac{1}{2} \mathbf{z}^{\mathrm{T}} \underbrace{\begin{bmatrix} c_1 + c_{12} & -c_{12} \\ -c_{12} & c_2 + c_{12} \end{bmatrix}}_{\mathbf{K}} \mathbf{z} \end{aligned} \tag{3.74}$$

The matrix $\mathbf{K}$ is *symmetric* and *positive definite* and is also called the *stiffness matrix*. The symmetry of the stiffness matrix implies the integrability condition, so that a potential energy exists for $\mathbf{f}_F$.

3.5 Dissipative Forces

A *dissipative force* $\mathbf{f}_D$ is a force whose work is irreversibly converted into heat (dissipated), i.e. $\mathbf{f}_D(t) \cdot \mathbf{v}(t) \leq 0$ for all times t. These can be forces acting over a volume, such as in an eddy current brake, or forces acting over a surface, as occurs when a rigid body moves through a fluid due to friction.

3.5.1 Motion of a Rigid Body through a Fluid

If one considers a rigid body moving uniformly with the velocity $\mathbf{v}$ without rotation through a (resting) fluid medium, then the surface distributed forces exerted by the fluid on the body can be expressed by a resultant force $\mathbf{f}_R$ and a resultant torque $\boldsymbol{\tau}_R^{(Z)}$ with respect to an arbitrarily chosen point Z (see also the previous explanations on the topic *general force system*). The resultant force $\mathbf{f}_R$ can be decomposed into a component $\mathbf{f}_A$

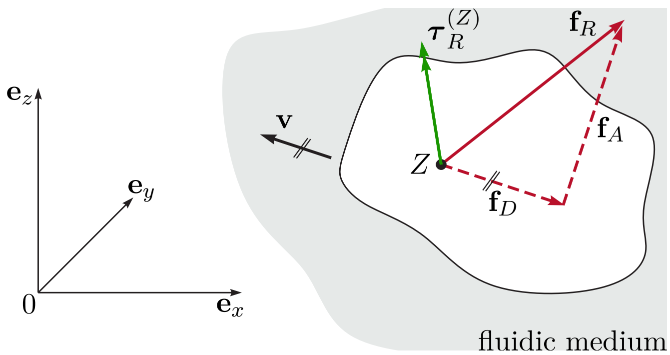


Figure 3.21: Moving rigid body in a fluid medium.

(*deflection force*) perpendicular to $\mathbf{v}$ and a component $\mathbf{f}_D$ (*drag force*) acting parallel in the opposite direction of $\mathbf{v}$, see Figure 3.21. The deflection force $\mathbf{f}_A$ is also called dynamic lift and is caused by the geometry of the rigid body. A simple relationship for the drag force $\mathbf{f}_D$ in a wide velocity range below the speed of sound is given by

$$\mathbf{f}_D = f_D \mathbf{e}_v = -c_W A \frac{\rho_f}{2} v^2 \mathbf{e}_v \tag{3.75}$$

with $v = \|\mathbf{v}\|_2$ and the direction vector of the velocity $\mathbf{e}_v$. Here, $c_W > 0$ denotes the (dimensionless) drag coefficient, A a suitable reference area and ρ_f the density of the fluid medium.

Exercise 3.9 (Free Fall). Create a mathematical model to describe the free fall of an object of mass m, cross-sectional area A and drag coefficient c_W in the Earth's atmosphere. Consider the change in air density as a function of altitude h in the form

$$\rho(h) = \rho_0 \exp\left(-\frac{h}{k}\right),$$

with the constants ρ_0 and k. Using MAPLE, determine a numerical solution of the model for the initial conditions $h(0) = 39\,\mathrm{km}$ and $v(0) = \dot{h}(0) = 0$ and the following parameters: $\rho_0 = 1.2\,\mathrm{kg/m^3}$, $k = 9100\,\mathrm{m}$, $A = 0.5\,\mathrm{m^2}$, $m = 100\,\mathrm{kg}$, $c_W = 0.5$, $g = 9.81\,\mathrm{m/s^2}$.

Solution of exercise 3.9.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}h(t) &= v(t)\\ m\frac{\mathrm{d}}{\mathrm{d}t}v(t) &= -mg + c_W A \rho_0 \exp\left(-\frac{h(t)}{k}\right)\frac{v(t)^2}{2}\end{aligned}$$

Solution in MAPLE: `Aufgabe_3_9.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

3.5.2 Friction between Solid Bodies

When two contacting solid bodies perform relative movements, then tangential friction forces acting over the area arise *due to the roughness of the surfaces* in the contact area. In the following, consider a mass m which is moved on a rough surface by an external horizontal force f_e, see Figure 3.22. If one cuts the mass free, then in addition to f_e, the

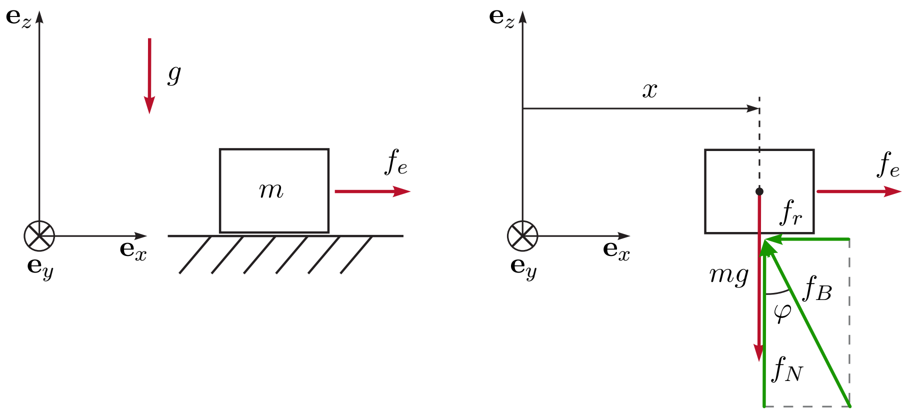


Figure 3.22: On static friction.

normal force f_N and the friction force f_r act on the mass. From experience one knows that the mass m only moves when the force f_e exceeds a certain value f_H, i.e. as long as the inequality[5] $|f_e| \leq f_H$ is satisfied, the static equilibrium conditions

$$
\begin{aligned}
\mathbf{e}_x &: f_e - f_r = 0 && (3.76a)\\
\mathbf{e}_z &: f_N - mg = 0 && (3.76b)
\end{aligned}
$$

apply and the mass remains stuck in place. In this context, f_r is therefore called the *static friction force* and represents a reaction force, as is already known from the cutting principle. In a first approximation, f_H can be expressed in the form

$$f_H = \mu_H f_N, \quad \text{for } f_N > 0, \tag{3.77}$$

where the *static friction coefficient* $\mu_H > 0$ depends only on the roughness of the contacting surfaces. If the external force f_e is increased so that the static friction is overcome, then the mass begins to move and the friction force $f_r = f_C$ due to *dry sliding friction* is

$$f_C = \mu_C f_N \operatorname{sgn}(\dot{x}), \quad \text{for } f_N > 0, \tag{3.78}$$

with the *sliding friction coefficient* $\mu_C > 0$. In this case, the equilibrium condition $f_N = mg$ still holds for the $\mathbf{e}_z$ direction and the law of conservation of momentum (3.33) for the $\mathbf{e}_x$ direction becomes

$$m\ddot{x} = f_e - \mu_C mg \operatorname{sgn}(\dot{x}). \tag{3.79}$$

The mathematical model of the mass of Figure 3.22 is therefore characterized by a *structural change*, i.e.

$$\textbf{Sticking}\text{: if } |f_e| \leq f_H \text{ and } \dot{x} = 0 \quad \begin{cases} \dot{x} &= 0 \\ \dot{v} &= 0 \end{cases} \tag{3.80a}$$

$$\textbf{Sliding}\text{: otherwise} \begin{cases} \dot{x} &= v \\ m\dot{v} &= f_e - \mu_C mg \operatorname{sgn}(v)\ . \end{cases} \tag{3.80b}$$

The friction law (3.77), (3.78) is also known as *Coulomb's law of friction* and is essentially considered as an elementary approximation theory for dry friction between solid bodies. The friction coefficients μ_H and μ_C generally have to be determined from experimental investigations. Typical values for some material pairings can be found in handbooks, see for example Table 3.1.

In the sticking state, one can introduce an angle φ according to Figure 3.22 in the form

$$\tan(\varphi) = \frac{f_r}{f_N}. \tag{3.81}$$

Substituting the limit value $f_H = \mu_H f_N$ for f_r, one obtains the relationship

$$\tan(\varphi_H) = \mu_H \tag{3.82}$$

with the angle of static friction φ_H. This allows a clear geometrical interpretation of static friction: If a body is subjected to an arbitrarily directed load, it remains at rest as long as the reaction force $\mathbf{f}_B$ at the contact surface lies within the so-called *friction cone*. The friction cone describes the cone of revolution around the normal $\mathbf{e}_n$ of the contact surfaces with the opening angle $2\varphi_H$, see Figure 3.23.

[5] Note that in general f_H can take different values for different signs of f_e, which is not considered here.

Table 3.1: Typical friction coefficients.

Material pairing	Static friction μ_H	Sliding friction μ_C
Bronze on bronze	0.18	0.2
Cast iron on bronze	0.28	0.2
Steel on steel	0.15	0.12
Pneumatic tire on asphalt	0.55	0.3
Oak on oak	0.54	0.34

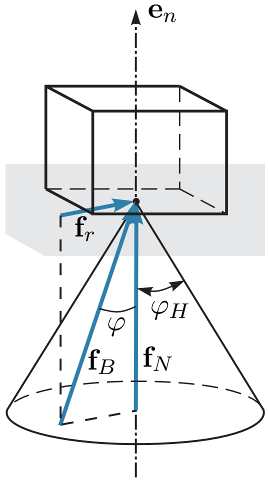


Figure 3.23: Friction cone.

Exercise 3.10. A mass m lies on an inclined plane and is pulled upwards by a person with the force f_S (see Figure 3.24). Calculate the necessary pulling force f_S as a function of the angles α and β as well as the mass m and the static friction coefficient μ_H, so that the mass can be moved.

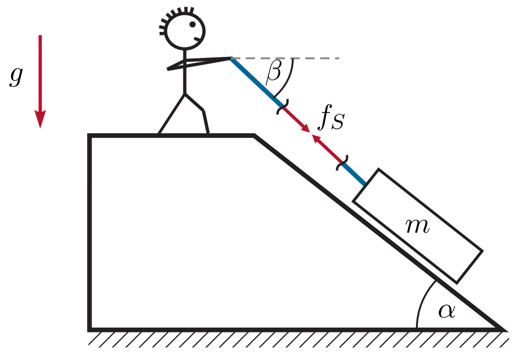


Figure 3.24: Mass on an inclined plane.

Solution of exercise 3.10.

$$f_S > \frac{mg(\mu_H \cos(\alpha) + \sin(\alpha))}{\cos(\beta - \alpha) + \mu_H \sin(\beta - \alpha)}$$

Solution in MAPLE: `Aufgabe_3_10.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Exercise 3.11. A person of mass m climbs a 21-step ladder of length l which is leaning against a wall, see Figure 3.25. How many steps can the person climb up the ladder without the ladder slipping away if the static friction coefficient between the ladder and the wall is zero and between the ladder and the floor is $\mu_H = 1/10$?

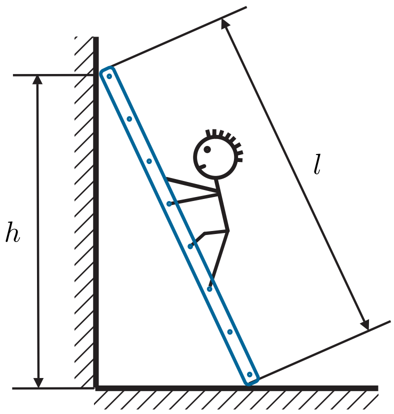


Figure 3.25: Person on a ladder.

Solution of exercise 3.11. The number of steps corresponds to the number rounded down to the nearest integer

$$\frac{20\mu_H}{\sqrt{\left(\frac{l}{h}\right)^2 - 1}} + 1 \; .$$

Note that $\tan(\arcsin(x)) = x/\sqrt{1-x^2}$ was used here.

Solution in MAPLE: `Aufgabe_3_11.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

If there is a continuous layer of lubricant between the two solid bodies, then the forces acting between the bodies essentially depend on the flow established in the gap between the two bodies. Very often, a simple model of the form

$$f_r = \mu_V \Delta v \tag{3.83}$$

is used in this context for the friction force f_r, with the *viscous friction coefficient* $\mu_V > 0$ and the relative velocity Δv of the two contacting surfaces of the rigid bodies. In the general case of mixed friction, Coulomb friction (3.77), (3.78) and viscous friction (3.83) are combined.

There are now components, called *dampers*, which realize a given (nonlinear) force-velocity characteristic $f_D(\Delta v)$ with $f_D(\Delta v)\Delta v > 0$ according to Figure 3.26. In the linear case, the damping force is given by $f_D = d\Delta v$ with the *damping coefficient* $d > 0$ *proportional to the velocity*.

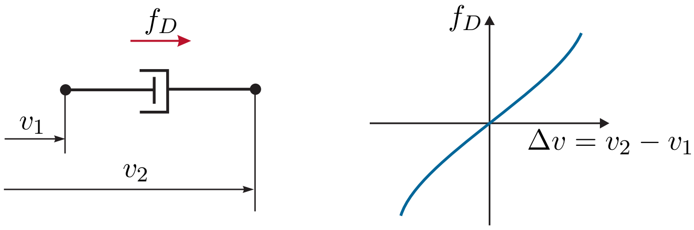


Figure 3.26: Nonlinear damper.

Example 3.9. A massless rope is guided around a stationary cylinder with a wrap angle α according to Figure 3.27, where $f_{S2} > f_{S1}$.

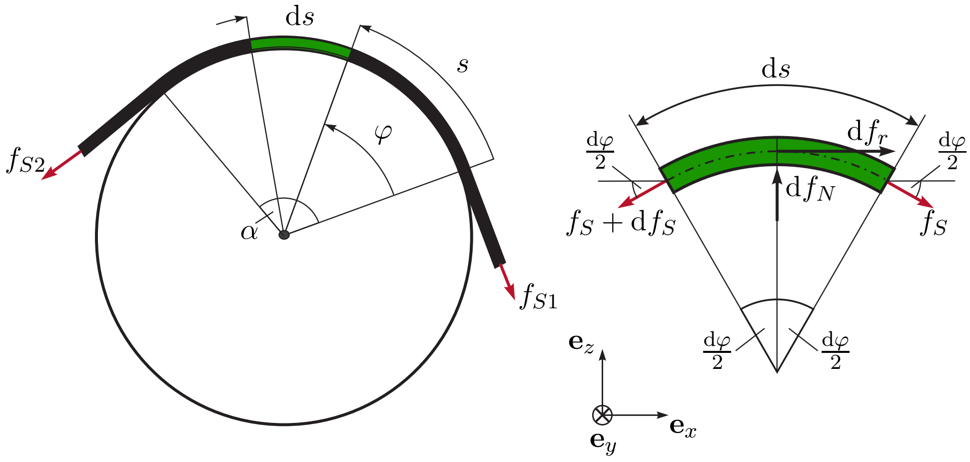


Figure 3.27: On rope friction.

If one now takes out an infinitesimal rope element, then the equilibrium conditions under the assumption of sufficiently small angles $\mathrm{d}\varphi/2$ (i.e. $\sin(\mathrm{d}\varphi/2) \approx \mathrm{d}\varphi/2$, $\cos(\mathrm{d}\varphi/2) \approx 1$) are

$$\mathbf{e}_x : f_S + \mathrm{d}f_r - (f_S + \mathrm{d}f_S) = 0 \tag{3.84a}$$

$$\mathbf{e}_z : \mathrm{d}f_N - f_S \frac{\mathrm{d}\varphi}{2} - (f_S + \mathrm{d}f_S)\frac{\mathrm{d}\varphi}{2} = 0 \tag{3.84b}$$

or, neglecting $\mathrm{d}f_S\,\mathrm{d}\varphi/2$, it follows

$$\mathrm{d}f_r = \mathrm{d}f_S \quad \text{and} \quad \mathrm{d}f_N = f_S\,\mathrm{d}\varphi\ . \tag{3.85}$$

With Coulomb's law of friction according to (3.77), (3.78), in particular $\mathrm{d}f_r = \mu\,\mathrm{d}f_N$, one obtains

$$\frac{\mathrm{d}f_S}{\mathrm{d}\varphi} = \mu f_S \tag{3.86}$$

or by integrating over the wrap angle from $\varphi = 0$ to $\varphi = \alpha$ one obtains the *rope friction equation* as

$$\int_{f_{S1}}^{f_{S2}} \frac{1}{f_S}\,\mathrm{d}f_S = \int_0^\alpha \mu\,\mathrm{d}\varphi \quad \text{or} \quad f_{S2} = f_{S1}\exp(\mu\alpha)\ . \tag{3.87}$$

For the case $f_{S1} > f_{S2}$, the relation $f_{S1} = f_{S2}\exp(\mu\alpha)$ can be derived analogously. If $\mu = \mu_H$ now denotes the static friction coefficient, then the system is in equilibrium as long as the inequality

$$f_{S1}\exp(-\mu_H\alpha) \le f_{S2} \le f_{S1}\exp(\mu_H\alpha) \tag{3.88}$$

is satisfied. The rope slips to the right for $f_{S1} > f_{S2} \exp(\mu_H \alpha)$ and to the left for $f_{S2} > f_{S1} \exp(\mu_H \alpha)$.

Exercise 3.12. A mass with the weight force mg hangs on a (massless) rope, which was wrapped once around a stationary cylinder (wrap angle 360°) and can be held in equilibrium with a force of 10 N. How many times do you have to wrap the rope around the cylinder so that 10 times the mass can also be held in equilibrium by the static friction of the rope with a force of 10 N?

Solution of exercise 3.12. The desired wrap angle α is

$$\alpha = 2\pi \frac{\ln\left(\frac{10mg}{10}\right)}{\ln\left(\frac{mg}{10}\right)} \ .$$

Solution in Maple: `Aufgabe_3_12.mw`
`https://www.acin.tuwien.ac.at/bachelor/modellbildung/`.

3.5.3 Rolling Friction

If a rigid wheel rolls on a rigid surface without slipping, then theoretically there is no *rolling resistance*. In reality, however, every rolling process is accompanied by deformations, which are associated with partial sliding processes in the contact area. Figure 3.28 shows the respective force ratios for a running wheel and a driven wheel. In the case of the

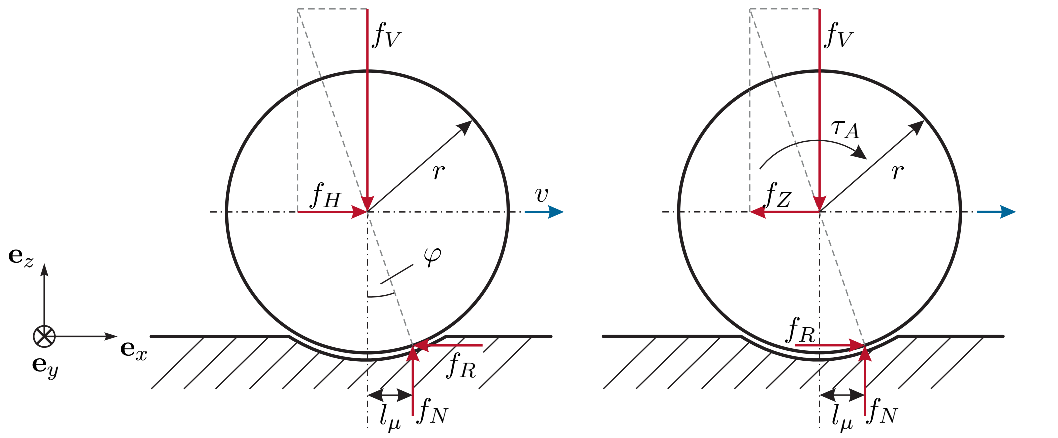


Figure 3.28: Running wheel and driven wheel.

running wheel, the horizontally acting force f_H must be introduced into the wheel via the axle in order to compensate for the rolling resistance. From the equilibrium conditions for very small angles $\varphi = \arctan(f_H/f_V)$

$$f_H - f_R = 0, \quad f_N - f_V = 0 \quad \text{and} \quad r f_R - l_\mu f_N = 0 \tag{3.89}$$

the rolling resistance force f_R follows to

$$f_R = \frac{l_\mu}{r} f_N = \mu_R f_V \tag{3.90}$$

with the *rolling friction coefficient* $\mu_R = l_\mu/r > 0$. For the same material pairing, the rolling friction coefficient is significantly smaller than the sliding friction coefficient.

Exercise 3.13. Show that for the driven wheel, the driving torque $\tau_A = l_\mu f_V$ must be applied to overcome the rolling resistance and that the pulling force is calculated as $f_Z = \tau_A/r - \mu_R f_V$.

Remark: The equilibrium conditions for the driven wheel under the assumption of very small angles $\varphi = \arctan(f_Z/f_V)$ are

$$f_Z - f_R = 0, \quad f_N - f_V = 0 \quad \text{and} \quad \tau_A - r f_R - l_\mu f_N = 0 \ .$$

3.6 Spring-Mass-Damper System

Many real technical systems can be described as a combination of rigid bodies with springs and dampers (e.g. wheel suspensions in vehicles, the micro-mechanical gyroscope from Example 1.3). Based on the previous results, the equations of motion of such spring-mass-damper systems can already be derived. To this end, consider the following example.

Example 3.10. Consider the spring-mass-damper system of Figure 3.29 with the masses m_1, m_2 and m_3, the linear damper elements with the positive damping constants d_{11}, d_{22} and d_{13} as well as the linear spring elements with the positive spring constants c_{11}, c_{22}, c_{13} and c_{23} and the relaxed lengths s_{011}, s_{022}, s_{013} and s_{023}. Furthermore, let the force f_L act on the mass m_3 and let g denote the acceleration due to gravity.

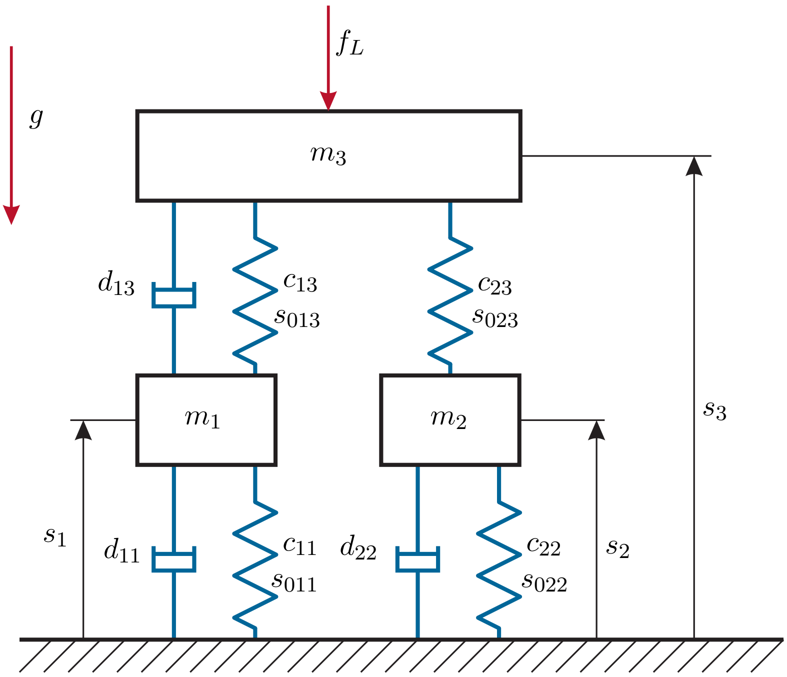


Figure 3.29: Spring-mass-damper system with three masses.

Applying the law of conservation of momentum (3.33) for each mass, one obtains three second-order differential equations

$$m_1\ddot{s}_1 = -m_1 g - c_{11}(s_1 - s_{011}) - d_{11}\dot{s}_1 + c_{13}(s_3 - s_1 - s_{013}) - d_{13}(\dot{s}_1 - \dot{s}_3) \tag{3.91a}$$

$$m_2\ddot{s}_2 = -m_2 g - c_{22}(s_2 - s_{022}) - d_{22}\dot{s}_2 + c_{23}(s_3 - s_2 - s_{023}) \tag{3.91b}$$

$$m_3\ddot{s}_3 = -m_3 g - c_{13}(s_3 - s_1 - s_{013}) + d_{13}(\dot{s}_1 - \dot{s}_3) - c_{23}(s_3 - s_2 - s_{023}) - f_L\ . \tag{3.91c}$$

The mathematical model (3.91c) can also be written more compactly in *matrix notation* in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{k} + \mathbf{b}f_L \tag{3.92}$$

with $\mathbf{q} = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}^{\mathrm{T}}$, the *symmetric, positive definite mass matrix* $\mathbf{M} = \mathrm{diag}(m_1, m_2, m_3)$, the *symmetric, positive (semi-)definite damping matrix*

$$\mathbf{D} = \begin{bmatrix} d_{11} + d_{13} & 0 & -d_{13} \\ 0 & d_{22} & 0 \\ -d_{13} & 0 & d_{13} \end{bmatrix}, \tag{3.93}$$

the *symmetric, positive definite stiffness matrix*

$$\mathbf{K} = \begin{bmatrix} c_{11} + c_{13} & 0 & -c_{13} \\ 0 & c_{22} + c_{23} & -c_{23} \\ -c_{13} & -c_{23} & c_{13} + c_{23} \end{bmatrix}, \tag{3.94}$$

the constant vector $\mathbf{k}$ and the constant input vector $\mathbf{b}$

$$\mathbf{k} = \begin{bmatrix} -m_1 g + c_{11} s_{011} - c_{13} s_{013} \\ -m_2 g + c_{22} s_{022} - c_{23} s_{023} \\ -m_3 g + c_{13} s_{013} + c_{23} s_{023} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}. \tag{3.95}$$

Exercise 3.14. Show the definiteness properties of the matrices $\mathbf{K}$ and $\mathbf{D}$.

To calculate the *equilibrium position* $\mathbf{q}_R$ for $f_L = 0$, one sets $\dot{\mathbf{q}} = \ddot{\mathbf{q}} = \mathbf{0}$ in (3.92) and solves the resulting linear system of equations $\mathbf{K}\mathbf{q}_R = \mathbf{k}$ for $\mathbf{q}_R$. Due to the positive definiteness, $\mathbf{K}$ is invertible and it follows

$$\mathbf{q}_R = \mathbf{K}^{-1}\mathbf{k} . \tag{3.96}$$

Introducing the deviation $\Delta\mathbf{q}$ of $\mathbf{q}$ from the equilibrium position (rest position) $\mathbf{q}_R$, i.e. $\Delta\mathbf{q} = \mathbf{q} - \mathbf{q}_R$, then the equation of motion (3.92) follows in the form

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + \mathbf{D}\Delta\dot{\mathbf{q}} + \mathbf{K}\Delta\mathbf{q} + \underbrace{\mathbf{K}\mathbf{q}_R}_{\mathbf{k}} = \mathbf{k} + \mathbf{b}f_L . \tag{3.97}$$

The numerical simulation of this spring-mass-damper system in MATLAB/SIMULINK is shown in `Beispiel_3_10.zip`, which can be downloaded from https://www.acin.tuwien.ac.at/bachelor/modellbildung/. Here, among other things, the influence of the parameters of the system on the solution properties can be analyzed.

The result of the previous example can be generalized in that every linear spring-mass-damper system can be written in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{f}_e \tag{3.98}$$

with the vector of position coordinates $\mathbf{q}$ (relative to the equilibrium position), the *symmetric, positive definite mass matrix* $\mathbf{M}$, the *symmetric, positive semi-definite damping matrix* $\mathbf{D}$, the *symmetric, positive definite stiffness matrix* $\mathbf{K}$, the input matrix $\mathbf{B}$ and the vector of external forces $\mathbf{f}_e$.

The energy stored in the system consists of the kinetic energy

$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}} \tag{3.99}$$

and the potential energy stored in the springs

$$V = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}. \tag{3.100}$$

If one now calculates the temporal change of the total energy $E = T + V$ along a solution trajectory of (3.98), then it follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E &= \frac{1}{2}\ddot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{K}\mathbf{q} + \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}\dot{\mathbf{q}} = \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\ddot{\mathbf{q}} + \mathbf{q}^{\mathrm{T}}\mathbf{K}\dot{\mathbf{q}}\\ &= \dot{\mathbf{q}}^{\mathrm{T}}(-\mathbf{D}\dot{\mathbf{q}} - \mathbf{K}\mathbf{q} + \mathbf{B}\mathbf{f}_e) + \mathbf{q}^{\mathrm{T}}\mathbf{K}\dot{\mathbf{q}} = -\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{D}\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{B}\mathbf{f}_e\ .\end{aligned} \tag{3.101}$$

The first term $-\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{D}\dot{\mathbf{q}} \le 0$ indicates the power dissipated in the damper elements and the second term $\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{B}\mathbf{f}_e$ describes the energy flows to or from the system due to the external forces $\mathbf{f}_e$.

> *Exercise* 3.15. Show that the change of the total energy is calculated as in (3.101) also for the spring-mass-damper system according to (3.92).

3.7 Conservation of Angular Momentum

In (3.10) it was shown that the torque $\boldsymbol{\tau}^{(0)}$ of a force $\mathbf{f}$ with the position vector $\mathbf{r}$ is calculated as $\boldsymbol{\tau}^{(0)} = \mathbf{r} \times \mathbf{f}$. If one now considers a point mass with mass m, the position vector $\mathbf{r}(t)$ from the origin of the inertial system $(0xyz)$ and the velocity $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$, then the *angular momentum* is defined as

$$\mathbf{l}^{(0)} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}. \tag{3.102}$$

Forming the cross product of both sides of the law of conservation of momentum (3.33) with the position vector $\mathbf{r}$, one obtains

$$\mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{r} \times \mathbf{f} = \boldsymbol{\tau}^{(0)}\ . \tag{3.103}$$

With

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \underbrace{\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}}_{\mathbf{v}} \times \underbrace{\mathbf{p}}_{m\mathbf{v}}}_{=\mathbf{0}} + \mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} \tag{3.104}$$

it follows from (3.103) that the *law of conservation of angular momentum (theorem of angular momentum)* is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{l}^{(0)} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \boldsymbol{\tau}^{(0)}\ , \tag{3.105}$$

i.e., the temporal change of angular momentum $\mathbf{l}^{(0)}$ with respect to an *arbitrary fixed point in space* 0 is equal to the torque $\boldsymbol{\tau}^{(0)}$ of the resultant force $\mathbf{f}$ acting on the point mass with respect to the same point 0.

Example 3.11. If the force vector always points to a point 0 (the center) during a motion, then it is called a *central motion*. This is the case, for example, in planetary motion, where the Sun forms the center. Since the torque $\boldsymbol{\tau}^{(0)}$ with respect to the center vanishes in a central motion, the angular momentum $\mathbf{l}^{(0)}$ must be constant according to (3.105).

The area swept by the position vector $\mathbf{r}$ in time $\mathrm{d}t$ can be described by the area vector $\mathrm{d}\mathbf{A}^{(0)} = \mathbf{n}_A \,\mathrm{d}A = \frac{1}{2}\mathbf{r} \times \mathrm{d}\mathbf{r}$, where $\mathbf{n}_A$ describes the normal vector and $\mathrm{d}A = \frac{1}{2}\|\mathbf{r} \times \mathrm{d}\mathbf{r}\|_2$ the corresponding size of the area element. If one now introduces the so-called *vectorial areal velocity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{(0)} = \frac{1}{2}\mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \frac{1}{2}\mathbf{r} \times \mathbf{v}, \tag{3.106}$$

then the angular momentum (3.102) can also be written in the form

$$\mathbf{l}^{(0)} = 2m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{(0)}. \tag{3.107}$$

From $\mathbf{l}^{(0)}$ =constant it follows according to (3.107) that the areal velocity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{(0)}$ is also constant for a central motion. This statement corresponds to *Kepler's second law*. This states that a 'radius vector' drawn from the Sun to the planet sweeps out equal areas in equal times, see Figure 3.30.

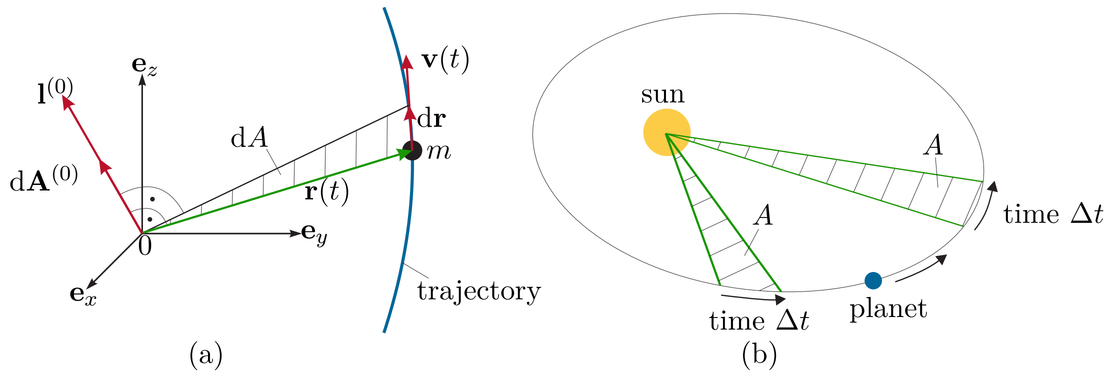


Figure 3.30: On the conservation of angular momentum (a) and Kepler's second law (b).

Example 3.12. Consider the mathematical pendulum of Figure 3.31 with the point mass m and the massless rigid pendulum of length l under the influence of gravity with the acceleration due to gravity g in the negative $\mathbf{e}_z$ direction.

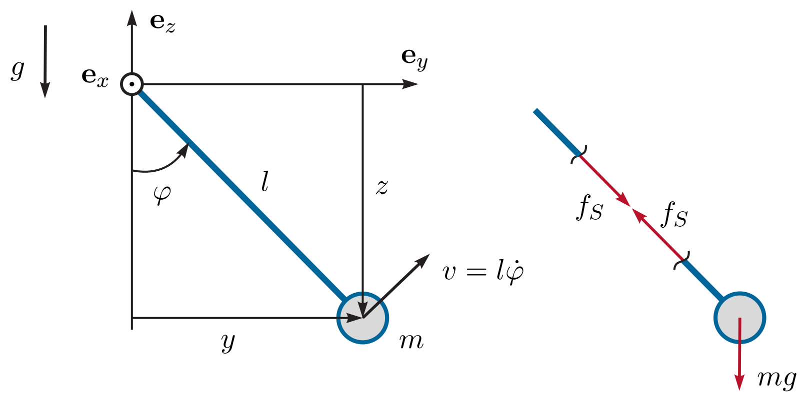


Figure 3.31: Mathematical pendulum.

If one cuts the pendulum open and introduces the cutting force f_S, then the law of conservation of momentum for the mass m reads as

$$\mathbf{e}_y : m\ddot{y} = -f_S \sin(\varphi) \tag{3.108a}$$
$$\mathbf{e}_z : m\ddot{z} = -mg + f_S \cos(\varphi) \ . \tag{3.108b}$$

Substituting the relations

$$y = l\sin(\varphi), \quad \dot{y} = l\cos(\varphi)\dot{\varphi}, \quad \ddot{y} = -l\sin(\varphi)\dot{\varphi}^2 + l\cos(\varphi)\ddot{\varphi} \tag{3.109a}$$
$$z = -l\cos(\varphi), \quad \dot{z} = l\sin(\varphi)\dot{\varphi}, \quad \ddot{z} = l\cos(\varphi)\dot{\varphi}^2 + l\sin(\varphi)\ddot{\varphi} \tag{3.109b}$$

into (3.108), one obtains

$$m\Big(-l\sin(\varphi)\dot{\varphi}^2 + l\cos(\varphi)\ddot{\varphi}\Big) = -f_S\sin(\varphi) \tag{3.110a}$$
$$m\Big(l\cos(\varphi)\dot{\varphi}^2 + l\sin(\varphi)\ddot{\varphi}\Big) = -mg + f_S\cos(\varphi). \tag{3.110b}$$

From the two equations (3.110), a differential equation for φ

$$ml^2\ddot{\varphi} = -mgl\sin(\varphi) \tag{3.111}$$

and the cutting force f_S in the form

$$f_S = mg\cos(\varphi) + ml\dot{\varphi}^2 \tag{3.112}$$

can now be calculated. The differential equation (3.111) can also be obtained directly via the law of conservation of angular momentum (3.105) with respect to the origin 0 of the coordinate system $(0xyz)$. The corresponding angular momentum $\mathbf{l}^{(0)}$ according to (3.102) is (see also Figure 3.31)

$$\mathbf{l}^{(0)} = \mathbf{r} \times m\mathbf{v} = \begin{bmatrix} 0 \\ l\sin(\varphi) \\ -l\cos(\varphi) \end{bmatrix} \times m \begin{bmatrix} 0 \\ l\dot{\varphi}\cos(\varphi) \\ l\dot{\varphi}\sin(\varphi) \end{bmatrix} = \begin{bmatrix} ml^2\dot{\varphi} \\ 0 \\ 0 \end{bmatrix} \tag{3.113}$$

and thus the law of conservation of angular momentum with respect to the $\mathbf{e}_x$ axis becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} l_x^{(0)} = ml^2\ddot{\varphi} = \tau_x^{(0)} = -mgl\sin(\varphi) \ . \tag{3.114}$$

The quantity

$$I_{xx}^{(0)} = ml^2 \tag{3.115}$$

is also called the *moment of inertia*.

Solution in MAPLE: `Beispiel_3_12.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

The numerical simulation of the mathematical pendulum in MATLAB/SIMULINK is shown in `Beispiel_3_12.zip` and can be downloaded from https://www.acin.tuwien.ac.at/bachelor/modellbildung/. Investigate here the influence of the initial conditions, the mass m and the chosen integration method on the solution behavior.

The previous example can now be easily extended to the rotation of a rigid body with the angular velocity $\omega = \dot{\varphi}$ about a fixed axis of rotation $\mathbf{e}_\omega$ (in the present case $\mathbf{e}_\omega = \mathbf{e}_z$), see Figure 3.32.

If one writes down the temporal change of the angular momentum about the axis of rotation for a mass element $\mathrm{d}m = \rho(x,y,z)\,\mathrm{d}\mathcal{V}$ with the volume element $\mathrm{d}\mathcal{V}$ and the density $\rho(x,y,z)$, which is located at a distance $r(x,y,z)$ from the axis of rotation, with

$$\mathbf{r} = \begin{bmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -r\sin(\varphi)\omega \\ r\cos(\varphi)\omega \\ 0 \end{bmatrix}, \tag{3.116}$$

one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathrm{d}m\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} -z\,\mathrm{d}mr\cos(\varphi)\omega \\ -z\,\mathrm{d}mr\sin(\varphi)\omega \\ r^2\,\mathrm{d}m\omega \end{bmatrix}. \tag{3.117}$$

For the description of the rotation about the axis of rotation $\mathbf{e}_\omega$, only the corresponding part of (3.117) about this axis is of interest in the following, i.e., the part

$$\mathbf{e}_\omega \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathrm{d}m\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}t} r^2\,\mathrm{d}m\omega. \tag{3.118}$$

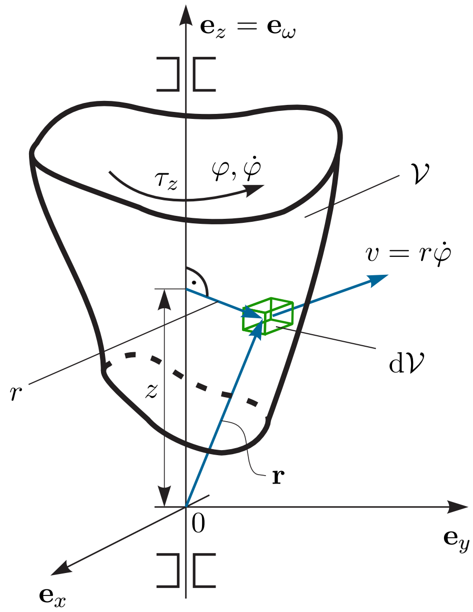


Figure 3.32: On the moment of inertia.

By integrating (3.118) over the entire rigid body volume $\mathcal{V}$, the *theorem of angular momentum* follows as

$$I_{zz}\dot{\omega} = I_{zz}\ddot{\varphi} = \tau_z \tag{3.119}$$

with the total external torque τ_z acting about the $\mathbf{e}_z$ axis and the moment of inertia

$$I_{zz} = \int_{\mathcal{V}} r^2 \,\mathrm{d}m = \int_{\mathcal{V}} \left(x^2 + y^2\right) \mathrm{d}m \ . \tag{3.120}$$

The *rotational kinetic energy* stored in the rotating mass is

$$T_r = \frac{1}{2} I_{zz}\dot{\varphi}^2 \ . \tag{3.121}$$

Example 3.13. The moment of inertia of a cylinder with radius R, constant density ρ and length l is (see Figure 3.33)

$$I_{zz} = \int_0^l \int_0^{2\pi} \int_0^R r^2 \rho r \,\mathrm{d}r \,\mathrm{d}\varphi \,\mathrm{d}z = \rho \frac{R^4\pi}{2} l = \frac{1}{2} m R^2 \ . \tag{3.122}$$

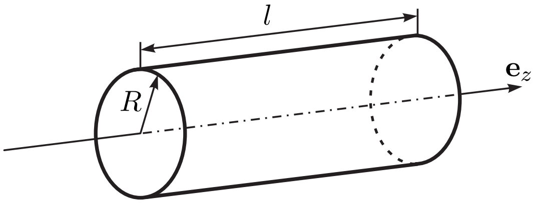


Figure 3.33: On the moment of inertia of a cylinder.

Exercise 3.16. Calculate the moment of inertia I of a homogeneous sphere with radius R and density ρ about an axis through the center of the sphere.

Solution of exercise 3.16.

$$I = \frac{8}{15}\pi\rho R^5 = \frac{2}{5}mR^2 \tag{3.123}$$

Solution in MAPLE: `Aufgabe_3_16.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

In the following, consider the rigid body of Figure 3.34. The origin S of the coordinate

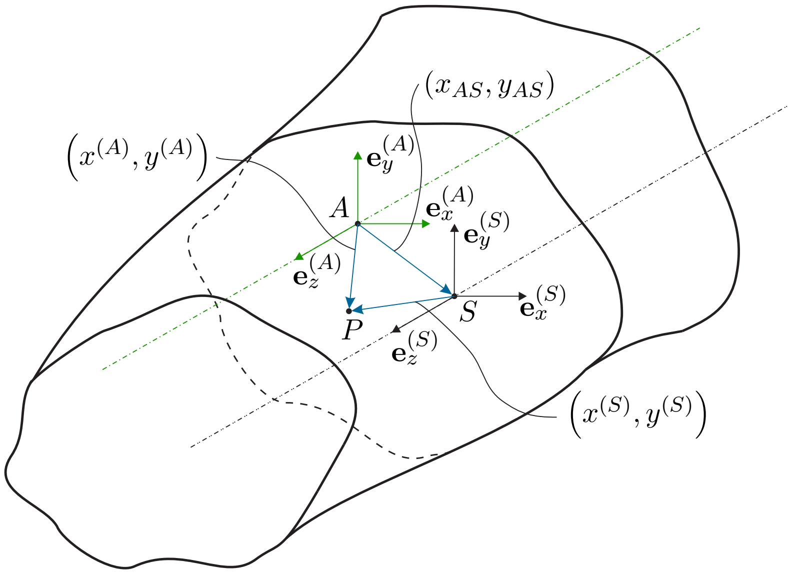


Figure 3.34: On Steiner's Theorem.

system $\left(Sx^{(S)}y^{(S)}z^{(S)}\right)$ describes the center of gravity of the body (see also (3.28)) and the moment of inertia about the $\mathbf{e}_z^{(S)}$ axis can be calculated using the relation

$$I_{zz}^{(S)} = \int_{\mathcal{V}} \left(r^{(S)}\right)^2 \mathrm{d}m = \int_{\mathcal{V}} \left(\left(x^{(S)}\right)^2 + \left(y^{(S)}\right)^2\right) \mathrm{d}m. \tag{3.124}$$

If one now wants to calculate the moment of inertia $I_{zz}^{(A)}$ of the same body with respect to the parallel $\mathbf{e}_z^{(A)}$ axis of the coordinate system $\left(Ax^{(A)}y^{(A)}z^{(A)}\right)$ (see Figure 3.34), then

$$I_{zz}^{(A)} = \int_{\mathcal{V}} \left(r^{(A)}\right)^2 \mathrm{d}m = \int_{\mathcal{V}} \left(\left(x^{(A)}\right)^2 + \left(y^{(A)}\right)^2\right) \mathrm{d}m \tag{3.125}$$

or with $x^{(A)} = x_{AS} + x^{(S)}$ and $y^{(A)} = y_{AS} + y^{(S)}$ one obtains

$$\begin{aligned} I_{zz}^{(A)} = & \int_{\mathcal{V}} \left((x_{AS})^2 + (y_{AS})^2\right) \mathrm{d}m + 2\int_{\mathcal{V}} \left(x_{AS}x^{(S)} + y_{AS}y^{(S)}\right) \mathrm{d}m \\ & + \int_{\mathcal{V}} \left(\left(x^{(S)}\right)^2 + \left(y^{(S)}\right)^2\right) \mathrm{d}m \\ = & \left((x_{AS})^2 + (y_{AS})^2\right)m + 2x_{AS}\underbrace{\int_{\mathcal{V}} x^{(S)}\,\mathrm{d}m}_{=0} + 2y_{AS}\underbrace{\int_{\mathcal{V}} y^{(S)}\,\mathrm{d}m}_{=0} + I_{zz}^{(S)}\ . \end{aligned} \tag{3.126}$$

Equation (3.126) shows that the moment of inertia $I_{zz}^{(A)}$ with respect to the $\mathbf{e}_z^{(A)}$ axis results from the sum of the moment of inertia $I_{zz}^{(S)}$ about the $\mathbf{e}_z^{(S)}$ axis through the center of gravity S and the multiplication of the total mass m by the squared distance $(x_{AS})^2 + (y_{AS})^2$ from the axis $\mathbf{e}_z^{(A)}$ to the axis $\mathbf{e}_z^{(S)}$. This relationship can also be found in the literature under the name *Steiner's Theorem*.

Example 3.14. Figure 3.35 shows a rigid body consisting of four symmetrically arranged solid cylinders, each with mass m and radius R, whose centers are located at a distance H from the axis of rotation $\mathbf{e}_z$.

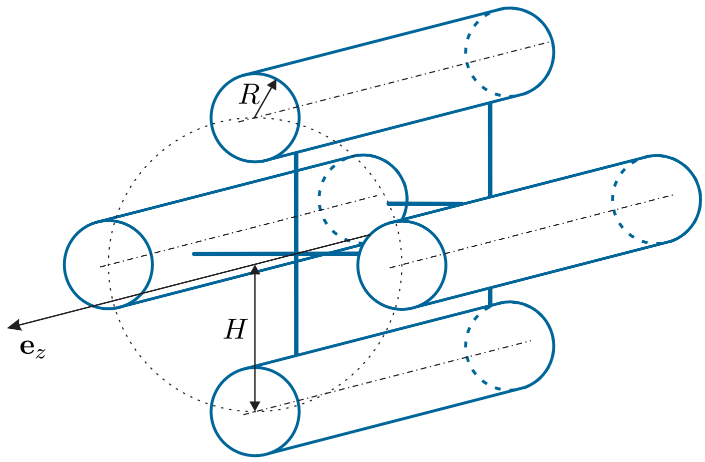


Figure 3.35: Rigid body consisting of four symmetrical cylinders.

It is assumed that the connecting bars between the cylinders are massless. The moment of inertia of a solid cylinder with respect to the $\mathbf{e}_z$ axis through the center of gravity is $I_{zz}^{(S)} = \frac{1}{2}mR^2$ according to (3.122). According to Steiner's Theorem, one thus obtains for the moment of inertia of the entire body

$$I_{zz} = 4\frac{1}{2}mR^2 + 4H^2m = 2m\Big(R^2 + 2H^2\Big) \; . \tag{3.127}$$

Example 3.15. Figure 3.36 shows a frictionlessly mounted, cuboid pendulum rod with homogeneous density ρ_S and geometrical dimensions length l_S, width b_S and height h_S.

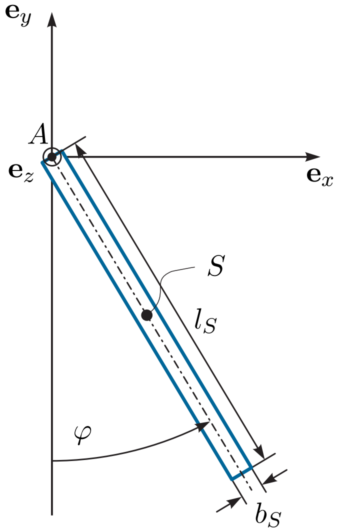


Figure 3.36: Pendulum rod.

Two variants will be presented in the following for calculating the kinetic energy. In the first variant, one calculates the moment of inertia $I_{S,zz}^{(A)}$ of the pendulum rod about the axis of rotation ($\mathbf{e}_z$ axis)

$$I_{S,zz}^{(A)} = \rho_S \int_{-h_S/2}^{h_S/2} \int_{-l_S}^{0} \int_{-b_S/2}^{b_S/2} \Big(x^2 + y^2\Big) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \rho_S \left(\frac{1}{3} l_S^3 b_S h_S + \frac{1}{12} b_S^3 l_S h_S\right) \tag{3.128}$$

and thus the kinetic energy is calculated according to (3.121) as

$$T = \frac{1}{2} I_{S,zz}^{(A)} \dot{\varphi}^2 \; . \tag{3.129}$$

In the second variant, one first sets up the position vector $\mathbf{r}_S$ from the origin 0 of the inertial system $(0xyz)$ to the center of gravity S of the pendulum rod

$$\mathbf{r}_S = \begin{bmatrix} l_S/2\sin(\varphi) \\ -l_S/2\cos(\varphi) \\ 0 \end{bmatrix} \tag{3.130}$$

and calculates the translational part of the kinetic energy according to (3.60) as

$$T_t = \frac{1}{2} m_S \dot{\mathbf{r}}_S^{\mathrm{T}} \dot{\mathbf{r}}_S = \frac{1}{2} m_S \frac{l_S^2}{4} \dot{\varphi}^2 \tag{3.131}$$

with the pendulum mass $m_S = \rho_S l_S b_S h_S$. If one now supplements the translational part of the kinetic energy T_t with the rotational part of the kinetic energy according to (3.121), one has to note that now the moment of inertia $I_{S,zz}^{(S)}$ must be calculated with respect to the center of gravity S (i.e., with respect to an axis of rotation parallel to the $\mathbf{e}_z$ axis through the center of gravity S)

$$I_{S,zz}^{(S)} = \rho_S \int_{-h_S/2}^{h_S/2} \int_{-l_S/2}^{l_S/2} \int_{-b_S/2}^{b_S/2} \left(x^2 + y^2 \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \rho_S \left(\frac{1}{12} l_S^3 b_S h_S + \frac{1}{12} b_S^3 l_S h_S \right) \tag{3.132}$$

and thus the rotational part of the kinetic energy follows as

$$T_r = \frac{1}{2} I_{S,zz}^{(S)} \dot{\varphi}.^2 \tag{3.133}$$

The kinetic energy of the pendulum rod is therefore

$$\begin{aligned} T &= T_t + T_r \\ &= \frac{1}{8} \rho_S b_S h_S l_S^3 \dot{\varphi}^2 + \frac{1}{2} \rho_S \left(\frac{1}{12} l_S^3 b_S h_S + \frac{1}{12} b_S^3 l_S h_S \right) \dot{\varphi}^2 \\ &= \frac{1}{2} \underbrace{\left(\frac{1}{3} \rho_S b_S h_S l_S^3 + \frac{1}{12} \rho_S b_S^3 l_S h_S \right)}_{= I_{S,zz}^{(A)}} \dot{\varphi}^2 \ . \end{aligned} \tag{3.134}$$

It should be noted that the relation

$$I_{S,zz}^{(A)} = I_{S,zz}^{(S)} + m_S \frac{l_S^2}{4} \tag{3.135}$$

corresponds exactly to Steiner's Theorem, see (3.126).

In general, it should be noted that when calculating the kinetic energy as the sum of a translational and a rotational part, the moment of inertia must always be used with respect to the axis of rotation shifted parallel to the center of gravity. This is of essential importance, especially in the following derivation of the equations of motion using the Euler-Lagrange equations!

Example 3.16. As an example, consider the drive train of Figure 3.37.

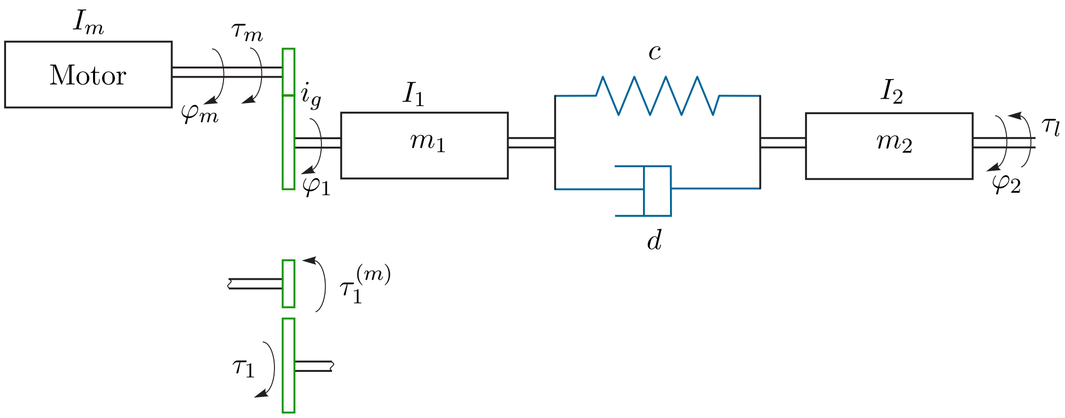


Figure 3.37: Drive train.

A motor with moment of inertia I_m generates a torque τ_m and drives a mass with moment of inertia I_1 via a lossless gear with gear ratio i_g (ratio of input speed to output speed)

$$\dot{\varphi}_1 = \frac{1}{i_g}\dot{\varphi}_m. \tag{3.136}$$

This mass is connected via a linear torsional spring with spring constant $c > 0$ and a rotational damper proportional to the angular velocity with damping constant $d > 0$ to another mass with moment of inertia I_2, on which the load torque τ_l acts. If one cuts the gear open (see Figure 3.37), the torque $\tau_1^{(m)}$ acts on the primary side. Since the gear was assumed to be lossless, the torque on the output side is

$$\tau_1\dot{\varphi}_1 = \tau_1^{(m)}\dot{\varphi}_m \quad \text{or} \quad \tau_1 = \tau_1^{(m)} i_g, \tag{3.137}$$

due to the gear ratio. Applying the law of conservation of angular momentum (3.105) separately for the two masses and the motor, it follows that

$$I_m\ddot{\varphi}_m = \tau_m - \tau_1^{(m)} \tag{3.138a}$$
$$I_1\ddot{\varphi}_1 = \tau_1 - c(\varphi_1 - \varphi_2) - d(\dot{\varphi}_1 - \dot{\varphi}_2) \tag{3.138b}$$
$$I_2\ddot{\varphi}_2 = c(\varphi_1 - \varphi_2) + d(\dot{\varphi}_1 - \dot{\varphi}_2) - \tau_l \tag{3.138c}$$

or, by eliminating $\tau_1^{(m)}$, τ_1 and φ_m, it follows

$$\tau_1 = \tau_1^{(m)} i_g = \tau_m i_g - I_m i_g^2 \ddot{\varphi}_1 \tag{3.139}$$

and

$$\left(I_1 + i_g^2 I_m\right)\ddot{\varphi}_1 = \tau_m i_g - c(\varphi_1 - \varphi_2) - d(\dot{\varphi}_1 - \dot{\varphi}_2) \tag{3.140a}$$
$$I_2\ddot{\varphi}_2 = c(\varphi_1 - \varphi_2) + d(\dot{\varphi}_1 - \dot{\varphi}_2) - \tau_l\ . \tag{3.140b}$$

In matrix notation, (3.140) can be written compactly in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{b}_e \tau_l + \mathbf{b}_u \tau_m \tag{3.141}$$

with $\mathbf{q} = [\varphi_1, \varphi_2]^{\mathrm{T}}$, $\mathbf{M} = \mathrm{diag}\Big(I_1 + i_g^2 I_m, I_2\Big)$ and

$$\mathbf{K} = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}, \mathbf{D} = \begin{bmatrix} d & -d \\ -d & d \end{bmatrix}, \mathbf{b}_e = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{b}_u = \begin{bmatrix} i_g \\ 0 \end{bmatrix} \tag{3.142}$$

according to (3.98).

Solution in Maple: `Beispiel_3_16.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Exercise 3.17. A sphere of mass m with radius R rolls down an inclined plane, see Figure 3.38. Give the equation of motion for φ neglecting rolling friction and determine the static friction coefficient μ_H for which rolling is possible.

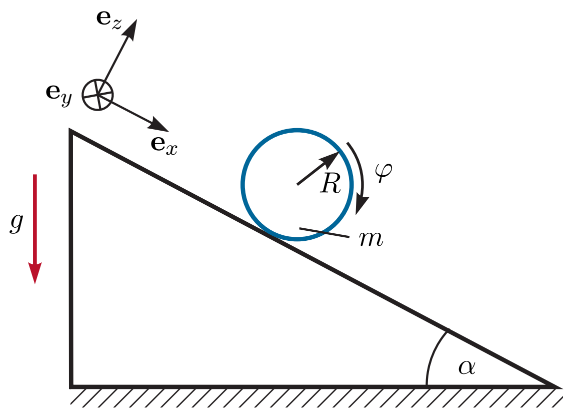


Figure 3.38: Rolling sphere.

Solution of exercise 3.17.

$$\ddot{\varphi} = \frac{5}{7R} g \sin(\alpha) \quad \text{for} \quad \mu_H \geq \frac{2}{7} \tan(\alpha)$$

Remark: Cut the sphere free and set up the law of conservation of momentum in the $\mathbf{e}_x$- and $\mathbf{e}_z$-direction of the indicated coordinate system as well as the law of conservation of angular momentum about the center of the sphere.

Solution in MAPLE: `Aufgabe_3_17.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

All solutions to the examples and Exercises in MAPLE and MATLAB/SIMULINK can also be downloaded collectively in `Kapitel_3.zip` from https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

3.8 References

[3.0] O. Bruhns and T. Lehmann, *Elemente der Mechanik I: Einführung, Statik*. Braunschweig/Wiesbaden: Vieweg, 1993.

[3.0] O. Bruhns and T. Lehmann, *Elemente der Mechanik II: Elastostatik*. Braunschweig/Wiesbaden: Vieweg, 1994.

[3.0] O. Bruhns and T. Lehmann, *Elemente der Mechanik III: Kinetik*. Braunschweig/Wiesbaden: Vieweg, 1994.

[3.0] M. Calkin, *Lagrangian and Hamiltonian Mechanics*. Singapore: World Scientific, 1996.

[3.0] D. Gross, W. Hauger, J. Schröder, and W. A. Wall, *Technische Mechanik 1: Statik*. Berlin Heidelberg: Springer, 2006.

[3.0] D. Gross, W. Hauger, J. Schröder, and W. A. Wall, *Technische Mechanik 2: Elastostatik*. Berlin Heidelberg: Springer, 2007.

[3.0] D. Gross, W. Hauger, J. Schröder, and W. A. Wall, *Technische Mechanik 3: Kinetik*. Berlin Heidelberg: Springer, 2008.

[3.0] D. Gross, W. Hauger, J. Schröder, and E. A. Werner, *Technische Mechanik 4: Hydromechanik, Elemente der Höheren Mechanik, Numerische Methoden*. Berlin Heidelberg: Springer, 2008.

[3.0] D. Gross, W. Ehlers, and P. Wriggers, *Formeln und Aufgaben zur Technischen Mechanik 1: Statik*. Berlin Heidelberg: Springer, 2008.

[3.0] D. Gross, W. Ehlers, and P. Wriggers, *Formeln und Aufgaben zur Technischen Mechanik 2: Elastostatik, Hydrostatik*. Berlin Heidelberg: Springer, 2007.

[3.0] D. Gross, W. Ehlers, and P. Wriggers, *Formeln und Aufgaben zur Technischen Mechanik 3: Kinetik, Hydromechanik*. Berlin Heidelberg: Springer, 2007.

[3.0] H. Goldstein, *Klassische Mechanik*. Wiesbaden: AULA-Verlag, 1991.

[3.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 1: Theoretical Basis*. Berlin: Springer, 2002.

[3.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 2: Applications*. Berlin: Springer, 2002.

[3.0] B. Heimann, W. Gerth, and K. Popp, *Mechatronik: Komponenten - Methoden - Beispiele*. Leipzig: Fachbuchverlag, 2003.

[3.0] R. Isermann, *Mechatronische Systeme: Grundlagen*. Berlin Heidelberg: Springer, 2008.

[3.0] L. Meirovitch, *Dynamics and Control of Structures*. New York: John Wiley & Sons, 1990.

[3.0] A. Shabana, *Dynamics of Multibody Systems*. Cambridge: Cambridge University Press, 1998.

[3.0] W. Schiehlen and P. Eberhard, *Technische Dynamik: Modelle für Regelung und Simulation*. Wiesbaden: Teubner, 2004.

[3.0] M. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: John Wiley & Sons, 1989.

[3.0] P. Wellstead, *Physical System Modelling*. London: Academic Press, 1979.

[3.0] F. Ziegler, *Mechanics of Solids and Fluids*. New York: Springer, 1995.

4 Rigid Body Kinematics

Chapter 2 described the kinematics of point masses, and Chapter 3 extended this to the description of planar motion. This chapter deals with the basics of the kinematics of general rigid body motion, which is used, for example, to describe the motion of robots. In doing so, any rigid body motion can be described as a combination of *translational* and *rotational motion*.

4.1 Rotation

Figure 4.1 shows a rigid body S with a body-fixed coordinate system $(0_1 x_1 y_1 z_1)$ and a spatially fixed coordinate system (inertial system) $(0_0 x_0 y_0 z_0)$. It is assumed that

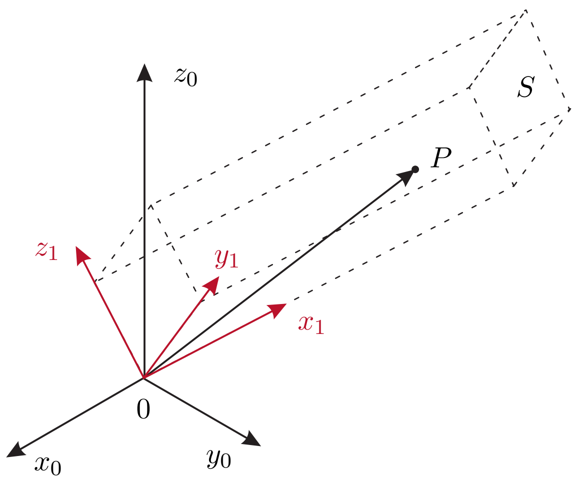


Figure 4.1: On the rotation matrix.

$\{\mathbf{e}_{x_1}, \mathbf{e}_{y_1}, \mathbf{e}_{z_1}\}$ and $\{\mathbf{e}_{x_0}, \mathbf{e}_{y_0}, \mathbf{e}_{z_0}\}$ each represent an orthonormal basis according to (2.1). The vector from the common origin of the coordinate systems to a point P of the rigid body can now be expressed either in the body-fixed coordinate system in the form

$$\mathbf{p}_1 = p_{1x}\mathbf{e}_{x_1} + p_{1y}\mathbf{e}_{y_1} + p_{1z}\mathbf{e}_{z_1} \tag{4.1}$$

or in the spatially fixed coordinate system by [1]

$$\mathbf{p}_0 = p_{0x}\mathbf{e}_{x_0} + p_{0y}\mathbf{e}_{y_0} + p_{0z}\mathbf{e}_{z_0}. \tag{4.2}$$

[1]Note that in this and the following chapter, the variable $\mathbf{p}$ is used to describe the position of points of a rigid body. Wherever necessary, it will be explicitly indicated that this does not refer to momentum.

Since $\mathbf{p}_1$ and $\mathbf{p}_0$ represent the same vector in different coordinate systems, the following relationship holds for their components

$$p_{0x} = \mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{p}_0 = \mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{p}_1 = p_{1x}\mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{e}_{x_1} + p_{1y}\mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{e}_{y_1} + p_{1z}\mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{e}_{z_1}\ . \tag{4.3}$$

If this is done analogously for p_{0y} and p_{0z}, one obtains the relation for a *pure rotation of the coordinate system*

$$\underbrace{\begin{bmatrix} p_{0x} \\ p_{0y} \\ p_{0z} \end{bmatrix}}_{\mathbf{p}_0} = \underbrace{\begin{bmatrix} \mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{e}_{x_1} & \mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{e}_{y_1} & \mathbf{e}_{x_0}^{\mathrm{T}}\mathbf{e}_{z_1} \\ \mathbf{e}_{y_0}^{\mathrm{T}}\mathbf{e}_{x_1} & \mathbf{e}_{y_0}^{\mathrm{T}}\mathbf{e}_{y_1} & \mathbf{e}_{y_0}^{\mathrm{T}}\mathbf{e}_{z_1} \\ \mathbf{e}_{z_0}^{\mathrm{T}}\mathbf{e}_{x_1} & \mathbf{e}_{z_0}^{\mathrm{T}}\mathbf{e}_{y_1} & \mathbf{e}_{z_0}^{\mathrm{T}}\mathbf{e}_{z_1} \end{bmatrix}}_{\mathbf{R}_0^1} \underbrace{\begin{bmatrix} p_{1x} \\ p_{1y} \\ p_{1z} \end{bmatrix}}_{\mathbf{p}_1}\ . \tag{4.4}$$

The (3×3) matrix $\mathbf{R}_0^1$ gives the transformation of the coordinates of a vector in the coordinate system $(0_1x_1y_1z_1)$ (superscript in $\mathbf{R}_0^1$) to the coordinates in the coordinate system $(0_0x_0y_0z_0)$ (subscript in $\mathbf{R}_0^1$). Analogously, the following applies

$$p_{1x} = \mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{p}_1 = \mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{p}_0 = p_{0x}\mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{x_0} + p_{0y}\mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{y_0} + p_{0z}\mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{z_0} \tag{4.5}$$

or

$$\underbrace{\begin{bmatrix} p_{1x} \\ p_{1y} \\ p_{1z} \end{bmatrix}}_{\mathbf{p}_1} = \underbrace{\begin{bmatrix} \mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{x_0} & \mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{y_0} & \mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{z_0} \\ \mathbf{e}_{y_1}^{\mathrm{T}}\mathbf{e}_{x_0} & \mathbf{e}_{y_1}^{\mathrm{T}}\mathbf{e}_{y_0} & \mathbf{e}_{y_1}^{\mathrm{T}}\mathbf{e}_{z_0} \\ \mathbf{e}_{z_1}^{\mathrm{T}}\mathbf{e}_{x_0} & \mathbf{e}_{z_1}^{\mathrm{T}}\mathbf{e}_{y_0} & \mathbf{e}_{z_1}^{\mathrm{T}}\mathbf{e}_{z_0} \end{bmatrix}}_{\mathbf{R}_1^0} \underbrace{\begin{bmatrix} p_{0x} \\ p_{0y} \\ p_{0z} \end{bmatrix}}_{\mathbf{p}_0}\ . \tag{4.6}$$

Now obviously the following must hold

$$\mathbf{p}_0 = \mathbf{R}_0^1\mathbf{p}_1 = \mathbf{R}_0^1\mathbf{R}_1^0\mathbf{p}_0 \quad \text{or} \quad \mathbf{p}_1 = \mathbf{R}_1^0\mathbf{p}_0 = \mathbf{R}_1^0\mathbf{R}_0^1\mathbf{p}_1 \tag{4.7}$$

and because of the commutativity of the dot product $\mathbf{e}_{x_1}^{\mathrm{T}}\mathbf{e}_{y_0} = \mathbf{e}_{y_0}^{\mathrm{T}}\mathbf{e}_{x_1}$ follows the *orthogonality* of the matrix $\mathbf{R}_1^0$, i.e.

$$\mathbf{R}_0^1 = \left(\mathbf{R}_1^0\right)^{-1} = \left(\mathbf{R}_1^0\right)^{\mathrm{T}}. \tag{4.8}$$

If one now assumes a right-handed coordinate system, then additionally $\det\left(\mathbf{R}_0^1\right) = +1$ holds. In this context, all orthogonal (3×3) matrices with determinant $+1$ are called *rotation matrices* of $\mathbb{R}^3$. Frequently, the notation $SO(3)$ is used for *special orthogonal group of order* 3.

Now there are three *elementary rotation matrices*, each describing the rotation about one of the three coordinate axes. Figure 4.2 shows the position of the coordinate systems $(0_1x_1y_1z_1)$ and $(0_0x_0y_0z_0)$ for a rotation about the z_0 axis by the angle ϕ. The corresponding rotation matrix $\mathbf{R}_0^1$ is

$$\mathbf{R}_0^1 = \mathbf{R}_{z,\phi} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}_\phi & -\mathrm{s}_\phi & 0 \\ \mathrm{s}_\phi & \mathrm{c}_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.9}$$

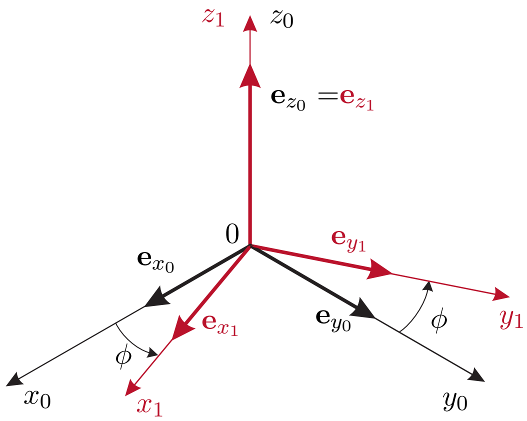


Figure 4.2: Elementary rotation about the z_0 axis with the angle ϕ.

Analogously, for the elementary rotations about the y_0- and x_0-axes with the respective angles θ and ψ, one obtains the rotation matrices

$$\mathbf{R}_0^1 = \mathbf{R}_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \mathrm{c}_\theta & 0 & \mathrm{s}_\theta \\ 0 & 1 & 0 \\ -\mathrm{s}_\theta & 0 & \mathrm{c}_\theta \end{bmatrix} \tag{4.10a}$$

$$\mathbf{R}_0^1 = \mathbf{R}_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) \\ 0 & \sin(\psi) & \cos(\psi) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_\psi & -\mathrm{s}_\psi \\ 0 & \mathrm{s}_\psi & \mathrm{c}_\psi \end{bmatrix} . \tag{4.10b}$$

Exercise 4.1 (Elementary Rotation Matrices). Calculate the elementary rotation matrices using (4.4).

Solution of exercise 4.1.

Solution in MAPLE: `Aufgabe_4_1.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Now consider three coordinate systems $(0_0 x_0 y_0 z_0)$, $(0_1 x_1 y_1 z_1)$ and $(0_2 x_2 y_2 z_2)$, which are connected by rotation. The vector $\mathbf{p}$ from the common origin of the coordinate systems to a point P can be represented in the coordinates of the respective coordinate systems (denoted by $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$). The following relationships apply

$$\mathbf{p}_0 = \mathbf{R}_0^1 \mathbf{p}_1 \quad \text{and} \quad \mathbf{p}_1 = \mathbf{R}_1^2 \mathbf{p}_2 \tag{4.11}$$

and for the concatenation of two rotations one obtains

$$\mathbf{p}_0 = \mathbf{R}_0^1\mathbf{R}_1^2\mathbf{p}_2 = \mathbf{R}_0^2\mathbf{p}_2 \quad \text{or} \quad \mathbf{R}_0^2 = \mathbf{R}_0^1\mathbf{R}_1^2 \ . \tag{4.12}$$

Furthermore, it is easy to see that the following relations also hold for the inverse

$$\mathbf{p}_2 = \mathbf{R}_2^0\mathbf{p}_0 \quad \text{with} \quad \mathbf{R}_2^0 = \left(\mathbf{R}_0^2\right)^{\mathrm{T}} = \left(\mathbf{R}_0^1\mathbf{R}_1^2\right)^{\mathrm{T}} = \left(\mathbf{R}_1^2\right)^{\mathrm{T}}\left(\mathbf{R}_0^1\right)^{\mathrm{T}} = \mathbf{R}_2^1\mathbf{R}_1^0. \tag{4.13}$$

Note that the concatenation of rotations is not a commutative operation, i.e. in general $\mathbf{R}_A\mathbf{R}_B \neq \mathbf{R}_B\mathbf{R}_A$ holds for two rotation matrices $\mathbf{R}_A$ and $\mathbf{R}_B$. As an example, consider the concatenation of two elementary rotations, first by the angle ϕ about the instantaneous z-axis and then by the angle θ about the (rotated) instantaneous y-axis. The rotation matrix in this case is

$$\mathbf{R}_{zy} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta} = \begin{bmatrix} \mathrm{c}_\phi & -\mathrm{s}_\phi & 0 \\ \mathrm{s}_\phi & \mathrm{c}_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathrm{c}_\theta & 0 & \mathrm{s}_\theta \\ 0 & 1 & 0 \\ -\mathrm{s}_\theta & 0 & \mathrm{c}_\theta \end{bmatrix} = \begin{bmatrix} \mathrm{c}_\theta\mathrm{c}_\phi & -\mathrm{s}_\phi & \mathrm{c}_\phi\mathrm{s}_\theta \\ \mathrm{s}_\phi\mathrm{c}_\theta & \mathrm{c}_\phi & \mathrm{s}_\phi\mathrm{s}_\theta \\ -\mathrm{s}_\theta & 0 & \mathrm{c}_\theta \end{bmatrix} . \tag{4.14}$$

If one changes the order, i.e., first a rotation by the angle θ about the instantaneous y-axis and then a rotation by the angle ϕ about the (rotated) instantaneous z-axis, one obtains

$$\mathbf{R}_{yz} = \mathbf{R}_{y,\theta}\mathbf{R}_{z,\phi} = \begin{bmatrix} \mathrm{c}_\theta & 0 & \mathrm{s}_\theta \\ 0 & 1 & 0 \\ -\mathrm{s}_\theta & 0 & \mathrm{c}_\theta \end{bmatrix}\begin{bmatrix} \mathrm{c}_\phi & -\mathrm{s}_\phi & 0 \\ \mathrm{s}_\phi & \mathrm{c}_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}_\theta\mathrm{c}_\phi & -\mathrm{c}_\theta\mathrm{s}_\phi & \mathrm{s}_\theta \\ \mathrm{s}_\phi & \mathrm{c}_\phi & 0 \\ -\mathrm{s}_\theta\mathrm{c}_\phi & \mathrm{s}_\theta\mathrm{s}_\phi & \mathrm{c}_\theta \end{bmatrix} . \tag{4.15}$$

It can be seen directly from (4.14) and (4.15) that $\mathbf{R}_{yz} \neq \mathbf{R}_{zy}$ holds, see Figure 4.3.

The concatenation of two rotations according to (4.14) or (4.15) implies that the second rotation is always performed with respect to the *already rotated* coordinate system. In comparison, assume now that the coordinate system $(0_0x_0y_0z_0)$ is rotated by the angle ϕ about the z_0-axis and the resulting rotated coordinate system $(0_1x_1y_1z_1)$ is rotated by the angle θ about the y_0-axis (in contrast to (4.14), where the rotation was about the y_1-axis), resulting in the coordinate system $(0_2x_2y_2z_2)$. If one denotes by $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$ one and the same vector in the different coordinate systems, then the following relations hold

$$\mathbf{p}_0 = \mathbf{R}_{z,\phi}\mathbf{p}_1 \quad \text{and} \quad \mathbf{p}_1 = \mathbf{R}_{z,-\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{z,\phi}\mathbf{p}_2 \neq \mathbf{R}_{y,\theta}\mathbf{p}_2 \tag{4.16}$$

or

$$\mathbf{p}_0 = \mathbf{R}_{y,\theta}\mathbf{R}_{z,\phi}\mathbf{p}_2 \ . \tag{4.17}$$

The expression $\mathbf{R}_{z,-\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{z,\phi}$ in (4.16) shows that first the rotation by the angle $-\phi$ is performed about the z_1-axis (which is identical to the z_0-axis) (this brings us back to the original $(0_0x_0y_0z_0)$ system), then the rotation by the angle θ about the y_0-axis, and finally the rotation back by the angle ϕ.

4.2 Parameterization of a Rotation

The nine entries of a (3×3) rotation matrix $\mathbf{R}$ are of course not linearly independent. Rather, the orientation of a rigid body can be determined by three rotational degrees of freedom, which is why the rotation matrix is generally characterized by only three linearly independent quantities. In the following, two commonly used parameterizations of a rotation are given.

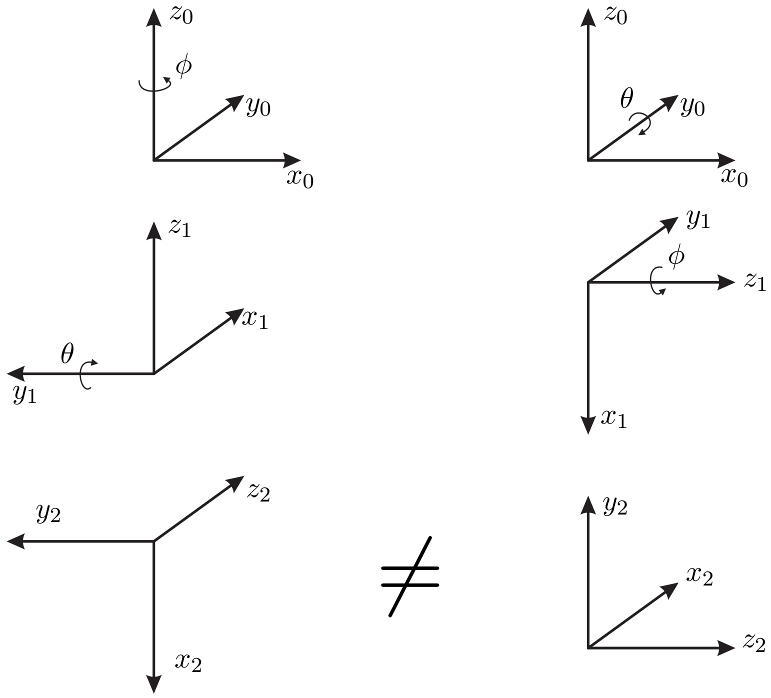


Figure 4.3: On the concatenation of rotations: $\mathbf{R}_{yz} \neq \mathbf{R}_{zy}$.

4.2.1 Euler Angles

To this end, consider the body-fixed coordinate system $(0_1 x_1 y_1 z_1)$ rotated with respect to a spatially fixed coordinate system (inertial system) $(0_0 x_0 y_0 z_0)$. In the *Euler angle parameterization*, the orientation of the coordinate system $(0_1 x_1 y_1 z_1)$ with respect to $(0_0 x_0 y_0 z_0)$ is represented by three successive rotations with the angles (ϕ, θ, ψ): First, a rotation about the z_0-axis by the angle ϕ, then a rotation about the (rotated) instantaneous y-axis by the angle θ, and finally a rotation about the (rotated) instantaneous z-axis by the angle ψ. Thus, the rotation matrix is

$$
\begin{aligned}
\mathbf{R}_0^1 = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{z,\psi} &= \begin{bmatrix} \mathrm{c}_\phi & -\mathrm{s}_\phi & 0 \\ \mathrm{s}_\phi & \mathrm{c}_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}_\theta & 0 & \mathrm{s}_\theta \\ 0 & 1 & 0 \\ -\mathrm{s}_\theta & 0 & \mathrm{c}_\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}_\psi & -\mathrm{s}_\psi & 0 \\ \mathrm{s}_\psi & \mathrm{c}_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{c}_\phi \mathrm{c}_\theta \mathrm{c}_\psi - \mathrm{s}_\phi \mathrm{s}_\psi & -\mathrm{c}_\phi \mathrm{c}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \\ \mathrm{s}_\phi \mathrm{c}_\theta \mathrm{c}_\psi + \mathrm{c}_\phi s_\psi & -\mathrm{s}_\phi \mathrm{c}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \\ -\mathrm{s}_\theta \mathrm{c}_\psi & \mathrm{s}_\theta \mathrm{s}_\psi & \mathrm{c}_\theta \end{bmatrix} .
\end{aligned}
\tag{4.18}
$$

4.2.2 Roll-Pitch-Yaw Angles

A parameterization of the rotation matrix in terms of rotation angles (ϕ, θ, ψ) about the coordinate axes of the spatially fixed coordinate system $(0_0 x_0 y_0 z_0)$ such that a rotation by the angle ψ is performed first about the x_0-axis, then a rotation by the angle θ about the y_0-axis, and finally a rotation by the angle ϕ about the z_0-axis (see Figure 4.4), results in the rotation matrix

$$
\begin{aligned}
\mathbf{R}_0^1 = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\psi} &= \begin{bmatrix} \mathrm{c}_\phi & -\mathrm{s}_\phi & 0 \\ \mathrm{s}_\phi & \mathrm{c}_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}_\theta & 0 & \mathrm{s}_\theta \\ 0 & 1 & 0 \\ -\mathrm{s}_\theta & 0 & \mathrm{c}_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_\psi & -\mathrm{s}_\psi \\ 0 & \mathrm{s}_\psi & \mathrm{c}_\psi \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{c}_\phi\mathrm{c}_\theta & -\mathrm{s}_\phi\mathrm{c}_\psi + \mathrm{c}_\phi\mathrm{s}_\theta\mathrm{s}_\psi & \mathrm{s}_\phi\mathrm{s}_\psi + \mathrm{c}_\phi\mathrm{s}_\theta\mathrm{c}_\psi \\ \mathrm{s}_\phi\mathrm{c}_\theta & \mathrm{c}_\phi\mathrm{c}_\psi + \mathrm{s}_\phi\mathrm{s}_\theta\mathrm{s}_\psi & -\mathrm{c}_\phi\mathrm{s}_\psi + \mathrm{s}_\phi\mathrm{s}_\theta\mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{c}_\theta\mathrm{s}_\psi & \mathrm{c}_\theta\mathrm{c}_\psi \end{bmatrix} .
\end{aligned}
\tag{4.19}
$$

Here, ψ is called the *roll angle*, θ the *pitch angle*, and ϕ the *yaw angle*.

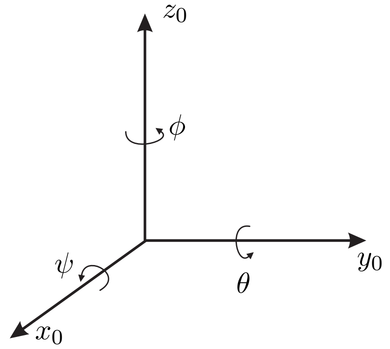


Figure 4.4: Parameterization of the rotation matrix using roll-pitch-yaw angles.

4.3 Translation

In addition to the pure rotation of a coordinate system discussed so far, the next step is to address the *pure translation of a coordinate system*. Consider the two coordinate systems $(0_0x_0y_0z_0)$ and $(0_1x_1y_1z_1)$ of Figure 4.5, which are not rotated with respect to each other, whose coordinate origins 0_0 and 0_1 are connected by the vector $\mathbf{d}_0^1$.

> *Note* 4.1 (Notation). The vector $\mathbf{d}_0^1$ describes the translational displacement of the coordinate system $(0_1x_1y_1z_1)$ with respect to the coordinate system $(0_0x_0y_0z_0)$ expressed in the coordinate system $(0_0x_0y_0z_0)$. For all further considerations, it holds that the quantities (displacement vector, vector of angular velocities, etc.) are always represented with respect to the coordinate system indicated by the lower right index.

Furthermore, let $\mathbf{p}_0$ and $\mathbf{p}_1$ denote the vectors from the origins 0_0 and 0_1 of the coordinate systems $(0_0x_0y_0z_0)$ and $(0_1x_1y_1z_1)$ to a point P. For a pure translational displacement $\mathbf{d}_0^1$ of the two coordinate systems, the following relation holds

$$
\mathbf{p}_0 = \mathbf{p}_1 + \mathbf{d}_0^1 . \tag{4.20}
$$

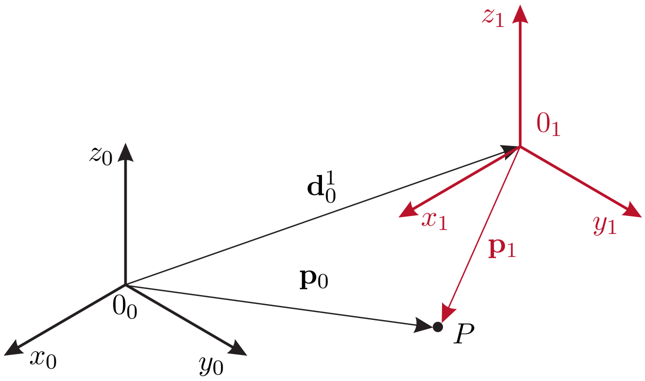


Figure 4.5: On the translational displacement.

4.4 Combined Rotation and Translation

As mentioned in the introduction, a rigid body motion generally consists of rotational and translational movements. Combining (4.7) with (4.20), one obtains the following relation for a combined translational and rotational motion

$$\mathbf{p}_0 = \mathbf{R}_0^1 \mathbf{p}_1 + \mathbf{d}_0^1 \ . \tag{4.21}$$

It can be shown directly that for the coordinate systems $(0_0 x_0 y_0 z_0)$, $(0_1 x_1 y_1 z_1)$ and $(0_2 x_2 y_2 z_2)$ shown in Figure 4.6, the relations

$$\mathbf{p}_0 = \mathbf{R}_0^1 \mathbf{p}_1 + \mathbf{d}_0^1 \quad \text{and} \quad \mathbf{p}_1 = \mathbf{R}_1^2 \mathbf{p}_2 + \mathbf{d}_1^2 \tag{4.22}$$

or

$$\mathbf{p}_0 = \mathbf{R}_0^1 \left(\mathbf{R}_1^2 \mathbf{p}_2 + \mathbf{d}_1^2 \right) + \mathbf{d}_0^1 = \mathbf{R}_0^2 \mathbf{p}_2 + \mathbf{R}_0^1 \mathbf{d}_1^2 + \mathbf{d}_0^1 \tag{4.23}$$

apply.

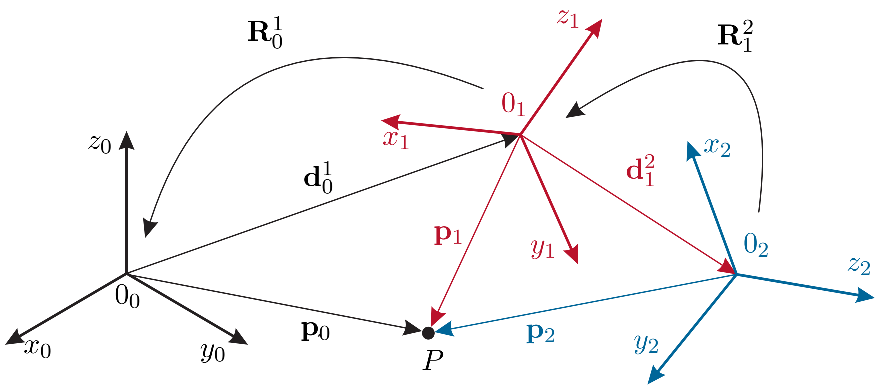


Figure 4.6: On the combined translational and rotational motion.

If one wants to describe the motion of a rigid body with respect to the inertial system $(0_0 x_0 y_0 z_0)$, one attaches a coordinate system $(0_1 x_1 y_1 z_1)$ rigidly to the rigid body (*body-fixed coordinate system*). Thus, the motion of the rigid body is described equivalently by the transformation (4.21) of the coordinate system. Therefore, a transformation of the form (4.21) with an orthogonal matrix $\mathbf{R}_0^1$ is also called a *rigid body motion*. The location of the body-fixed coordinate system is arbitrary. However, it is often useful to place the body-fixed coordinate system at the center of rotation or the center of gravity of the rigid body.

Rigid body motions can be efficiently represented in terms of *homogeneous transformations* of the form

$$\mathbf{H}_0^1 = \begin{bmatrix} \mathbf{R}_0^1 & \mathbf{d}_0^1 \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{with} \quad \mathbf{R}_0^1 \in SO(3). \tag{4.24}$$

Consider a configuration $\mathbf{P}_0$ of a rigid body

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix}, \tag{4.25}$$

then by rotation by $\mathbf{R}_0^1$ and translation by $\mathbf{d}_0^1$ one obtains the configuration

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{p}_1 \\ 1 \end{bmatrix} \tag{4.26}$$

in the form (compare with (4.21))

$$\mathbf{P}_0 = \mathbf{H}_0^1 \mathbf{P}_1 \ . \tag{4.27}$$

It is easy to see that the configuration of the rigid body resulting from another rigid body motion (rotation by $\mathbf{R}_1^2$ and translation by $\mathbf{d}_1^2$)

$$\mathbf{P}_2 = \begin{bmatrix} \mathbf{p}_2 \\ 1 \end{bmatrix} \tag{4.28}$$

satisfies the following relation

$$\mathbf{P}_1 = \mathbf{H}_1^2 \mathbf{P}_2 = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{d}_1^2 \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{P}_2. \tag{4.29}$$

To prove this, one combines (4.27) with (4.29). One obtains with

$$\underbrace{\begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix}}_{\mathbf{P}_0} = \underbrace{\begin{bmatrix} \mathbf{R}_0^1 & \mathbf{d}_0^1 \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{H}_0^1} \underbrace{\begin{bmatrix} \mathbf{R}_1^2 & \mathbf{d}_1^2 \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{H}_1^2} \underbrace{\begin{bmatrix} \mathbf{p}_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} \mathbf{R}_0^1 \mathbf{R}_1^2 & \mathbf{R}_0^1 \mathbf{d}_1^2 + \mathbf{d}_0^1 \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{H}_0^2} \underbrace{\begin{bmatrix} \mathbf{p}_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} \tag{4.30}$$

directly the result of (4.23). Furthermore, for the inverse homogeneous transformation we have

$$\mathbf{H}_1^0 = \left(\mathbf{H}_0^1\right)^{-1} = \begin{bmatrix} \left(\mathbf{R}_0^1\right)^{\mathrm{T}} & -\left(\mathbf{R}_0^1\right)^{\mathrm{T}} \mathbf{d}_0^1 \\ \mathbf{0} & 1 \end{bmatrix}. \tag{4.31}$$

The description of a rigid body motion by means of homogeneous transformations proves to be particularly advantageous for kinematic chains, as they occur frequently in robots, for example.

Example 4.1 (Planar Manipulator). Figure 4.7 shows a simple planar manipulator consisting of two links. Link 1 (length l_1, distance to the center of gravity l_{s1}) is rotatably mounted at one end about the z-axis (angle φ_1). At the other end, link 2 with length l_2 and distance l_{s2} to the center of gravity is rotatably mounted about the z-axis. The angle φ_2 denotes the relative rotation of link 2 with respect to link 1.

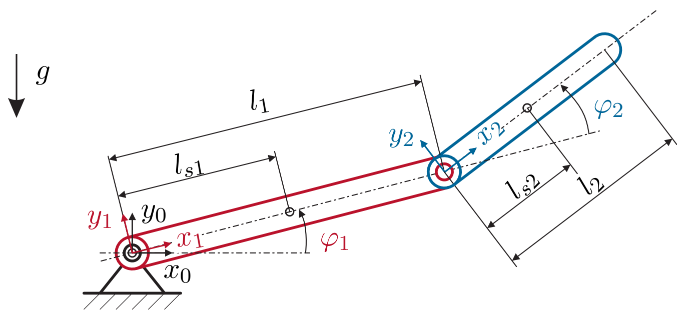


Figure 4.7: Planar manipulator with 2 degrees of freedom.

To calculate the position of the centers of gravity as well as the end-effector attached to the end of link 2, the rotation matrices are calculated to describe the rotations of the coordinate systems $(0_1x_1y_1z_1)$ and $(0_2x_2y_2z_2)$ with respect to the inertial coordinate system $(0_0x_0y_0z_0)$. These follow directly from the elementary rotation matrices according to (4.9) to

$$\mathbf{R}_0^1 = \begin{bmatrix} c_{\varphi_1} & -s_{\varphi_1} & 0 \\ s_{\varphi_1} & c_{\varphi_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{R}_1^2 = \begin{bmatrix} c_{\varphi_2} & -s_{\varphi_2} & 0 \\ s_{\varphi_2} & c_{\varphi_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.32}$$

The vectors of displacement of the coordinate systems with respect to each other are calculated as

$$\mathbf{d}_0^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{d}_1^2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \tag{4.33}$$

and the positions of the centers of gravity as well as the end-effector in the respective body-fixed coordinate systems are given by

$$\mathbf{p}_1^{s1} = \begin{bmatrix} l_{s1} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2^{s2} = \begin{bmatrix} l_{s2} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2^{e2} = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}. \tag{4.34}$$

The absolute position of these points relative to the chosen inertial coordinate system $(0_0x_0y_0z_0)$ (and expressed in this inertial system) is calculated according to

(4.21) to

$$\mathbf{p}_0^{s1} = \mathbf{R}_0^1 \mathbf{p}_1^{s1} + \mathbf{d}_0^1 = \begin{bmatrix} l_{s1}\cos(\varphi_1) \\ l_{s1}\sin(\varphi_1) \\ 0 \end{bmatrix} \tag{4.35a}$$

$$\mathbf{p}_0^{s2} = \mathbf{R}_0^1 \left(\mathbf{d}_1^2 + \mathbf{R}_1^2 \mathbf{p}_2^{s2} \right) + \mathbf{d}_0^1 = \begin{bmatrix} l_1\cos(\varphi_1) + l_{s2}\cos(\varphi_1+\varphi_2) \\ l_1\sin(\varphi_1) + l_{s2}\sin(\varphi_1+\varphi_2) \\ 0 \end{bmatrix} \tag{4.35b}$$

$$\mathbf{p}_0^{e2} = \mathbf{R}_0^1 \left(\mathbf{d}_1^2 + \mathbf{R}_1^2 \mathbf{p}_2^{e2} \right) + \mathbf{d}_0^1 = \begin{bmatrix} l_1\cos(\varphi_1) + l_{e2}\cos(\varphi_1+\varphi_2) \\ l_1\sin(\varphi_1) + l_{e2}\sin(\varphi_1+\varphi_2) \\ 0 \end{bmatrix} . \tag{4.35c}$$

The MAPLE file `Planar_Manipulator.mw` on https://www.acin.tuwien.ac.at/bachelor/modellbildung/ shows the solution of this example using homogeneous transformations.

Note 4.2. To simplify the representation of rigid body kinematics, a symbolic description is often used in the literature. Figure 4.8 shows a common representation of translational and rotational joints of a rigid body kinematic.

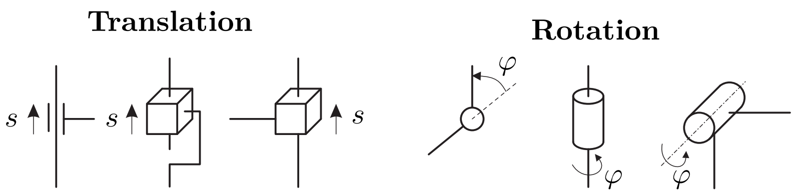


Figure 4.8: Symbolic representation of translational and rotational joints.

Example 4.2 (Tower Crane). The planar manipulator represents a mechanical rigid body system whose motion is described solely by rotations about the axes of rotation. In the case of the tower crane sketched in Figure 4.9, the motion of the load results from a combination of rotations (angles φ_1, φ_3) with a translation (displacement s_2).

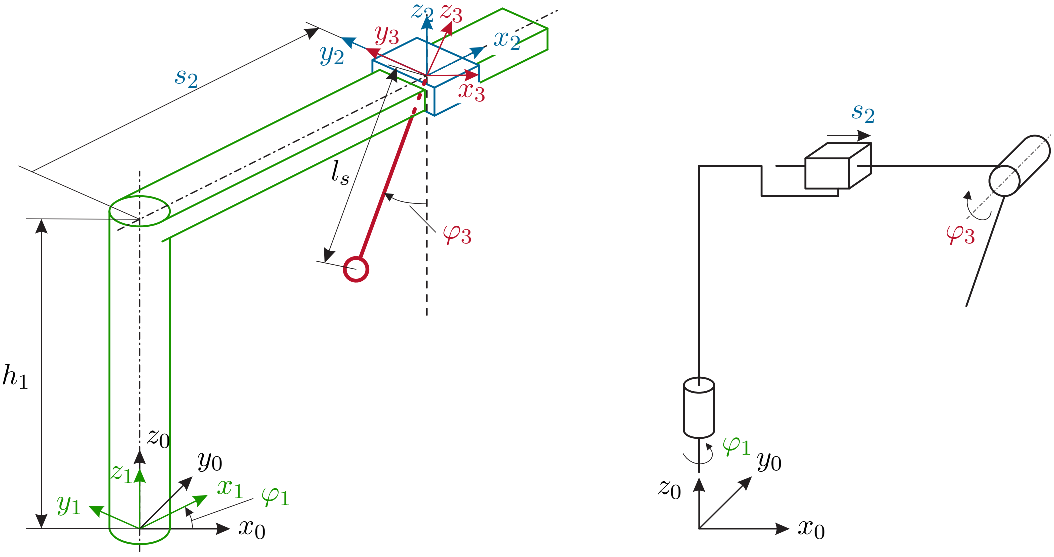


Figure 4.9: Sketch and kinematics of a tower crane for Example 4.2.

The system has 3 degrees of freedom: a rotation of the tower about the z-axis of the inertial coordinate system $(0_0 x_0 y_0 z_0)$ by the angle φ_1, a translation of the trolley along the x-axis of the coordinate system $(0_1 x_1 y_1 z_1)$ by s_2, and a rotation of the cable about the y-axis of the coordinate system $(0_2 x_2 y_2 z_2)$ by the angle φ_3. For simplicity, it has been assumed that the cable length l_s is constant and that only a rotation of the cable with the load about the y-axis of the coordinate system $(0_2 x_2 y_2 z_2)$ occurs.

The rotation matrices describing the rotations of the coordinate systems in this example are

$$\mathbf{R}_0^1 = \begin{bmatrix} \mathrm{c}_{\varphi_1} & -\mathrm{s}_{\varphi_1} & 0 \\ \mathrm{s}_{\varphi_1} & \mathrm{c}_{\varphi_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}_1^2 = \mathbf{E}, \quad \mathbf{R}_2^3 = \begin{bmatrix} \mathrm{c}_{\varphi_3} & 0 & \mathrm{s}_{\varphi_3} \\ 0 & 1 & 0 \\ -\mathrm{s}_{\varphi_3} & 0 & \mathrm{c}_{\varphi_3} \end{bmatrix}, \tag{4.36}$$

with the identity matrix $\mathbf{E}$. The displacements of the coordinate systems relative to each other are described by

$$\mathbf{d}_0^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{d}_1^2 = \begin{bmatrix} s_2 \\ 0 \\ h_1 \end{bmatrix}, \quad \mathbf{d}_2^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.37}$$

and the vector $\mathbf{p}_3^L$ to the load is calculated in the coordinate system $(0_3x_3y_3z_3)$ to

$$\mathbf{p}_3^L = \begin{bmatrix} 0 \\ 0 \\ -l_s \end{bmatrix} . \tag{4.38}$$

This immediately allows to calculate the position of the load in the inertial coordinate system by applying

$$\mathbf{p}_0^L = \mathbf{d}_0^1 + \mathbf{R}_0^1\Big(\mathbf{d}_1^2 + \mathbf{R}_1^2\Big(\mathbf{d}_2^3 + \mathbf{R}_2^3\mathbf{p}_3^L\Big)\Big) \tag{4.39}$$

to

$$\mathbf{p}_0^L = \begin{bmatrix} \mathrm{c}_{\varphi_1}(s_2 - \mathrm{s}_{\varphi_3}l_s) \\ \mathrm{s}_{\varphi_1}(s_2 - \mathrm{s}_{\varphi_3}l_s) \\ h_1 - \mathrm{c}_{\varphi_3}l_s \end{bmatrix} . \tag{4.40}$$

Note that the position s_2, as well as the angles φ_1 and φ_3 are time-dependent variables.

Solution in MAPLE: `Turmdrehkran_einfach.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise 4.2 (Tower Crane 2). Extend the kinematic model of the tower crane to include the change of the cable length as well as an additional rotation of the cable about the x_3-axis, see Figure 4.10. Calculate the position $\mathbf{p}_0^L$ of the load as a function of the degrees of freedom of the system.

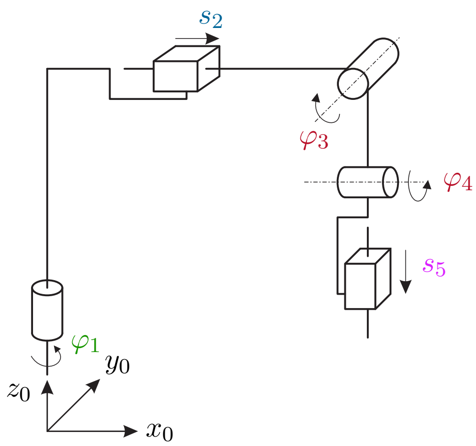


Figure 4.10: Schematic representation of the kinematics of the tower crane with 5 degrees of freedom.

Solution of exercise 4.2. The position of the load is calculated as

$$\mathbf{p}_0^L = \begin{bmatrix} s_2 \mathrm{c}_{\varphi_1} + s_5(-\mathrm{s}_{\varphi_1}\mathrm{s}_{\varphi_4} - \mathrm{c}_{\varphi_1}\mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4}) \\ s_2 \mathrm{s}_{\varphi_1} + s_5(\mathrm{c}_{\varphi_1}\mathrm{s}_{\varphi_4} - \mathrm{s}_{\varphi_1}\mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4}) \\ h_1 - s_5 \mathrm{c}_{\varphi_3}\mathrm{c}_{\varphi_4} \end{bmatrix} . \tag{4.41}$$

Solution in MAPLE: `Turmdrehkran.mw`
`https://www.acin.tuwien.ac.at/bachelor/modellbildung/`

4.5 Angular Velocity

The elements of the rotation matrices are functions of the rotation angles, which in turn are generally functions of time, see for example (4.9) and (4.10). If one calculates the total time derivative of a rotation matrix $\mathbf{R} \in SO(3)$ and assumes for simplicity that the rotation matrix depends only on one rotation angle $\theta(t)$, then it follows

$$\dot{\mathbf{R}}(\theta) = \frac{\partial}{\partial \theta}\mathbf{R}\dot{\theta} \, . \tag{4.42}$$

Due to the orthogonality of the rotation matrix $\mathbf{R}$, the following relations hold

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{E} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}\mathbf{R}^{\mathrm{T}}\right) = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} + \mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} = \mathbf{0} \, . \tag{4.43}$$

This shows that the matrix

$$\mathbf{S} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} \tag{4.44}$$

is a skew-symmetric (3×3) matrix and thus there always exists a unique *vector of angular velocities* $\boldsymbol{\omega}^{\mathrm{T}} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}$ such that $\mathbf{S}$ can be written in the form

$$\mathbf{S}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} . \tag{4.45}$$

For the elementary rotation matrices (4.9) and (4.10), one obtains for example

$$\mathbf{S}_{z,\phi} = \dot{\mathbf{R}}_{z,\phi}\mathbf{R}_{z,\phi}^{\mathrm{T}} = \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.46}$$

or

$$\mathbf{S}_{y,\theta} = \dot{\mathbf{R}}_{y,\theta}\mathbf{R}_{y,\theta}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & \dot{\theta} \\ 0 & 0 & 0 \\ -\dot{\theta} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_{x,\psi} = \dot{\mathbf{R}}_{x,\psi}\mathbf{R}_{x,\psi}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\psi} \\ 0 & \dot{\psi} & 0 \end{bmatrix} . \tag{4.47}$$

Combining (4.43) and (4.44), then

$$\dot{\mathbf{R}}(\theta) = \dot{\mathbf{R}}\underbrace{\mathbf{R}^{\mathrm{T}}\mathbf{R}}_{\mathbf{E}} = \mathbf{S}\mathbf{R} \ . \tag{4.48}$$

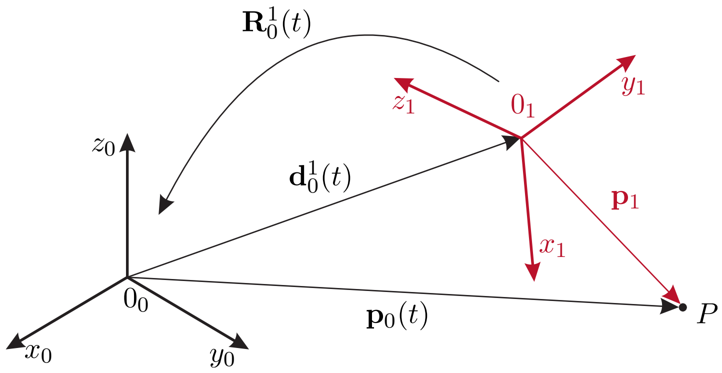


Figure 4.11: On the angular velocity.

Assume now that $\mathbf{p}_1$ represents a fixed (time-invariant) vector from the origin of the coordinate system $(0_1x_1y_1z_1)$ to a point P, see Figure 4.11. The coordinate system $(0_1x_1y_1z_1)$ performs translational and rotational motions with respect to a spatially fixed coordinate system (inertial system) $(0_0x_0y_0z_0)$. The velocity $\dot{\mathbf{p}}_0$ of the point P measured in the inertial system is then given by the relation (compare (4.21))

$$\dot{\mathbf{p}}_0 = \dot{\mathbf{R}}_0^1\mathbf{p}_1 + \mathbf{R}_0^1\underbrace{\dot{\mathbf{p}}_1}_{=\mathbf{0}} + \dot{\mathbf{d}}_0^1 = \mathbf{S}\left(\boldsymbol{\omega}_0^1\right)\mathbf{R}_0^1\mathbf{p}_1 + \dot{\mathbf{d}}_0^1 \tag{4.49}$$

or

$$\dot{\mathbf{p}}_0 = \boldsymbol{\omega}_0^1 \times \left(\mathbf{R}_0^1\mathbf{p}_1\right) + \dot{\mathbf{d}}_0^1. \tag{4.50}$$

To show this, consider a general angular velocity vector $\boldsymbol{\omega}^{\mathrm{T}} = \begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}$ and a vector $\mathbf{r}^{\mathrm{T}} = \begin{bmatrix}r_x & r_y & r_z\end{bmatrix}$. Then it holds

$$\mathbf{S}(\boldsymbol{\omega})\mathbf{r} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \boldsymbol{\omega} \times \mathbf{r} \ . \tag{4.51}$$

The next step is to calculate how the angular velocities are calculated for multiple coordinate systems that are rotated relative to each other. To this end, consider the rotation matrix $\mathbf{R}_0^2$ according to the relation (4.12) with $\mathbf{R}_0^2 = \mathbf{R}_0^1\mathbf{R}_1^2$ and differentiate it with respect to time. On the one hand, according to (4.48), one obtains the relation

$$\dot{\mathbf{R}}_0^2 = \mathbf{S}\left(\boldsymbol{\omega}_0^2\right)\mathbf{R}_0^2 \tag{4.52}$$

and on the other hand, applying the product rule yields

$$\dot{\mathbf{R}}_0^2 = \dot{\mathbf{R}}_0^1\mathbf{R}_1^2 + \mathbf{R}_0^1\dot{\mathbf{R}}_1^2 = \mathbf{S}\left(\boldsymbol{\omega}_0^1\right)\mathbf{R}_0^1\mathbf{R}_1^2 + \mathbf{R}_0^1\mathbf{S}\left(\boldsymbol{\omega}_1^2\right)\underbrace{\left(\mathbf{R}_0^1\right)^{\mathrm{T}}\mathbf{R}_0^1}_{\mathbf{E}}\mathbf{R}_1^2 \,. \tag{4.53}$$

To further simplify (4.53), one uses the following relation, which holds for any 3-dimensional vector $\mathbf{k}$

$$\begin{aligned}\mathbf{R}_0^1\mathbf{S}\left(\boldsymbol{\omega}_1^2\right)\left(\mathbf{R}_0^1\right)^{\mathrm{T}}\mathbf{k} &= \mathbf{R}_0^1\left(\boldsymbol{\omega}_1^2 \times \left(\mathbf{R}_0^1\right)^{\mathrm{T}}\mathbf{k}\right) = \left(\mathbf{R}_0^1\boldsymbol{\omega}_1^2\right) \times \mathbf{R}_0^1\left(\mathbf{R}_0^1\right)^{\mathrm{T}}\mathbf{k}\\ &= \left(\mathbf{R}_0^1\boldsymbol{\omega}_1^2\right) \times \mathbf{k} = \mathbf{S}\left(\mathbf{R}_0^1\boldsymbol{\omega}_1^2\right)\mathbf{k}\end{aligned} \tag{4.54}$$

as well as the addition property of two skew-symmetric matrices

$$\underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{\mathbf{S}(\mathbf{a})} + \underbrace{\begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}}_{\mathbf{S}(\mathbf{b})} = \underbrace{\begin{bmatrix} 0 & -a_z - b_z & a_y + b_y \\ a_z + b_z & 0 & -a_x - b_x \\ -a_y - b_y & a_x + b_x & 0 \end{bmatrix}}_{\mathbf{S}(\mathbf{a}+\mathbf{b})} \,. \tag{4.55}$$

With (4.54), (4.55) and $\mathbf{R}_0^2 = \mathbf{R}_0^1\mathbf{R}_1^2$, (4.53) reduces to

$$\dot{\mathbf{R}}_0^2 = \left(\mathbf{S}\left(\boldsymbol{\omega}_0^1\right) + \mathbf{S}\left(\mathbf{R}_0^1\boldsymbol{\omega}_1^2\right)\right)\mathbf{R}_0^2 = \mathbf{S}\left(\boldsymbol{\omega}_0^1 + \mathbf{R}_0^1\boldsymbol{\omega}_1^2\right)\mathbf{R}_0^2 \tag{4.56}$$

and comparing (4.52) with (4.56) yields the following relationship for the vector of angular velocities

$$\boldsymbol{\omega}_0^2 = \boldsymbol{\omega}_0^1 + \mathbf{R}_0^1\boldsymbol{\omega}_1^2 \,. \tag{4.57}$$

As can be seen in (4.57), it only makes sense to add the vectors of angular velocities if these vectors are expressed with respect to the same coordinate system. The expression $\mathbf{R}_0^1\boldsymbol{\omega}_1^2$ transforms the vector of angular velocity $\boldsymbol{\omega}_1^2$ into the coordinate system $(0_0x_0y_0z_0)$ and can only then be added to $\boldsymbol{\omega}_0^1$. The relations just derived can be consistently extended to the general case in the form

$$\dot{\mathbf{R}}_0^n = \mathbf{S}(\boldsymbol{\omega}_0^n)\mathbf{R}_0^n \quad \text{with} \quad \mathbf{R}_0^n = \mathbf{R}_0^1\mathbf{R}_1^2\ldots\mathbf{R}_{n-1}^n \tag{4.58}$$

and

$$\boldsymbol{\omega}_0^n = \boldsymbol{\omega}_0^1 + \mathbf{R}_0^1\boldsymbol{\omega}_1^2 + \mathbf{R}_0^2\boldsymbol{\omega}_2^3 + \ldots + \mathbf{R}_0^{n-1}\boldsymbol{\omega}_{n-1}^n. \tag{4.59}$$

4.6 Manipulator Jacobian Matrix

Assume that the homogeneous transformation

$$\mathbf{H}_k^l(\mathbf{q}) = \begin{bmatrix} \mathbf{R}_k^l(\mathbf{q}) & \mathbf{d}_k^l(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{with} \quad \mathbf{R}_k^l(\mathbf{q}) \in SO(3) \tag{4.60}$$

describes the motion of a coordinate system $(0_l x_l y_l z_l)$ relative to the coordinate system $(0_k x_k y_k z_k)$ and can be parameterized by the independent variables (generalized coordinates $\mathbf{q}$ in the form of angles and positions) q_j, $j = 1, \ldots, n$. The vector of angular velocities $\boldsymbol{\omega}_k^l$ associated with $\mathbf{H}_k^l$ can be obtained using (4.44), (4.45) from the following relation

$$\mathbf{S}\left(\boldsymbol{\omega}_k^l\right) = \dot{\mathbf{R}}_k^l(\mathbf{q})\mathbf{R}_k^l(\mathbf{q})^{\mathrm{T}} = \sum_{j=1}^{n}\left(\frac{\partial}{\partial q_j}\mathbf{R}_k^l(\mathbf{q})\right)\mathbf{R}_k^l(\mathbf{q})^{\mathrm{T}}\dot{q}_j \tag{4.61}$$

in the form

$$\boldsymbol{\omega}_k^l = (\mathbf{J}_{\boldsymbol{\omega}})_k^l(\mathbf{q})\dot{\mathbf{q}}. \tag{4.62}$$

Exercise 4.3. Calculate the matrix $(\mathbf{J}_{\boldsymbol{\omega}})_0^3(\mathbf{q})$ for the case that the rotation of the coordinate system $(0_3 x_3 y_3 z_3)$ with respect to the coordinate system $(0_0 x_0 y_0 z_0)$ is given by the concatenation of rotations by the angles ϕ, θ and ψ according to the definition of the Euler angles in Section 4.2.1.

Solution of exercise 4.3. The Jacobian matrix is calculated as

$$(\mathbf{J}_{\boldsymbol{\omega}})_0^3(\mathbf{q}) = \begin{bmatrix} 0 & -\mathrm{s}_\phi & \mathrm{c}_\phi \mathrm{s}_\theta \\ 0 & \mathrm{c}_\phi & \mathrm{s}_\phi \mathrm{s}_\theta \\ 1 & 0 & \mathrm{c}_\theta \end{bmatrix} \tag{4.63}$$

Solution in Maple: `Aufgabe_4_3.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/.

Analogously, the translational velocity $\mathbf{v}_k^l$ can also be parameterized by $\mathbf{q}$ and $\dot{\mathbf{q}}$ in the form

$$\mathbf{v}_k^l = \dot{\mathbf{d}}_k^l(\mathbf{q}) = \sum_{j=1}^{n}\left(\frac{\partial}{\partial q_j}\mathbf{d}_k^l(\mathbf{q})\right)\dot{q}_j = \underbrace{\begin{bmatrix} \frac{\partial}{\partial q_1}\mathbf{d}_k^l(\mathbf{q}) & \frac{\partial}{\partial q_2}\mathbf{d}_k^l(\mathbf{q}) & \ldots & \frac{\partial}{\partial q_n}\mathbf{d}_k^l(\mathbf{q}) \end{bmatrix}}_{(\mathbf{J}_{\mathbf{v}})_k^l(\mathbf{q})}\dot{\mathbf{q}}. \tag{4.64}$$

In summary, the vector of angular velocity $\boldsymbol{\omega}_k^l$ and the vector of translational velocity $\mathbf{v}_k^l$ can be written as follows

$$\begin{bmatrix} \mathbf{v}_k^l \\ \boldsymbol{\omega}_k^l \end{bmatrix} = \begin{bmatrix} (\mathbf{J}_{\mathbf{v}})_k^l(\mathbf{q}) \\ (\mathbf{J}_{\boldsymbol{\omega}})_k^l(\mathbf{q}) \end{bmatrix}\dot{\mathbf{q}} = \mathbf{J}_k^l(\mathbf{q})\dot{\mathbf{q}}. \tag{4.65}$$

The matrix $\mathbf{J}_k^l(\mathbf{q})$ is often referred to as the *Manipulator Jacobian Matrix* (Geometric Jacobian Matrix).

Remark: In the English literature, the term Manipulator Jacobian Matrix often refers to the Jacobian matrix associated with the velocities and angular velocities of the end-effector, while the Jacobian matrices associated with the components of the manipulator are often referred to as Body Jacobian Matrices. In this lecture, this distinction is not made, and therefore the Jacobian matrix of any component of a manipulator (end-effector, components) is always referred to as the Manipulator Jacobian Matrix.

Example 4.3 (Continuation Planar Manipulator). In Example 4.1, the kinematics of the planar manipulator shown in Figure 4.7 was calculated. In this example, the translational and rotational velocities of the centers of gravity of the links of the planar manipulator are to be calculated. These will be needed later, for example, for the calculation of the kinetic energy.

From the rotation matrices $\mathbf{R}_0^1$ and $\mathbf{R}_1^2$ according to (4.32), one obtains with (4.44) the matrices

$$\mathbf{S}_0^1 = \begin{bmatrix} 0 & -\omega_1 & 0 \\ \omega_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S}_1^2 = \begin{bmatrix} 0 & -\omega_2 & 0 \\ \omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.66}$$

where $\omega_1 = \dot{\varphi}_1$ and $\omega_2 = \dot{\varphi}_2$ was used. The corresponding vectors $\boldsymbol{\omega}_0^1$ and $\boldsymbol{\omega}_1^2$ are obtained from the components of $\mathbf{S}_0^1$ and $\mathbf{S}_1^2$, respectively, as $\boldsymbol{\omega}_0^1 = \begin{bmatrix} 0 & 0 & \omega_1 \end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{\omega}_1^2 = \begin{bmatrix} 0 & 0 & \omega_2 \end{bmatrix}^{\mathrm{T}}$. The angular velocity of link 2 can be calculated according to (4.57) by

$$\boldsymbol{\omega}_0^2 = \boldsymbol{\omega}_0^1 + \mathbf{R}_0^1 \boldsymbol{\omega}_1^2 \tag{4.67}$$

to $\boldsymbol{\omega}_0^2 = \begin{bmatrix} 0 & 0 & \omega_1 + \omega_2 \end{bmatrix}^{\mathrm{T}}$.

Remark: Note that the same result for $\boldsymbol{\omega}_0^2$ is obtained if one calculates $\mathbf{S}_0^2$ directly from $\mathbf{R}_0^2 = \mathbf{R}_0^1 \mathbf{R}_1^2$.

The Manipulator Jacobian Matrices are then obtained by partial differentiation of the angular velocity vectors with respect to the time derivatives $\dot{\mathbf{q}} = \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix}^{\mathrm{T}}$ of the degrees of freedom $\mathbf{q} = \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}^{\mathrm{T}}$

$$(\mathbf{J}_\omega)_0^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad (\mathbf{J}_\omega)_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{4.68}$$

It is easy to verify that $\boldsymbol{\omega}_0^1 = (\mathbf{J}_\omega)_0^1 \dot{\mathbf{q}}$ and $\boldsymbol{\omega}_0^2 = (\mathbf{J}_\omega)_0^2 \dot{\mathbf{q}}$ holds.

The translational velocities $\mathbf{v}_0^{s1}$ and $\mathbf{v}_0^{s2}$ of the centers of gravity required for further calculations are calculated according to (4.64) with the corresponding Manipulator

Jacobian Matrices

$$(\mathbf{J_v})_0^{s1} = \begin{bmatrix} -\mathrm{s}_{\varphi_1} l_{s1} & 0 \\ \mathrm{c}_{\varphi_1} l_{s1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\mathbf{J_v})_0^{s2} = \begin{bmatrix} -\mathrm{s}_{\varphi_1} l_1 - \mathrm{s}_{\varphi_1+\varphi_2} l_{s2} & -\mathrm{s}_{\varphi_1+\varphi_2} l_{s2} \\ \mathrm{c}_{\varphi_1} l_{s1} + \mathrm{c}_{\varphi_1+\varphi_2} l_{s2} & \mathrm{c}_{\varphi_1+\varphi_2} l_{s2} \\ 0 & 0 \end{bmatrix}. \tag{4.69}$$

Furthermore, the translational velocity of the end effector $\mathbf{v}_0^{e2}$ can be obtained using the Manipulator Jacobian Matrix

$$(\mathbf{J_v})_0^{e2} = \begin{bmatrix} -\mathrm{s}_{\varphi_1} l_1 - \mathrm{s}_{\varphi_1+\varphi_2} l_2 & -\mathrm{s}_{\varphi_1+\varphi_2} l_2 \\ \mathrm{c}_{\varphi_1} l_{s1} + \mathrm{c}_{\varphi_1+\varphi_2} l_2 & \mathrm{c}_{\varphi_1+\varphi_2} l_2 \\ 0 & 0 \end{bmatrix}. \tag{4.70}$$

Solution in MAPLE: `Planarer_Manipulator.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise 4.4 (Continuation Tower Crane 2). In this exercise we again consider the tower crane from Exercise 4.2. For the later calculation of the equations of motion of the system, the velocities and angular velocities, or the corresponding Manipulator Jacobian Matrices of the center of gravity of the tower, the trolley, and the load are required. The position of the center of gravity of the tower $\mathbf{p}_1^{sT}$ in the coordinate system $(0_1 x_1 y_1 z_1)$ is given by $\mathbf{p}_1^{sT} = \begin{bmatrix} l_{xsT} & 0 & l_{zsT} \end{bmatrix}^{\mathrm{T}}$. The centers of gravity of the trolley and the (point-shaped) load lie at the coordinate origins of the corresponding coordinate systems, i.e. $\mathbf{p}_2^{sK} = \mathbf{p}_5^{L} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. Calculate the Manipulator Jacobian Matrices for these three components of the tower crane.

Solution of exercise 4.4. The Manipulator Jacobian Matrices of the center of gravity of the tower are calculated as

$$(\mathbf{J_v})_0^{sT} = \begin{bmatrix} -\mathrm{s}_{\varphi_1} l_{xsT} & 0 & 0 & 0 & 0 \\ \mathrm{c}_{\varphi_2} l_{xsT} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\mathbf{J}_\omega)_0^{sT} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.71}$$

The Manipulator Jacobian Matrices of the trolley are

$$(\mathbf{J_v})_0^{sK} = \begin{bmatrix} -\mathrm{s}_{\varphi_1} s_2 & \mathrm{c}_{\varphi_1} & 0 & 0 & 0 \\ \mathrm{c}_{\varphi_2} s_2 & \mathrm{s}_{\varphi_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\mathbf{J}_\omega)_0^{sK} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.72}$$

Finally, the Manipulator Jacobian Matrices of the load are given by

$$
(\mathbf{J_v})_0^{sL} =
$$
$$
\begin{bmatrix}
\left(\mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4}s_5 - s_2\right)\mathrm{s}_{\varphi_1} - \mathrm{c}_{\varphi_1}\mathrm{s}_{\varphi_4}s_5 & \mathrm{c}_{\varphi_1} & -\mathrm{c}_{\varphi_1}\mathrm{c}_{\varphi_3}\mathrm{c}_{\varphi_4}s_5 & \left(\mathrm{c}_{\varphi_1}\mathrm{s}_{\varphi_3}\mathrm{s}_{\varphi_4} - \mathrm{s}_{\varphi_1}\mathrm{c}_{\varphi_4}\right)s_5 & -\mathrm{s}_{\varphi_1}\mathrm{s}_{\varphi_4} - \mathrm{c}_{\varphi_1}\mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4} \\
\left(-\mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4}s_5 + s_2\right)\mathrm{c}_{\varphi_1} - \mathrm{s}_{\varphi_1}\mathrm{s}_{\varphi_4}s_5 & \mathrm{s}_{\varphi_1} & -\mathrm{s}_{\varphi_1}\mathrm{c}_{\varphi_3}\mathrm{c}_{\varphi_4}s_5 & \left(\mathrm{s}_{\varphi_1}\mathrm{s}_{\varphi_3}\mathrm{s}_{\varphi_4} + \mathrm{c}_{\varphi_1}\mathrm{c}_{\varphi_4}\right)s_5 & \mathrm{c}_{\varphi_1}\mathrm{s}_{\varphi_4} - \mathrm{s}_{\varphi_1}\mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4} \\
0 & 0 & \mathrm{s}_{\varphi_3}\mathrm{c}_{\varphi_4}s_5 & \mathrm{c}_{\varphi_3}\mathrm{s}_{\varphi_4}s_5 & -\mathrm{c}_{\varphi_3}\mathrm{c}_{\varphi_4}
\end{bmatrix}
\tag{4.73}
$$

and

$$
(\mathbf{J}_{\boldsymbol{\omega}})_0^{sL} = \begin{bmatrix} 0 & 0 & -\mathrm{s}_{\varphi_1} & \mathrm{c}_{\varphi_1}\mathrm{c}_{\varphi_3} & 0 \\ 0 & 0 & \mathrm{c}_{\varphi_1} & \mathrm{s}_{\varphi_1}\mathrm{c}_{\varphi_3} & 0 \\ 1 & 0 & 0 & -\mathrm{s}_{\varphi_3} & 0 \end{bmatrix} . \tag{4.74}
$$

Lösung in MAPLE: `Turmdrehkran.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

4.7 References

[4.0] M. Calkin, *Lagrangian and Hamiltonian Mechanics*. Singapore: World Scientific, 1996.

[4.0] H. Goldstein, *Klassische Mechanik*. Wiesbaden: AULA-Verlag, 1991.

[4.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 1: Theoretical Basis*. Berlin: Springer, 2002.

[4.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 2: Applications*. Berlin: Springer, 2002.

[4.0] B. Heimann, W. Gerth, and K. Popp, *Mechatronik: Komponenten - Methoden - Beispiele*. Leipzig: Fachbuchverlag, 2003.

[4.0] R. Isermann, *Mechatronische Systeme: Grundlagen*. Berlin Heidelberg: Springer, 2008.

[4.0] A. Shabana, *Dynamics of Multibody Systems*. Cambridge: Cambridge University Press, 1998.

[4.0] B. Siciliano, L. Sciavicco, L. Villani, and G. Oriolo, *Robotics: Modelling, Planning and Control*. London, UK: Springer, 2009.

[4.0] M. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: John Wiley & Sons, 1989.

[4.0] P. Wellstead, *Physical System Modelling*. London: Academic Press, 1979.

5 Rigid Body Dynamics

5.1 Kinetic and Potential Energy

In Chapter 3.4 the translational kinetic energy and the potential energy were introduced. In this section these results are systematically extended for general rigid bodies. The resulting formulation of the kinetic and potential energy forms the basis for determining the equations of motion of rigid body systems using the Euler-Lagrange equations in the next section.

Consider now a rigid body S according to Figure 5.1 with a mass density $\rho(x_0, y_0, z_0)$ in the inertial coordinate system $(0_0x_0y_0z_0)$. The velocity of a point P of the rigid body

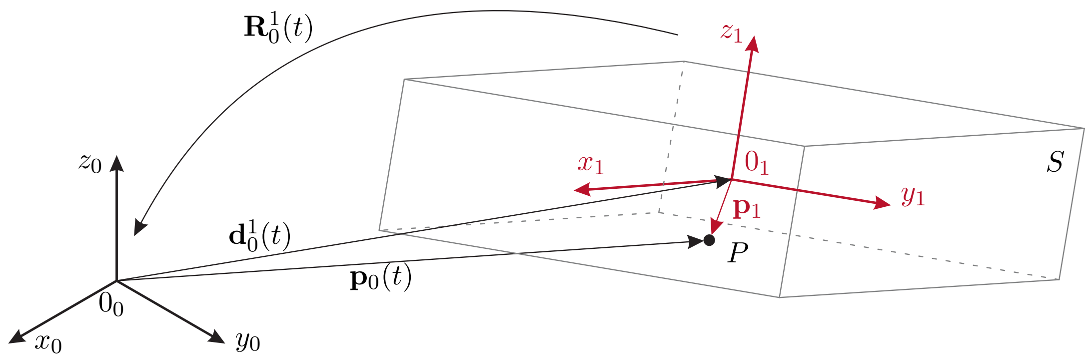


Figure 5.1: Calculation of the kinetic energy.

is given by $\dot{\mathbf{p}}_0(x_0, y_0, z_0)$. Then the total kinetic energy is obtained by integration over the volume $\mathcal{V}$ of the rigid body S in the form

$$T = \frac{1}{2}\int_{\mathcal{V}} \dot{\mathbf{p}}_0^{\mathrm{T}}(x_0, y_0, z_0)\dot{\mathbf{p}}_0(x_0, y_0, z_0)\underbrace{\rho(x_0, y_0, z_0)\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0}_{\mathrm{d}m} \ . \tag{5.1}$$

Compare this to the considerations in Chapter 3.4. This expression can be simplified by additionally defining a body-fixed coordinate system $(0_1x_1y_1z_1)$ whose origin 0_1 coincides with the center of mass of the rigid body S. The position of the center of mass $\mathbf{p}_0^S$ in the coordinate system $(0_0x_0y_0z_0)$ is calculated according to (3.28) via the relation

$$\mathbf{p}_0^S = \frac{1}{m}\int_{\mathcal{V}} \mathbf{p}_0 \mathrm{d}m, \tag{5.2}$$

where the mass of the rigid body S is defined by $m = \int_{\mathcal{V}} \mathrm{d}m$. According to Figure 5.1 and (4.21), the following applies to the chosen position of the origin 0_1

$$\mathbf{p}_0 = \mathbf{R}_0^1\mathbf{p}_1 + \mathbf{d}_0^1 = \mathbf{R}_0^1\mathbf{p}_1 + \mathbf{p}_0^S \ . \tag{5.3}$$

With (4.49) and (4.50) follows for the velocity $\dot{\mathbf{p}}_0$

$$\dot{\mathbf{p}}_0 = \mathbf{S}\Big(\boldsymbol{\omega}_0^1\Big)\mathbf{R}_0^1\mathbf{p}_1 + \dot{\mathbf{p}}_0^S = \mathbf{S}\Big(\boldsymbol{\omega}_0^1\Big)\mathbf{r} + \mathbf{v}_0^S = \boldsymbol{\omega}_0^1 \times \mathbf{r} + \mathbf{v}_0^S \tag{5.4}$$

with the abbreviations $\mathbf{r} = \mathbf{R}_0^1\mathbf{p}_1$ and $\mathbf{v}_0^S = \dot{\mathbf{p}}_0^S$. In order to keep the equations clear for the further derivation, $\boldsymbol{\omega}_0^1 = \boldsymbol{\omega}$ is used with the components $\boldsymbol{\omega}^{\mathrm{T}} = \begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}$. Substituting the expression (5.4) for $\dot{\mathbf{p}}_0$ into the kinetic energy (5.1) gives

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{V}}\Big(\mathbf{S}(\boldsymbol{\omega})\mathbf{r} + \mathbf{v}_0^S\Big)^{\mathrm{T}}\Big(\mathbf{S}(\boldsymbol{\omega})\mathbf{r} + \mathbf{v}_0^S\Big)\mathrm{d}m \\ &= \underbrace{\frac{1}{2}\int_{\mathcal{V}}\mathbf{r}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}(\boldsymbol{\omega})\mathbf{S}(\boldsymbol{\omega})\mathbf{r}\mathrm{d}m}_{T_r} + \underbrace{\frac{1}{2}\int_{\mathcal{V}}\Big(\mathbf{r}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}(\boldsymbol{\omega})\mathbf{v}_0^S + \Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{S}(\boldsymbol{\omega})\mathbf{r}\Big)\mathrm{d}m}_{T_k} + \underbrace{\frac{1}{2}\int_{\mathcal{V}}\Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{v}_0^S\mathrm{d}m}_{T_t}\ . \end{aligned} \tag{5.5}$$

If we now consider that $\mathbf{S}(\boldsymbol{\omega})$ and $\mathbf{v}_0^S$ are pure time functions and do not depend on the integration variables, then the expressions of (5.5) can be further simplified.

The third term T_t of (5.5) then reads

$$T_t = \frac{1}{2}\int_{\mathcal{V}}\Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{v}_0^S\mathrm{d}m = \frac{1}{2}\Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{v}_0^S\int_{\mathcal{V}}\mathrm{d}m = \frac{1}{2}m\Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{v}_0^S \tag{5.6}$$

and describes the *translational part of the kinetic energy.* This expression can be interpreted in such a way that the total mass m of the rigid body is combined into a point mass at the center of gravity $\mathbf{p}_0^S$, cf. (3.60).

The second term T_k in (5.5) vanishes identically for the chosen position of the coordinate system $(0_1x_1y_1z_1)$ in the center of gravity. To show this, one uses the relation $\mathbf{r} = \mathbf{p}_0 - \mathbf{p}_0^S$ (compare (5.3)) and T_k simplifies due to the definition of the center of gravity (5.2) to

$$\begin{aligned} T_k &= \frac{1}{2}\int_{\mathcal{V}}\Big(\mathbf{r}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}(\boldsymbol{\omega})\mathbf{v}_0^S + \Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{S}(\boldsymbol{\omega})\mathbf{r}\Big)\mathrm{d}m = \Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{S}(\boldsymbol{\omega})\int_{\mathcal{V}}\Big(\mathbf{p}_0 - \mathbf{p}_0^S\Big)\mathrm{d}m \\ &= \Big(\mathbf{v}_0^S\Big)^{\mathrm{T}}\mathbf{S}(\boldsymbol{\omega})\left(\underbrace{\int_{\mathcal{V}}\mathbf{p}_0\mathrm{d}m}_{=\mathbf{p}_0^S m \text{ because of } (5.2)} - \mathbf{p}_0^S\underbrace{\int_{\mathcal{V}}\mathrm{d}m}_{=m}\right) = 0\ . \end{aligned} \tag{5.7}$$

It should be pointed out again that this expression only disappears because the body-fixed coordinate system $(0_1x_1y_1z_1)$ has been placed at the center of gravity of the rigid body.

For the simplification of the term T_r of (5.5), one needs the relationship

$$\mathbf{h}^{\mathrm{T}}\mathbf{h} = \begin{bmatrix}h_x & h_y & h_z\end{bmatrix}\begin{bmatrix}h_x \\ h_y \\ h_z\end{bmatrix} = \mathrm{spur}\Big(\mathbf{h}\mathbf{h}^{\mathrm{T}}\Big) = \mathrm{spur}\begin{bmatrix}h_x^2 & h_xh_y & h_xh_z \\ h_yh_x & h_y^2 & h_yh_z \\ h_zh_x & h_zh_y & h_z^2\end{bmatrix}. \tag{5.8}$$

Thus T_r from (5.5) can be equivalently expressed in the form

$$T_r = \frac{1}{2}\int_{\mathcal{V}}\mathbf{r}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}(\boldsymbol{\omega})\mathbf{S}(\boldsymbol{\omega})\mathbf{r}\mathrm{d}m = \frac{1}{2}\mathrm{spur}\Big(\mathbf{S}(\boldsymbol{\omega})\int_{\mathcal{V}}\mathbf{r}\mathbf{r}^{\mathrm{T}}\mathrm{d}m\mathbf{S}^{\mathrm{T}}(\boldsymbol{\omega})\Big). \tag{5.9}$$

With the matrix

$$\mathbf{J} = \int_{\mathcal{V}} \mathbf{r}\mathbf{r}^{\mathrm{T}} \mathrm{d}m = \begin{bmatrix} \int_{\mathcal{V}} r_x^2 \mathrm{d}m & \int_{\mathcal{V}} r_x r_y \mathrm{d}m & \int_{\mathcal{V}} r_x r_z \mathrm{d}m \\ \int_{\mathcal{V}} r_x r_y \mathrm{d}m & \int_{\mathcal{V}} r_y^2 \mathrm{d}m & \int_{\mathcal{V}} r_y r_z \mathrm{d}m \\ \int_{\mathcal{V}} r_x r_z \mathrm{d}m & \int_{\mathcal{V}} r_y r_z \mathrm{d}m & \int_{\mathcal{V}} r_z^2 \mathrm{d}m \end{bmatrix} \tag{5.10}$$

and the simplification

$$\begin{aligned} \mathrm{spur}\Big(\mathbf{S}(\boldsymbol{\omega})\mathbf{J}\mathbf{S}^{\mathrm{T}}(\boldsymbol{\omega})\Big) &= \mathrm{spur}\left(\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix}\begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}\right) \\ &= \omega_x^2(J_{yy}+J_{zz}) + \omega_y^2(J_{xx}+J_{zz}) + \omega_z^2(J_{xx}+J_{yy}) - 2\omega_x\omega_y J_{xy} - 2\omega_y\omega_z J_{yz} - 2\omega_x\omega_z J_{xz} \\ &= [\omega_x, \omega_y, \omega_z]\begin{bmatrix} J_{yy}+J_{zz} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{xx}+J_{zz} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{xx}+J_{yy} \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}_0\boldsymbol{\omega} \end{aligned} \tag{5.11}$$

the *rotational part of the kinetic energy* results in

$$T_r = \frac{1}{2}\left(\boldsymbol{\omega}_0^1\right)^{\mathrm{T}}\mathbf{I}_0\boldsymbol{\omega}_0^1 \tag{5.12}$$

with

$$\mathbf{I}_0 = \begin{bmatrix} \int_{\mathcal{V}}\left(r_y^2+r_z^2\right)\mathrm{d}m & -\int_{\mathcal{V}} r_x r_y \mathrm{d}m & -\int_{\mathcal{V}} r_x r_z \mathrm{d}m \\ -\int_{\mathcal{V}} r_x r_y \mathrm{d}m & \int_{\mathcal{V}}\left(r_x^2+r_z^2\right)\mathrm{d}m & -\int_{\mathcal{V}} r_y r_z \mathrm{d}m \\ -\int_{\mathcal{V}} r_x r_z \mathrm{d}m & -\int_{\mathcal{V}} r_y r_z \mathrm{d}m & \int_{\mathcal{V}}\left(r_x^2+r_y^2\right)\mathrm{d}m \end{bmatrix}. \tag{5.13}$$

Here, $\mathbf{I}_0$ describes the so-called *inertia matrix*.

Note that for the calculation of $\mathbf{I}_0$ according to (5.13), the vector $\mathbf{r} = \mathbf{p}_0 - \mathbf{p}_0^S$ is represented in the coordinate system $(0_0 x_0 y_0 z_0)$. This can make the calculation of the integrals in the definition of $\mathbf{I}_0$ very complex, especially the definition of the integration limits. A simpler form of calculation of $\mathbf{I}_0$ and thus T_r is obtained if one uses the definition $\mathbf{r} = \mathbf{R}_0^1\mathbf{p}_1$ in $\mathbf{J}$ according to (5.10)

$$\mathbf{J} = \int_{\mathcal{V}} \mathbf{r}\mathbf{r}^{\mathrm{T}}\mathrm{d}m = \int_{\mathcal{V}} \mathbf{R}_0^1\mathbf{p}_1\mathbf{p}_1^{\mathrm{T}}\left(\mathbf{R}_0^1\right)^{\mathrm{T}}\mathrm{d}m = \mathbf{R}_0^1\int_{\mathcal{V}}\mathbf{p}_1\mathbf{p}_1^{\mathrm{T}}\mathrm{d}m\left(\mathbf{R}_0^1\right)^{\mathrm{T}}. \tag{5.14}$$

If one applies the same steps as in (5.11) to this representation of $\mathbf{J}$, one obtains

$$\mathrm{spur}\left(\mathbf{S}\left(\boldsymbol{\omega}_0^1\right)\mathbf{J}\left(\mathbf{S}\left(\boldsymbol{\omega}_0^1\right)\right)^{\mathrm{T}}\right) = \left(\boldsymbol{\omega}_0^1\right)^{\mathrm{T}}\underbrace{\mathbf{R}_0^1\mathbf{I}_1\left(\mathbf{R}_0^1\right)^{\mathrm{T}}}_{\mathbf{I}_0}\boldsymbol{\omega}_0^1 . \tag{5.15}$$

Therein, the representation of the inertia matrix $\mathbf{I}_1$ in the body-fixed coordinate system is calculated from

$$\mathbf{I}_1 = \begin{bmatrix} \int_{\mathcal{V}} \left(p_{1,y}^2 + p_{1,z}^2\right) \mathrm{d}m & -\int_{\mathcal{V}} p_{1,x} p_{1,y} \mathrm{d}m & -\int_{\mathcal{V}} p_{1,x} p_{1,z} \mathrm{d}m \\ -\int_{\mathcal{V}} p_{1,x} p_{1,y} \mathrm{d}m & \int_{\mathcal{V}} \left(p_{1,x}^2 + p_{1,z}^2\right) \mathrm{d}m & -\int_{\mathcal{V}} p_{1,y} p_{1,z} \mathrm{d}m \\ -\int_{\mathcal{V}} p_{1,x} p_{1,z} \mathrm{d}m & -\int_{\mathcal{V}} p_{1,y} p_{1,z} \mathrm{d}m & \int_{\mathcal{V}} \left(p_{1,x}^2 + p_{1,y}^2\right) \mathrm{d}m \end{bmatrix} . \tag{5.16}$$

This calculation is often much easier than the direct calculation of $\mathbf{I}_0$. $\mathbf{I}_1$ is independent of the motion of the rigid body and thus a constant matrix. With a suitable choice of the body-fixed coordinate system $(0_1 x_1 y_1 z_1)$ in the direction of the so-called principal axes of inertia, $\mathbf{I}_1$ simplifies to a diagonal matrix. In this case the so-called *products of inertia* (off-diagonal elements of $\mathbf{I}_1$) vanish.

Remark: The expression

$$\left(\mathbf{R}_0^1\right)^{\mathrm{T}} \boldsymbol{\omega}_0^1 = \mathbf{R}_1^0 \boldsymbol{\omega}_0^1 = {}_1\boldsymbol{\omega}_0^1 \tag{5.17}$$

corresponds to the transformation of the vector of angular velocities from the inertial coordinate system to the body-fixed coordinate system. The rotational part T_r of the kinetic energy can therefore also be represented in the form

$$T_r = \frac{1}{2}\left({}_1\boldsymbol{\omega}_0^1\right)^{\mathrm{T}} \mathbf{I}_1 \left({}_1\boldsymbol{\omega}_0^1\right) . \tag{5.18}$$

Note that in the definition of $\boldsymbol{\omega}_0^1$ it was implicitly used that this vector is described in the coordinate system $(0_0 x_0 y_0 z_0)$, i.e. $\boldsymbol{\omega}_0^1 = {}_0\boldsymbol{\omega}_0^1$. However, in order to avoid ambiguities in the notation, the representations ${}_0\boldsymbol{\omega}_0^1$ and ${}_1\boldsymbol{\omega}_0^1$ will not be used further in the remainder of the script.

In total, the *kinetic energy* of a rigid body is given by

$$T = \frac{1}{2} m \left(\mathbf{v}_0^S\right)^{\mathrm{T}} \mathbf{v}_0^S + \frac{1}{2}\left(\boldsymbol{\omega}_0^1\right)^{\mathrm{T}} \mathbf{I}_0 \boldsymbol{\omega}_0^1 . \tag{5.19}$$

It should be pointed out again that this formulation of the kinetic energy of a rigid body assumes that $\mathbf{v}_0^S$ is the velocity of the center of gravity and $\mathbf{I}_0$ is the inertia matrix about this center of gravity. The *potential energy* due to the gravitational field in $\mathbf{e}_g$ direction with the gravitational constant g is easily obtained using (5.2) in the form

$$V = -\int_{\mathcal{V}} g \mathbf{e}_g^{\mathrm{T}} \mathbf{p}_0 \mathrm{d}m = -mg\mathbf{e}_g^{\mathrm{T}} \mathbf{p}_0^S . \tag{5.20}$$

In Chapter 4.6, the translational velocity $\mathbf{v}_0^l$ and the angular velocity $\boldsymbol{\omega}_0^l$ of a point of the rigid body S_l were represented as a function of the generalized coordinates $\mathbf{q}$ and their time derivatives $\dot{\mathbf{q}}$ with the help of the Jacobian matrices $(\mathbf{J}_{\mathbf{v}})_0^l(\mathbf{q})$ and $(\mathbf{J}_{\boldsymbol{\omega}})_0^l(\mathbf{q})$, see (4.65). This representation can now be used in (5.19) to obtain

$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left(\left((\mathbf{J_v})_0^1\right)^{\mathrm{T}} m (\mathbf{J_v})_0^1 + \left((\mathbf{J}_{\boldsymbol{\omega}})_0^1\right)^{\mathrm{T}} \mathbf{I}_0 (\mathbf{J}_{\boldsymbol{\omega}})_0^1\right)\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \ . \tag{5.21}$$

Therein $\mathbf{M}(\mathbf{q})$ denotes the so-called *mass matrix*. One can now show that the mass matrix is positive definite, i.e. $\mathbf{M}(\mathbf{q}) > 0$. This leads directly to the fact that the kinetic energy T is positive for every $\dot{\mathbf{q}} \neq \mathbf{0}$.

Remark: The following procedure is recommended for setting up the kinetic and potential energies of a system of rigid bodies:

(A) Define an inertial coordinate system $(0_0 x_0 y_0 z_0)$ and a body-fixed coordinate system $(0_l x_l y_l z_l)$ for each rigid body S_l. If reasonable and possible, the body-fixed coordinate systems should be aligned with the respective principal axes of inertia.

(B) Define the generalized coordinates $\mathbf{q}$. Determine the rotation matrices and displacement vectors that connect the coordinate systems. Write down the position vectors $\mathbf{p}_0^{S_l}$ from the origin of the inertial system to the centers of mass of the rigid bodies as functions of the generalized coordinates.

(C) Determine the translational and rotational parts of the kinetic energy according to (5.6) and (5.13). Calculate the mass matrix $\mathbf{M}(\mathbf{q})$ as the sum of the mass matrices of the individual rigid bodies according to (5.21). Determine the potential energy due to a gravitational field according to (5.20). Calculate the potential energy of other potential forces such as spring elements according to (3.69).

Example 5.1 (Continuation Planar Manipulator). The systematics for the calculation of the kinetic and potential energy just presented will be applied in this example to the planar manipulator from Examples 4.1 and 4.3. As can be seen in Figure 5.2, the structure has been extended by a linear spring (spring stiffness c_e, relaxed length s_{0e}) and a linear damper (damping coefficient d_e), which are installed between the end effector and the fixed bearing A.

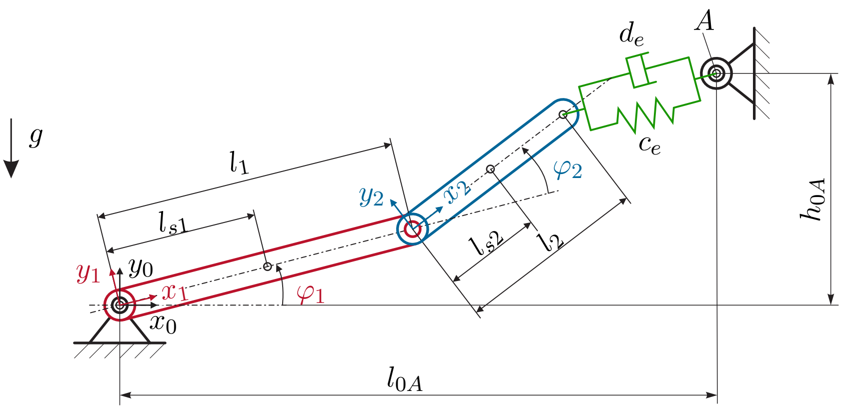


Figure 5.2: Sketch of the planar manipulator in Example 5.1.

In the first step, the translational parts T_t of the kinetic energy are determined according to (5.6). This yields for the first rod

$$T_t^{s1} = \frac{1}{2} m_1 \left(\dot{\mathbf{p}}_0^{s1}\right)^{\mathrm{T}} \dot{\mathbf{p}}_0^{s1} = \frac{1}{2} m_1 l_{s1}^2 \omega_1^2, \tag{5.22}$$

where $\omega_1 = \dot{\varphi}_1$ and $\mathbf{p}_0^{s1}$ is defined in (4.35a). Similarly, we calculate

$$T_t^{s2} = \frac{1}{2} m_2 \left(\dot{\mathbf{p}}_0^{s2}\right)^{\mathrm{T}} \dot{\mathbf{p}}_0^{s2} = \frac{1}{2} m_2 \Big(l_1^2 \omega_1^2 + 2 l_1 l_{s2} \omega_1 (\omega_1 + \omega_2) \cos(\varphi_2) + l_{s2}^2 (\omega_1 + \omega_2)^2\Big), \tag{5.23}$$

with $\omega_2 = \dot{\varphi}_2$ and $\mathbf{p}_0^{s2}$ from (4.35b).

To calculate the rotational part of the kinetic energy, in the first step the inertia matrices $\mathbf{I}_1^{s1}$ and $\mathbf{I}_2^{s2}$ of the rods around the respective center of gravity $s1$ and $s2$ and in the respective body-fixed coordinate systems are defined. Due to the chosen orientation of the body-fixed coordinate systems in the direction of the principal axes of inertia, these result in the form of the diagonal matrices

$$\mathbf{I}_1^{s1} = \begin{bmatrix} I_{xx,1}^{s1} & 0 & 0 \\ 0 & I_{yy,1}^{s1} & 0 \\ 0 & 0 & I_{zz,1}^{s1} \end{bmatrix}, \quad \mathbf{I}_2^{s2} = \begin{bmatrix} I_{xx,2}^{s2} & 0 & 0 \\ 0 & I_{yy,2}^{s2} & 0 \\ 0 & 0 & I_{zz,2}^{s2} \end{bmatrix}. \tag{5.24}$$

The formulation of the inertia matrices $\mathbf{I}_0^{s1}$, $\mathbf{I}_0^{s2}$ in the inertial system required in (5.12) are calculated using the respective rotation matrices $\mathbf{R}_0^1$ or $\mathbf{R}_0^2$ according to (5.15). This gives

$$\mathbf{I}_0^{s1} = \mathbf{R}_0^1 \mathbf{I}_1^{s1} \left(\mathbf{R}_0^1\right)^{\mathrm{T}} = \begin{bmatrix} I_{xx,1}^{s1} \mathrm{c}_{\varphi_1}^2 + I_{yy,1}^{s1} \mathrm{s}_{\varphi_1}^2 & \left(I_{xx,1}^{s1} - I_{yy,1}^{s1}\right) \mathrm{s}_{\varphi_1} \mathrm{c}_{\varphi_1} & 0 \\ \left(I_{xx,1}^{s1} - I_{yy,1}^{s1}\right) \mathrm{s}_{\varphi_1} \mathrm{c}_{\varphi_1} & I_{xx,1}^{s1} \mathrm{s}_{\varphi_1}^2 + I_{yy,1}^{s1} \mathrm{c}_{\varphi_1}^2 & 0 \\ 0 & 0 & I_{zz,1}^{s1} \end{bmatrix}. \tag{5.25}$$

The resulting inertia matrix $\mathbf{I}_0^{s2} = \mathbf{R}_0^2 \mathbf{I}_2^{s2} \left(\mathbf{R}_0^2\right)^{\mathrm{T}}$ is relatively large and is therefore not shown here for reasons of space. With $\mathbf{I}_0^{s1}$, $\mathbf{I}_0^{s2}$ as well as $\boldsymbol{\omega}_0^1$ and $\boldsymbol{\omega}_0^2$ from Example 4.3, one obtains

$$T_r^{s1} = \frac{1}{2}\left(\boldsymbol{\omega}_0^1\right)^{\mathrm{T}} \mathbf{I}_0^{s1} \boldsymbol{\omega}_0^1 = \frac{1}{2} I_{zz,1}^{s1} \omega_1^2 \tag{5.26a}$$

$$T_r^{s2} = \frac{1}{2}\left(\boldsymbol{\omega}_0^2\right)^{\mathrm{T}} \mathbf{I}_0^{s2} \boldsymbol{\omega}_0^2 = \frac{1}{2} I_{zz,2}^{s2} (\omega_1 + \omega_2)^2 \ . \tag{5.26b}$$

An equivalent calculation of the kinetic energy can be obtained using the mass matrix from (5.21). For the system under consideration, this is calculated to be

$$\mathbf{M} = \begin{bmatrix} m_1 l_{s1}^2 + m_2\left(l_1^2 + 2 l_1 l_{s2} \mathrm{c}_{\varphi_2} + l_{s2}^2\right) + I_{zz,1}^{s1} + I_{zz,2}^{s2} & m_2 l_{s2}\left(l_{s2} + l_1 \mathrm{c}_{\varphi_2}\right) + I_{zz,2}^{s2} \\ m_2 l_{s2}\left(l_{s2} + l_1 \mathrm{c}_{\varphi_2}\right) + I_{zz,2}^{s2} & m_2 l_{s2}^2 + I_{zz,2}^{s2} \end{bmatrix}. \tag{5.27}$$

A simple calculation shows that the expression for T determined with (5.21) and (5.27) corresponds to the sum $T = T_t^{s1} + T_t^{s2} + T_r^{s1} + T_r^{s2}$.

The potential energy components V^{s1}, V^{s2} due to gravity are given by (5.20) in the form

$$V^{s1} = -m_1 g \mathbf{e}_g^{\mathrm{T}} \mathbf{p}_0^{s1} = m_1 g l_{s1} \sin(\varphi_1) \tag{5.28a}$$

$$V^{s2} = -m_2 g \mathbf{e}_g^{\mathrm{T}} \mathbf{p}_0^{s2} = m_2 g\left(l_1 \sin(\varphi_1) + l_{s2} \sin(\varphi_1 + \varphi_2)\right), \tag{5.28b}$$

where $\mathbf{e}_g^{\mathrm{T}} = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}$ describes the unit vector in the direction of gravity and g the acceleration due to gravity.

To calculate the potential energy of the linear spring, the length of the spring is needed. For this purpose, the vector $\mathbf{p}_0^A$ is drawn from the origin of the inertial system to the bearing A and the vector $\mathbf{r}_e$ is obtained from the end effector $e2$ to the bearing A

$$\mathbf{r}_e = \mathbf{p}_0^A - \mathbf{p}_0^{e2} = \begin{bmatrix} l_{0A} - l_1 \cos(\varphi_1) - l_2 \cos(\varphi_1 + \varphi_2) \\ h_{0A} - l_1 \sin(\varphi_1) - l_2 \sin(\varphi_1 + \varphi_2) \\ 0 \end{bmatrix} . \tag{5.29}$$

Thus, the length s_e of the spring is $s_e = \sqrt{\mathbf{r}_e^{\mathrm{T}} \mathbf{r}_e}$ and the potential energy of the spring is calculated according to (3.70) to

$$V_e = \frac{1}{2} c_2 (s_e - s_{0e})^2 \ . \tag{5.30}$$

Solution in MAPLE: `Planarer_Manipulator.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Example 5.2 (Continuation Tower Crane 2). In this example the kinetic and potential energy are calculated for the tower crane from Exercises 4.2 and 4.4. The first rigid body of the tower crane is the tower with the firmly attached jib. It is assumed that the tower is in the form of a solid cylinder (height h_1, radius r_1, density ρ_1), while the jib is approximated by a cuboid (dimensions l_{1x}, l_{1y} l_{1z}, density ρ_1).

The centers of gravity of the two rigid bodies in the body-fixed coordinate system $(0_1x_1y_1z_1)$ are

$$\mathbf{p}_1^{sZ} = \begin{bmatrix} 0 \\ 0 \\ \frac{h_1}{2} \end{bmatrix}, \quad \mathbf{p}_1^{sQ} = \begin{bmatrix} \frac{l_{1x}}{2} \\ 0 \\ l_{1z} \end{bmatrix} \tag{5.31}$$

and the masses result from $m_Z = r_1^2 \pi h_1 \rho_1$ or $m_Q = l_{1x} l_{1y} l_{1z} \rho_1$. The inertia matrices about the respective centers of gravity in the body-fixed coordinate system can be taken from a formula collection in the form

$$\mathbf{I}_1^{sZ} = \begin{bmatrix} I_{xx,1}^{sZ} & 0 & 0 \\ 0 & I_{yy,1}^{sZ} & 0 \\ 0 & 0 & I_{zz,1}^{sZ} \end{bmatrix}, \quad \mathbf{I}_1^{sQ} = \begin{bmatrix} I_{xx,1}^{sQ} & 0 & 0 \\ 0 & I_{yy,1}^{sQ} & 0 \\ 0 & 0 & I_{zz,1}^{sQ} \end{bmatrix}, \tag{5.32}$$

with $I_{xx,1}^{sZ} = I_{yy,1}^{sZ} = (3r_1^2 + h_1^2) m_Z/12$, $I_{zz,1}^{sZ} = r_1^2 m_z/2$, $I_{xx,1}^{sQ} = \left(l_{1y}^2 + l_{1z}^2\right) m_Q/12$, $I_{yy,1}^{sQ} = (l_{1x}^2 + l_{1z}^2) m_Q/12$ and $I_{zz,1}^{sQ} = \left(l_{1x}^2 + l_{1y}^2\right) m_Q/12$.

There are now two possible approaches to calculate the kinetic energy of the tower: The first possibility is to calculate the common center of gravity $\mathbf{p}_1^{sT}$ of the two rigid bodies and to determine the total inertia matrix $\mathbf{I}_1^{sT}$ about the common center of gravity using Steiner's Theorem. The second, much simpler possibility is to determine the kinetic energy for each of the two rigid bodies separately. In the solution in Maple both approaches are presented. Here, the steps for the second possibility, i.e. the separate determination of the kinetic energy of the two rigid bodies, are presented. Starting from the position $\mathbf{p}_1^{sZ}$ of the center of gravity of the cylinder in the body-fixed coordinate system, the position $\mathbf{p}_0^{sZ}$ in the inertial system can be determined by applying the homogeneous transformation $\mathbf{H}_0^1$ from Exercise 4.2. The corresponding velocity $\mathbf{v}_0^{sZ} = \dot{\mathbf{p}}_0^{sZ}$ is immediately obtained with the Jacobian matrix $(\mathbf{J_v})_0^{sZ}$ to

$$\mathbf{v}_0^{sZ} = (\mathbf{J_v})_0^{sZ} \dot{\mathbf{q}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ v_2 \\ \omega_3 \\ \omega_4 \\ v_5 \end{bmatrix} = \mathbf{0} \tag{5.33}$$

and thus $T_t^{sZ} = 0$. The rotational part T_r^{sZ} results in the form

$$T_r^{sZ} = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\Big((\mathbf{J}_{\boldsymbol{\omega}})_0^{sZ}\Big)^{\mathrm{T}}\mathbf{R}_0^1\mathbf{I}_1^{sZ}\Big(\mathbf{R}_0^1\Big)^{\mathrm{T}}(\mathbf{J}_{\boldsymbol{\omega}})_0^{sZ}\dot{\mathbf{q}} = \frac{1}{2}I_{zz,1}^{sZ}\omega_1^2 \tag{5.34}$$

cf. (5.12). Therein, the Jacobian matrix $(\mathbf{J}_{\boldsymbol{\omega}})_0^{sZ}$ corresponds to the Jacobian matrix $(\mathbf{J}_{\boldsymbol{\omega}})_0^{sT}$ calculated in Exercise 4.4, since, of course, the cylinder rotates at the same speed as the entire tower with jib.

In an analogous way, the components of the kinetic energy of the cuboid can be determined to be

$$T_t^{sQ} = \frac{1}{2}m_Q\left(\frac{l_{1x}}{2}\right)^2\omega_1^2 \tag{5.35a}$$

$$T_r^{sQ} = \frac{1}{2}I_{zz,1}^{sQ}\omega_1^2 \ . \tag{5.35b}$$

A further representation of the kinetic energy is possible by the mass matrix $\mathbf{M}^{sT}$ of the tower, see (5.21) This is calculated for the considered subsystem with

$$\begin{aligned}\mathbf{M}^{sT}(\mathbf{q}) = &\Big((\mathbf{J}_{\mathbf{v}})_0^{sZ}\Big)^{\mathrm{T}}(\mathbf{J}_{\mathbf{v}})_0^{sZ}m_Z + \Big((\mathbf{J}_{\mathbf{v}})_0^{sQ}\Big)^{\mathrm{T}}(\mathbf{J}_{\mathbf{v}})_0^{sQ}m_Q \\ &+ \Big((\mathbf{J}_{\boldsymbol{\omega}})_0^{sZ}\Big)^{\mathrm{T}}\mathbf{I}_0^{sZ}(\mathbf{J}_{\boldsymbol{\omega}})_0^{sZ} + \Big((\mathbf{J}_{\boldsymbol{\omega}})_0^{sQ}\Big)^{\mathrm{T}}\mathbf{I}_0^{sQ}(\mathbf{J}_{\boldsymbol{\omega}})_0^{sQ}\end{aligned} \tag{5.36}$$

to

$$\mathbf{M}^{sT}(\mathbf{q}) = \begin{bmatrix} \frac{l_{1x}^2}{4}m_Q + I_{zz,1}^{sZ} + I_{zz,1}^{sQ} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.37}$$

and $T^{sT} = T_t^{sZ} + T_t^{sQ} + T_r^{sZ} + T_r^{sQ} = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}^{sT}(\mathbf{q})\dot{\mathbf{q}}$.

For the trolley it is assumed that the center of gravity $\mathbf{p}_0^{sK}$ lies at the origin of the coordinate system $(0_2x_2y_2z_2)$. The inertia matrix in this body-fixed coordinate system is given by the diagonal matrix $\mathbf{I}_2^{sK} = \mathrm{diag}\Big(I_{xx,2}^{sK}, I_{xx,2}^{sK}, I_{xx,2}^{sK}\Big)$. The mass matrix $\mathbf{M}^{sK}(\mathbf{q})$ of the trolley is calculated analogously to (5.36) with

$$\mathbf{M}^{sK}(\mathbf{q}) = \Big((\mathbf{J}_{\mathbf{v}})_0^{sK}\Big)^{\mathrm{T}}(\mathbf{J}_{\mathbf{v}})_0^{sK}m_K + \Big((\mathbf{J}_{\boldsymbol{\omega}})_0^{sK}\Big)^{\mathrm{T}}\mathbf{I}_0^{sK}(\mathbf{J}_{\boldsymbol{\omega}})_0^{sK} \tag{5.38}$$

to

$$\mathbf{M}^{sK}(\mathbf{q}) = \begin{bmatrix} s_2^2 m_K + I_{zz,2}^{sK} & 0 & 0 & 0 & 0 \\ 0 & m_K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{5.39}$$

Thus, the kinetic energy can be calculated to

$$T^{sK} = \frac{1}{2} m_K v_2^2 + \frac{1}{2}\Big(m_K s_2^2 + I_{zz,2}^{sK}\Big)\omega_1^2 . \tag{5.40}$$

Finally, it is assumed for the load that it can be described as a point mass m_L. This means that the inertia matrix of the load vanishes ($\mathbf{I}_5^{sL} = \mathbf{0}$), whereby the rotational part of the kinetic energy of the load also becomes zero. The calculation of the mass matrix is done according to (5.21) by

$$\mathbf{M}^{sL}(\mathbf{q}) = \Big((\mathbf{J}_\mathbf{v})_0^{sL}\Big)^{\mathrm{T}} (\mathbf{J}_\mathbf{v})_0^{sL} m_L \tag{5.41}$$

and the kinetic energy is calculated from $T^{sL} = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}^{sL}(\mathbf{q})\dot{\mathbf{q}}$. The resulting expression is relatively extensive, which is why it is not given in the lecture notes.

Since the position of the centers of gravity $\mathbf{p}_0^{sT}$ and $\mathbf{p}_0^{sK}$ of the tower and the trolley do not change in z_0-direction (i.e. in the direction of gravity) due to the motion, their potential energy is constant and can be chosen to be zero, i.e. $V^{sT} = V^{sK} = 0$. To calculate the potential energy of the load, one defines the unit vector $\mathbf{e}_g^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$ in the direction of gravity and obtains according to (5.20)

$$V^{sL} = -m_L g \mathbf{e}_g^{\mathrm{T}} \mathbf{p}_0^{sL} = m_L g (h_1 - s_5 \cos(\varphi_3)\cos(\varphi_4)) . \tag{5.42}$$

Solution in MAPLE: `Turmdrehkran.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

5.2 Euler-Lagrange Equations

The Euler-Lagrange equations derived in this section allow the determination of the equations of motion of rigid body systems based on the kinetic and potential energy. The starting point of the derivation is the conservation of momentum (3.33) applied to a point mass m in the Cartesian inertial coordinate system $(0_0 x_0 y_0 z_0)$

$$m\ddot{\mathbf{p}} = \mathbf{f}, \tag{5.43}$$

where $\mathbf{f}^{\mathrm{T}} = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}$ denotes the sum of all forces acting on the point mass and $\mathbf{p}^{\mathrm{T}} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}$ denotes the position vector from the coordinate origin 0_0 to the point mass. The position of a point mass whose motion is not subject to any constraints is uniquely determined by specifying the three translational displacements p_x, p_y and p_z with respect to the inertial coordinate system. One then also says that the *point mass* has *3 degrees of freedom*. In contrast, the configuration of a freely movable *rigid body* is described by *6 degrees of freedom*, namely 3 degrees of freedom for the translational displacement and 3 degrees of freedom of rotation to describe the orientation of the rigid body relative to the inertial system.

Now, in general, the motion of a rigid body system is subject to *constraints* that must be taken into account. Consider, for example, the motion of a mass on an inclined plane according to Figure 5.3(a) with the constraint $p_y = a(1 - p_x/b)$ or the spherical pendulum according to Figure 5.3(b) with the constraint

$$p_x^2 + p_y^2 + p_z^2 = l^2 \ . \tag{5.44}$$

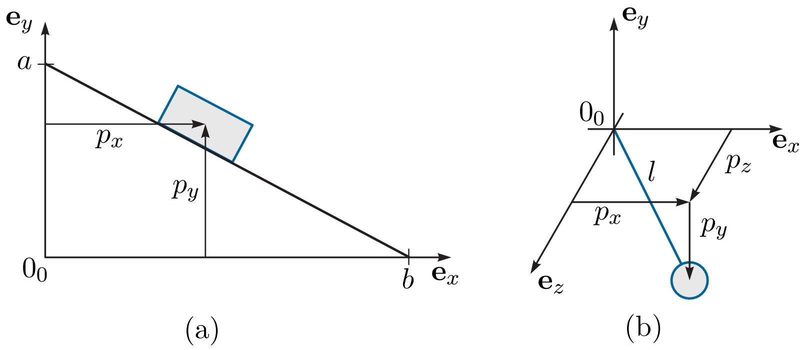


Figure 5.3: Constraints.

Another example are two mass particles i and j of a rigid body, which can be thought of as being connected by a line of fixed length l_{ij}. Thus, the positions $\mathbf{p}_i$ and $\mathbf{p}_j$ of the two mass particles must satisfy the constraint $\|\mathbf{p}_i - \mathbf{p}_j\|_2^2 = (\mathbf{p}_i - \mathbf{p}_j)^{\mathrm{T}}(\mathbf{p}_i - \mathbf{p}_j) = l_{ij}^2$.

If a constraint can be expressed in the form

$$f(\mathbf{p}_1, \mathbf{p}_2, \ldots, t) = 0 \ , \tag{5.45}$$

then one speaks of a *holonomic constraint*. Constraints that cannot be represented in this way are called *nonholonomic*. These include inequality constraints

$$f(\mathbf{p}_1, \mathbf{p}_2, \ldots, t) \geq 0 \ , \tag{5.46}$$

as they occur, for example, when a point mass moves in a spherical shell with radius a in the form $a^2 - \|\mathbf{p}\|_2^2 \geq 0$. Also, constraints that explicitly depend on the velocity and are not integrable, i.e,

$$f(\mathbf{p}_1, \mathbf{p}_2, \ldots, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \ldots, t) = 0 \ , \tag{5.47}$$

are nonholonomic. In some literature, constraints according to (5.45) and (5.47) are also classified as *geometric* and *kinematic constraints*. A typical case of a nonholonomic (kinematic) constraint is the rolling of a disk on a plane.

One can easily see that a system of N point masses that is free of constraint has $3N$ independent coordinates or degrees of freedom. If, for example, there are $(3N-n)$ holonomic constraints of the form

$$f_j(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N, t) = 0, \quad j = 1, \ldots, (3N-n) \;, \tag{5.48}$$

then it is immediately obvious that

(A) the coordinates are no longer linearly independent of each other, and

(B) constraint forces must occur in order to comply with the constraints, which are not known a priori.

With the help of the $(3N-n)$ holonomic constraints, it is now possible to eliminate $(3N-n)$ of the $3N$ coordinates or to introduce n new independent coordinates q_i, $i = 1, \ldots, n$, by which all (old) coordinates can be expressed in the form

$$\mathbf{p}_j = \mathbf{p}_j(q_1, q_2, \ldots, q_n, t) = \mathbf{p}_j(\mathbf{q}, t), \quad j = 1, \ldots, N \;. \tag{5.49}$$

One then also says that the system has n *degrees of freedom* and the n new independent coordinates q_i, $i = 1, \ldots, n$, or $\mathbf{q}^{\mathrm{T}} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ are called *generalized coordinates*.

If one decomposes the forces $\mathbf{f}_i$ acting on the mass particles into *(external) applied forces* $\mathbf{f}_i^{\mathrm{e}}$ and *constraint forces* $\mathbf{f}_i^{\mathrm{z}}$ according to (B), then the equations of motion (5.43) for the system of N point masses are

$$m_i \ddot{\mathbf{p}}_i = \mathbf{f}_i^{\mathrm{e}} + \mathbf{f}_i^{\mathrm{z}}, \quad i = 1, \ldots, N \;. \tag{5.50}$$

Note that (5.48) and (5.50) provide only $(6N-n)$ equations for determining the $6N$ unknowns $\mathbf{p}_i$ and $\mathbf{f}_i^{\mathrm{z}}$, $i = 1, \ldots, N$. If one considers, for example, the frictionlessly sliding mass on the inclined plane according to Figure 5.4(a), then one has two equations of motion and one constraint for the unknown quantities p_x, p_y and f_x^{z}, f_y^{z}. The missing equation is given by the fact that the constraint force $\mathbf{f}^{\mathrm{z}}$ is perpendicular to the inclined plane.

In general, the missing equations are obtained from the *principle of virtual work*. This states that the *sum of the work done by the constraint forces is equal to zero*. Note, however, that this statement is not valid if the constraints are time-dependent, e.g. the inclined plane changes with time. For this reason, we introduce the concept of *virtual displacement* of a system. The system is held fixed at a time t and in this fixed state an arbitrary infinitesimal displacement $\delta \mathbf{p}_i$ compatible with the constraints (5.45) is then performed. For the spherical pendulum of Figure 5.4(b), for example, this means that the following relation

$$(p_x + \delta p_x)^2 + (p_y + \delta p_y)^2 + (p_z + \delta p_z)^2 = l^2 \tag{5.51}$$

must be fulfilled. Taking into account (5.44) and neglecting second-order terms, i.e. $(\delta p_x)^2 = (\delta p_y)^2 = (\delta p_z)^2 = 0$, (5.51) follows to

$$p_x \delta p_x + p_y \delta p_y + p_z \delta p_z = 0 \ . \tag{5.52}$$

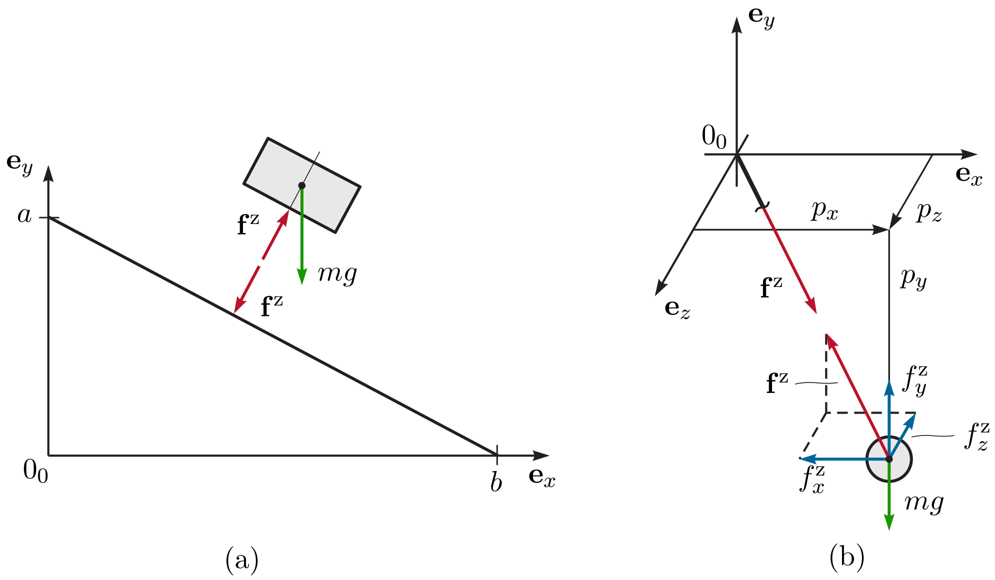


Figure 5.4: Constraint forces.

The *principle of virtual work* now states that the sum of the work δW^z done by the constraint forces $\mathbf{f}_i^z$ during a virtual displacement is equal to zero, i.e., for the system of N point masses

$$\delta W^{\mathrm{z}} = \sum_{i=1}^{N} (\mathbf{f}_i^{\mathrm{z}})^{\mathrm{T}} \delta \mathbf{p}_i = 0 \ . \tag{5.53}$$

Considering again the spherical pendulum of Figure 5.4(b), then according to (5.53) obviously the condition

$$f_x^{\mathrm{z}} \delta p_x + f_y^{\mathrm{z}} \delta p_y + f_z^{\mathrm{z}} \delta p_z = 0 \tag{5.54}$$

must be fulfilled. Now, assuming $p_z \neq 0$, solve (5.52) for δp_z and substitute this into (5.54), this yields

$$\left(f_x^{\mathrm{z}} - \frac{p_x}{p_z} f_z^{\mathrm{z}} \right) \delta p_x + \left(f_y^{\mathrm{z}} - \frac{p_y}{p_z} f_z^{\mathrm{z}} \right) \delta p_y = 0 \tag{5.55}$$

and because of the independence of δp_x and δp_y the conditions

$$f_x^{\mathrm{z}} = \frac{p_x}{p_z} f_z^{\mathrm{z}} \quad \text{and} \quad f_y^{\mathrm{z}} = \frac{p_y}{p_z} f_z^{\mathrm{z}} \tag{5.56}$$

must hold. This means that the constraint force $\mathbf{f}^{\mathrm{z}} = \begin{bmatrix} f_x^{\mathrm{z}} & f_y^{\mathrm{z}} & f_z^{\mathrm{z}} \end{bmatrix}^{\mathrm{T}}$ must point in the direction of the massless rod of length l, compare Figure 5.4(b). In an analogous way, one can show that the constraint force in the case of the frictionlessly sliding mass on the inclined plane is perpendicular to the plane, see Figure 5.4(a).

Usually, one is not interested in the constraint forces, which is why they are calculated from (5.50) and substituted into (5.53). This gives the so-called *D'Alembert's principle* in the form

$$\sum_{i=1}^{N}(m_i\ddot{\mathbf{p}}_i - \mathbf{f}_i^{\mathrm{e}})^{\mathrm{T}}\delta\mathbf{p}_i = 0\ . \tag{5.57}$$

Now, if we assume that the system has n degrees of freedom and can be described according to (5.49) by the generalized coordinates q_j, $j = 1, \ldots, n$, then for the virtual displacement (note that time t is kept constant during the virtual displacement)

$$\delta\mathbf{p}_i = \sum_{j=1}^{n}\frac{\partial\mathbf{p}_i}{\partial q_j}\delta q_j \tag{5.58}$$

and (5.57) follows to

$$\sum_{j=1}^{n}\sum_{i=1}^{N} m_i\ddot{\mathbf{p}}_i^{\mathrm{T}}\frac{\partial\mathbf{p}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} f_{q,j}\delta q_j \tag{5.59}$$

with

$$f_{q,j} = \sum_{i=1}^{N}(\mathbf{f}_i^{\mathrm{e}})^{\mathrm{T}}\frac{\partial\mathbf{p}_i}{\partial q_j}\ . \tag{5.60}$$

Here, $f_{q,j}$, $j = 1, \ldots, n$, denotes a component of the *generalized force* $\mathbf{f}_q = \begin{bmatrix} f_{q,1} & f_{q,2} & \cdots & f_{q,n}\end{bmatrix}^{\mathrm{T}}$. This does not necessarily have the dimension of a force, since the associated generalized coordinate q_j does not necessarily have the dimension of a length (hence the term generalized). However, the product $\dot{q}_j f_{q,j}$ must in any case result in a power.

Now applying the product rule of differentiation to the left side of (5.59) gives

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{p}}_i^{\mathrm{T}}\frac{\partial\mathbf{p}_i}{\partial q_j} = \sum_{i=1}^{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\mathbf{p}}_i^{\mathrm{T}}\frac{\partial\mathbf{p}_i}{\partial q_j}\right) - m_i\dot{\mathbf{p}}_i^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathbf{p}_i}{\partial q_j}\right)\ . \tag{5.61}$$

Using the velocities $\mathbf{v}_i$

$$\mathbf{v}_i = \dot{\mathbf{p}}_i = \sum_{j=1}^{n}\frac{\partial\mathbf{p}_i}{\partial q_j}\dot{q}_j + \frac{\partial\mathbf{p}_i}{\partial t} \tag{5.62}$$

or

$$\frac{\partial\mathbf{v}_i}{\partial\dot{q}_j} = \frac{\partial\mathbf{p}_i}{\partial q_j} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathbf{p}_i}{\partial q_j} = \sum_{k=1}^{n}\frac{\partial^2\mathbf{p}_i}{\partial q_j\partial q_k}\dot{q}_k + \frac{\partial^2\mathbf{p}_i}{\partial q_j\partial t} = \frac{\partial\mathbf{v}_i}{\partial q_j} \tag{5.63}$$

(5.61) follows to

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{p}}_i^{\mathrm{T}}\frac{\partial\mathbf{p}_i}{\partial q_j} = \sum_{i=1}^{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\mathbf{v}_i^{\mathrm{T}}\frac{\partial\mathbf{v}_i}{\partial\dot{q}_j}\right) - m_i\mathbf{v}_i^{\mathrm{T}}\frac{\partial\mathbf{v}_i}{\partial q_j}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_j}T - \frac{\partial}{\partial q_j}T \tag{5.64}$$

with the translational part of the *kinetic energy* T according to (5.6)

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_i^{\mathrm{T}} \mathbf{v}_i \ . \tag{5.65}$$

Substituting (5.64) into (5.59) yields

$$\sum_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}T - f_{q,j}\right)\delta q_j = 0 \ . \tag{5.66}$$

Since the virtual displacements δq_j, $j = 1, \ldots, n$, are independent of each other, one immediately obtains n ordinary differential equations of second order (*Euler-Lagrange equations*), which describe the motion of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}T = f_{q,j}, \quad j = 1, \ldots, n \tag{5.67}$$

with the *generalized coordinates* q_j and the *generalized velocities* $\dot{q}_j$.

> *Note* 5.1. This derivation of the Euler-Lagrange equations can be generalized to rigid body systems, i.e., for bodies performing both translational and rotational motion. Using the total kinetic energy of the rigid body system according to (5.19), one obtains the same expression as in (5.67).

The generalized forces $\mathbf{f}_q$ can be described as the sum of generalized forces, which can be derived from a *scalar potential function* $V(\mathbf{q})$ (see e.g. (5.20)), and from externally applied generalized forces as well as dissipative generalized forces (see Section 3.5), summarized in the vector $\mathbf{f}_q^{np}$. Thus

$$f_{q,j} = f_{q,j}^{np} - \frac{\partial}{\partial q_j}V \tag{5.68}$$

holds. The Euler-Lagrange equations (5.67) can thus be formulated in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j}L - \frac{\partial}{\partial q_j}L = f_{q,j}^{np}, \quad j = 1, \ldots, n, \tag{5.69}$$

with the *Lagrangian* $L = T - V$ (Lagrangian = kinetic energy minus potential energy).

In (5.60) the generalized force $\mathbf{f}_q$ of an external force $\mathbf{f}^{\mathrm{e}}$ was calculated. To generalize this formulation to external forces and torques, consider an external force $\mathbf{f}^{\mathrm{e}}$ or an external torque $\boldsymbol{\tau}^{\mathrm{e}}$. The power supplied by the force $\mathbf{f}^{\mathrm{e}}$ or the torque $\boldsymbol{\tau}^{\mathrm{e}}$ is calculated to

$$P_{\mathbf{f}} = (\mathbf{f}^{\mathrm{e}})^{\mathrm{T}}\mathbf{v}^{\mathrm{e}} \quad \text{or} \quad P_{\boldsymbol{\tau}} = (\boldsymbol{\tau}^{\mathrm{e}})^{\mathrm{T}}\boldsymbol{\omega}^{\mathrm{e}} \tag{5.70}$$

with the corresponding velocity vector $\mathbf{v}^{\mathrm{e}} = \dot{\mathbf{p}}^{\mathrm{e}}$ at the point of application $\mathbf{p}^{\mathrm{e}}$ of the force. Furthermore, $\boldsymbol{\omega}^{\mathrm{e}}$ denotes the vector of the angular velocity of the rigid body on which the external torque acts. Note that the components of $\mathbf{f}^{\mathrm{e}}$ and $\mathbf{v}^{\mathrm{e}}$ or of $\boldsymbol{\tau}^{\mathrm{e}}$ and $\boldsymbol{\omega}^{\mathrm{e}}$ must be expressed with respect to the identical coordinate system. In Section 4.6 it was shown that $\mathbf{v}^{\mathrm{e}}$ and $\boldsymbol{\omega}^{\mathrm{e}}$ can be written using the manipulator Jacobian matrices in the form

$$\mathbf{v}^{\mathrm{e}} = (\mathbf{J}_{\mathbf{v}})^{\mathrm{e}}(\mathbf{q})\dot{\mathbf{q}} \quad \text{and} \quad \boldsymbol{\omega}^{\mathrm{e}} = (\mathbf{J}_{\boldsymbol{\omega}})^{\mathrm{e}}(\mathbf{q})\dot{\mathbf{q}} \ , \tag{5.71}$$

see (4.65). Substituting (5.71) into (5.70), one obtains from

$$P_{\mathbf{f}} = \underbrace{(\mathbf{f}^{\mathrm{e}})^{\mathrm{T}}(\mathbf{J_v})^{\mathrm{e}}(\mathbf{q})}_{\mathbf{f}_{q,\mathbf{f}}^{\mathrm{T}}}\dot{\mathbf{q}} \quad \text{or} \quad P_{\boldsymbol{\tau}} = \underbrace{(\boldsymbol{\tau}^{\mathrm{e}})^{\mathrm{T}}(\mathbf{J}_{\boldsymbol{\omega}})^{\mathrm{e}}(\mathbf{q})}_{\mathbf{f}_{q,\boldsymbol{\tau}}^{\mathrm{T}}}\dot{\mathbf{q}} \tag{5.72}$$

immediately the generalized forces corresponding to $\mathbf{f}^{\mathrm{e}}$ and $\boldsymbol{\tau}^{\mathrm{e}}$

$$\mathbf{f}_{q,\mathbf{f}} = ((\mathbf{J_v})^{\mathrm{e}})^{\mathrm{T}}\mathbf{f}^{\mathrm{e}} \quad \text{and} \quad \mathbf{f}_{q,\boldsymbol{\tau}} = ((\mathbf{J}_{\boldsymbol{\omega}})^{\mathrm{e}})^{\mathrm{T}}\boldsymbol{\tau}^{\mathrm{e}} \ . \tag{5.73}$$

The Euler-Lagrange equations allow a very systematic calculation of the equations of motion based on the kinetic and potential energy of the rigid body system. In the context of control engineering questions, this formulation is also chosen because the energies play an important role in the (nonlinear) control design.

Note 5.2. The Euler-Lagrange equations (5.69) still lead to the correct equations of motion even if the generalized forces do not originate from a potential of the form $V(\mathbf{q})$, but from a generalized potential $\bar{V}(\mathbf{q}, \dot{\mathbf{q}})$ that satisfies the following condition

$$f_{q,j} = -\frac{\partial}{\partial q_j}\bar{V} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_j}\bar{V}\right) \ . \tag{5.74}$$

This is the case, for example, when describing electromagnetic forces on moving charges.

Note 5.3. The Euler-Lagrange equations (5.67) can also be derived via a variational principle, the *Hamilton's principle*. In its integral formulation for conservative systems, this states that the motion of a system between times t_1 and t_2 takes place in such a way that the line integral $\int_{t_1}^{t_2} L \,\mathrm{d}t$ with $L = T - V$ is an extremum for the path traversed or the variation of the integral vanishes. Although this will not be discussed further here, it should be pointed out that this formulation can be formally extended very elegantly to the case of distributed-parameter systems (systems with infinitely many degrees of freedom, described by partial differential equations).

Example 5.3 (Spherical Pendulum). As a simple example, consider the spherical pendulum of Figure 5.5 with the point mass m and the length l as well as an external force $\mathbf{f}^{\mathrm{e}}$ always acting in the direction of the negative $\mathbf{e}_x$ axis. The point mass has three degrees of freedom and the rigid rod of length l gives a holonomic constraint $p_x^2 + p_y^2 + p_z^2 = l^2$. Thus, the spherical pendulum has two degrees of freedom ($n = 2$) and the two angles θ and φ are chosen as generalized coordinates.

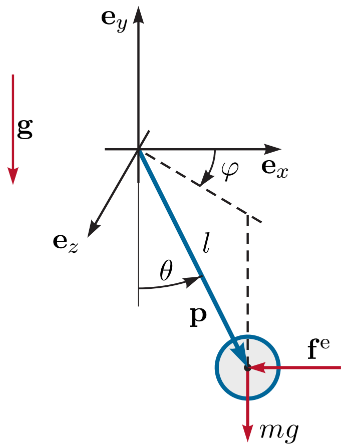


Figure 5.5: Spherical pendulum with external force $\mathbf{f}^{\mathrm{e}}$.

The position vector $\mathbf{p}$ from the origin 0 of the inertial coordinate system $(0_0 x_0 y_0 z_0)$ to the point mass m is calculated in the form

$$\mathbf{p} = \begin{bmatrix} l\sin(\theta)\cos(\varphi) & -l\cos(\theta) & l\sin(\theta)\sin(\varphi) \end{bmatrix}^{\mathrm{T}} . \tag{5.75}$$

The kinetic energy is then obtained according to (5.65) to

$$T = \frac{1}{2} m \dot{\mathbf{p}}^{\mathrm{T}} \dot{\mathbf{p}} = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta) \right) . \tag{5.76}$$

If one defines that for $\theta = 0$ the potential energy V is equal to zero, then with the acceleration due to gravity g the potential energy follows to

$$V = mgl(1 - \cos(\theta)) . \tag{5.77}$$

The external force is $\mathbf{f}^{\mathrm{e}} = \begin{bmatrix} -f_x^{\mathrm{e}} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and therefore the generalized forces according to (5.60) follow to

$$f_\theta = (\mathbf{f}^{\mathrm{e}})^{\mathrm{T}} \frac{\partial \mathbf{p}}{\partial \theta} = -f_x^{\mathrm{e}} l \cos(\theta)\cos(\varphi) \quad , \quad f_\varphi = (\mathbf{f}^{\mathrm{e}})^{\mathrm{T}} \frac{\partial \mathbf{p}}{\partial \varphi} = f_x^{\mathrm{e}} l \sin(\theta)\sin(\varphi) . \tag{5.78}$$

The Euler-Lagrange equations (5.69) can now be formulated with the help of the Lagrangian $L = T - V$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{\theta}} L - \frac{\partial}{\partial \theta} L = f_\theta \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{\varphi}} L - \frac{\partial}{\partial \varphi} L = f_\varphi \tag{5.79}$$

or

$$ml^2 \ddot{\theta} - ml^2 \dot{\varphi}^2 \cos(\theta)\sin(\theta) + mgl\sin(\theta) = -f_x^{\mathrm{e}} l \cos(\theta)\cos(\varphi) \tag{5.80a}$$

$$ml^2 \left(\ddot{\varphi} \sin^2(\theta) + 2\dot{\varphi}\dot{\theta}\cos(\theta)\sin(\theta) \right) = f_x^{\mathrm{e}} l \sin(\theta)\sin(\varphi). \tag{5.80b}$$

As a result of the Euler-Lagrange equations, ordinary differential equations of second order are always obtained which contain the second time derivatives of the generalized coordinates. For a *state space representation* of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbf{x}(t_0) = \mathbf{x}_0$ with the state $\mathbf{x}$ and the input $\mathbf{u}$ according to (1.5), one typically chooses the generalized coordinates q_j, $j = 1, \ldots, n$ and the generalized velocities $\dot{q}_j$, $j = 1, \ldots, n$ as state variables. For the example of the spherical pendulum, the state variables are given by $\mathbf{x} = \begin{bmatrix} \theta & \omega_\theta & \varphi & \omega_\varphi \end{bmatrix}^{\mathrm{T}}$, with $\omega_\theta = \dot{\theta}$, $\omega_\varphi = \dot{\varphi}$ and the input variable by $u = f_x^{\mathrm{e}}$. The system of explicit ordinary differential equations of first order equivalent to (5.80) is then

$$\dot{x}_1 = x_2 \tag{5.81a}$$
$$\dot{x}_2 = \frac{1}{ml^2}\Big(-ul\cos(x_1)\cos(x_3) + ml^2x_4^2\cos(x_1)\sin(x_1) - mgl\sin(x_1)\Big) \tag{5.81b}$$
$$\dot{x}_3 = x_4 \tag{5.81c}$$
$$\dot{x}_4 = \frac{1}{ml^2\sin^2(x_1)}\Big(ul\sin(x_1)\sin(x_3) - 2ml^2x_4x_2\cos(x_1)\sin(x_1)\Big)\ . \tag{5.81d}$$

Solution in MAPLE: `SphaerischesPendel.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Example 5.4 (Ball on Beam). Figure 5.6 shows a ball with mass m_K and radius r_K rolling on a rotatably mounted beam. The moment of inertia of the beam about the axis of rotation (z axis) is I_{zz}^B and the input variable is given by the external torque τ^{e} about the axis of rotation. The system has two mechanical degrees of freedom and the beam angle φ_1 and the distance r of the center of the ball from the y axis of the beam-fixed coordinate system $(0_1x_1y_1z_1)$ are chosen as generalized coordinates.

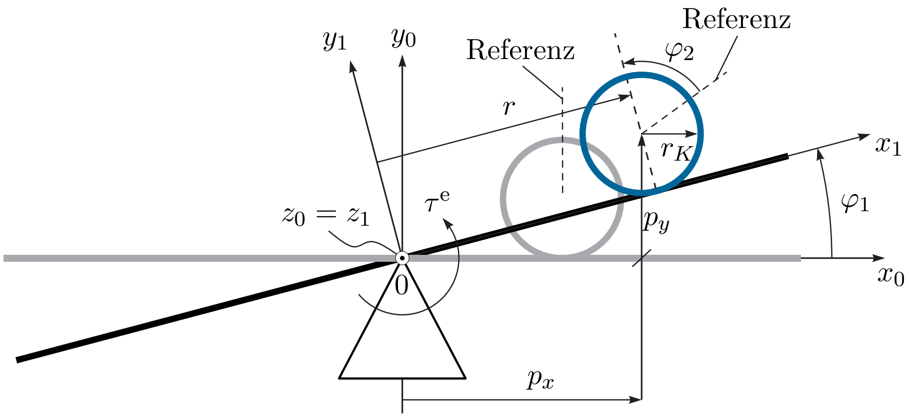


Figure 5.6: Ball on beam.

The kinetic energy of the system is composed of the translational part $T_{t,K}$ and the rotational part $T_{r,K}$ of the ball as well as the rotational part $T_{r,B}$ of the beam. To calculate $T_{t,K}$, the vector from the origin 0 of the inertial coordinate system $(0_0x_0y_0z_0)$ to the center of the ball (center of gravity) is first written in the form

$$\mathbf{p}^{sK} = \begin{bmatrix} r\cos(\varphi_1) - r_K\sin(\varphi_1) \\ r\sin(\varphi_1) + r_K\cos(\varphi_1) \\ 0 \end{bmatrix} . \tag{5.82}$$

The translational part of the kinetic energy of the ball $T_{t,K}$ is then calculated according to (5.6) to

$$T_{t,K} = \frac{1}{2}m_K\left(\dot{\mathbf{p}}^{sK}\right)^{\mathrm{T}}\dot{\mathbf{p}}^{sK} = \frac{1}{2}m_K\left(r^2\dot{\varphi}_1^2 + (\dot{r} - r_K\dot{\varphi}_1)^2\right) . \tag{5.83}$$

For the rotational part of the kinetic energy of the ball $T_{r,K}$, note that the moment of inertia of the ball I_{zz}^{sK} about the axis of rotation (parallel to the z-axis through the center of the ball) according to (3.123) is as follows

$$I_{zz}^{sK} = \frac{2}{5}m_K r_K^2 . \tag{5.84}$$

Now, to calculate the angular velocity of the ball about the axis of rotation (z_0-axis), note that due to the rolling motion of the ball, the relationship

$$\dot{r} = -r_K\dot{\varphi}_2 , \tag{5.85}$$

holds. With respect to the beam-fixed coordinate system $(0_0x_0y_0z_0)$, the ball rotates with the angular velocity $\dot{\varphi}_2$ about the z_1 axis. However, since the beam also rotates about the z_0-axis with the angular velocity $\dot{\varphi}_1$, the effective angular velocity of the ball results from the sum of both rotations to $\dot{\varphi}_1 + \dot{\varphi}_2$. The rotational part of the kinetic energy is then

$$T_{r,K} = \frac{1}{2}I_{zz}^{sK}(\dot{\varphi}_1 + \dot{\varphi}_2)^2 = \frac{1}{2}I_{zz}^{sK}\left(\dot{\varphi}_1 - \frac{\dot{r}}{r_K}\right)^2 . \tag{5.86}$$

The rotational part of the kinetic energy of the beam is calculated to be

$$T_{r,B} = \frac{1}{2}I_{zz}^{sB}\dot{\varphi}_1^2 . \tag{5.87}$$

Assuming that for $\varphi_1 = 0$ the potential energy V is equal to zero, then with the acceleration due to gravity g the potential energy follows to

$$V = m_K g(r\sin(\varphi_1) + r_K\cos(\varphi_1)) - m_K g r_K . \tag{5.88}$$

With the Lagrangian

$$L(\varphi_1, \dot{\varphi}_1, r, \dot{r}) = T_{t,K} + T_{r,K} + T_{r,B} - V \tag{5.89}$$

the equations of motion are obtained (see (5.69))

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{r}} L(\varphi_1, \dot{\varphi}_1, r, \dot{r})\right) - \frac{\partial}{\partial r} L(\varphi_1, \dot{\varphi}_1, r, \dot{r}) = 0 \tag{5.90a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{\varphi}_1} L(\varphi_1, \dot{\varphi}_1, r, \dot{r})\right) - \frac{\partial}{\partial \varphi_1} L(\varphi_1, \dot{\varphi}_1, r, \dot{r}) = \tau^{\mathrm{e}} \tag{5.90b}$$

or

$$\left(m_K + \frac{I_{zz}^{sK}}{r_K^2}\right)\ddot{r} - \left(\frac{I_{zz}^{sK}}{r_K} + m_K r_K\right)\ddot{\varphi}_1 - m_K r \dot{\varphi}_1^2 + m_K g \sin(\varphi_1) = 0 \tag{5.91a}$$

$$\begin{aligned} -\left(m_K r_K + \frac{I_{zz}^{sK}}{r_K}\right)\ddot{r} + \left(I_{zz}^{sK} + I_{zz}^{sB} + m_K\left(r^2 + r_K^2\right)\right)\ddot{\varphi}_1 \\ +2 m_K r \dot{r} \dot{\varphi}_1 + m_K g (r\cos(\varphi_1) - r_K \sin(\varphi_1)) = \tau^{\mathrm{e}} \ . \end{aligned} \tag{5.91b}$$

Solution in MAPLE: `KugelaufBalken.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise 5.1. Bring the system (5.91) into state space representation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbf{x}(t_0) = \mathbf{x}_0$ with the state $\mathbf{x} = \begin{bmatrix} \varphi_1 & \dot{\varphi}_1 = \omega_1 & r & \dot{r} = v \end{bmatrix}^{\mathrm{T}}$ and the input $u = \tau^{\mathrm{e}}$. Furthermore, calculate the stationary equilibrium points of the system.

Solution of exercise 5.1. The equilibrium points of the system are $\varphi_{1,R} = 0$, $\omega_{1,R} = 0$, r_R is arbitrary, $v_R = 0$ and $\tau_R^{\mathrm{e}} = g m_K r_R$.

Exercise 5.2 (Cart with Pendulum). Given is the mechanical system of Figure 5.7. The cart has the mass m_W, is driven by a driving force f^{e} and is attached to the inertial system with a linear spring (spring constant $c_W > 0$, relaxed length s_{W0}). Furthermore, assume that the friction can be approximately expressed by a velocity-proportional force $f_R = -d_R \dot{s}$, $d_R > 0$. The frictionlessly mounted pendulum rod is homogeneous with density ρ_S and cuboid-shaped with length l_S, width b_S and height h_S. Calculate the equations of motion using the Euler-Lagrange equations (5.69).

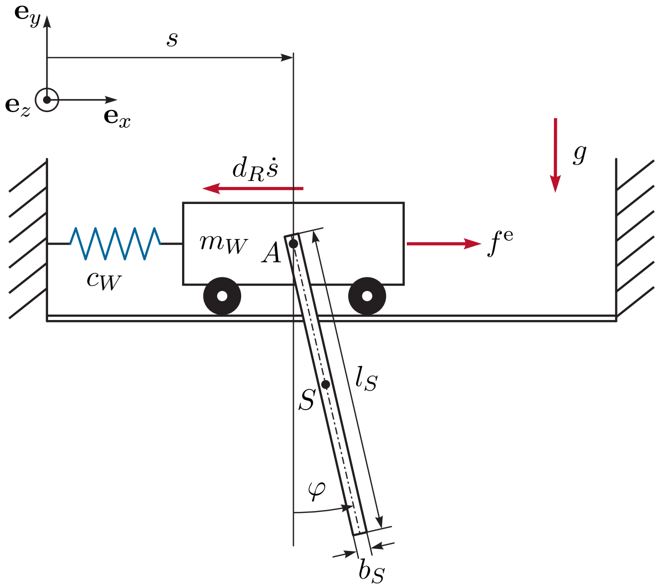


Figure 5.7: Cart with pendulum.

Solution of exercise 5.2. The mass of the pendulum is calculated to be $m_S = \rho_S l_S b_S h_S$ and the moment of inertia about the center of gravity S is $I_{zz}^S = \frac{1}{12} m_S (l_S^2 + b_S^2)$. The equations of motion are obtained to

$$(m_W + m_S)\ddot{s} + \frac{1}{2} m_S l_S \cos(\varphi)\ddot{\varphi} - \frac{1}{2} m_S l_S \sin(\varphi)\dot{\varphi}^2 + c_W(s - s_{W0}) = f^e - d_R\dot{s} \tag{5.92a}$$

$$\frac{1}{2} m_S l_S \cos(\varphi)\ddot{s} + \left(I_{zz}^S + \frac{1}{4} m_S l_S^2\right)\ddot{\varphi} + \frac{1}{2} m_S g l_S \sin(\varphi) = 0\ . \tag{5.92b}$$

Solution in MAPLE: `WagenmitPendel.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

So far it has been shown that the equations of motion of a mechanical rigid body system with the generalized coordinates $\mathbf{q}^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}$ can be calculated from the Euler-Lagrange equations (5.69). Furthermore, it has been shown that for rigid body systems the kinetic energy can be represented using the mass matrix $\mathbf{M}(\mathbf{q})$ in the form (5.21). Finally, the potential energy for rigid body systems is independent of $\dot{\mathbf{q}}$, i.e. $V = V(\mathbf{q})$.

Under these assumptions, the equations of motion (5.69) can be written in matrix notation in the form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{f}_q^{np} \tag{5.93}$$

or in component notation as follows

$$\sum_{j=1}^{n} M_{kj}(\mathbf{q})\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(\mathbf{q})\dot{q}_i\dot{q}_j + g_k(\mathbf{q}) = f_{q,k}^{np}, \quad k = 1,\dots,n \tag{5.94}$$

with the so-called *Christoffel symbols of the first kind*

$$c_{ijk}(\mathbf{q}) = \frac{1}{2}\left(\frac{\partial M_{kj}(\mathbf{q})}{\partial q_i} + \frac{\partial M_{ki}(\mathbf{q})}{\partial q_j} - \frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}\right) . \tag{5.95}$$

As can be seen from (5.93), the equations of motion on the left-hand side contain three different terms: (i) *acceleration terms* in which the second time derivative of the generalized coordinates appears, (ii) terms in which the product $\dot{q}_i\dot{q}_j$ occurs (*centrifugal terms* for $i = j$ and *Coriolis terms* for $i \neq j$) and (iii) the terms of the *potential forces* which only depend on $\mathbf{q}$. The potential forces are calculated directly from the potential energy $V(\mathbf{q})$ via the relationship

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} g_1(\mathbf{q}) & g_2(\mathbf{q}) & \dots & g_n(\mathbf{q}) \end{bmatrix}^{\mathrm{T}} \quad \text{with} \quad g_k(\mathbf{q}) = \frac{\partial V(\mathbf{q})}{\partial q_k} . \tag{5.96}$$

By comparing (5.94) with (5.93) one can see that the (k, j)th element C_{kj} of the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ can be calculated in the form

$$C_{kj}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{n} c_{ijk}(\mathbf{q})\dot{q}_i \tag{5.97}$$

from the Christoffel symbols of the first kind (5.95).

Note 5.4. For $\mathbf{f}_q^{np} = \mathbf{0}$ in (5.69) or (5.93), one speaks of a *conservative system*, a system in which the total energy $E = T + V$ does not change due to the motion or no dissipation occurs in the system. To show that for $\mathbf{f}_q^{np} = \mathbf{0}$ the total energy E is constant, one calculates the time derivative in the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E &= \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}}\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^{\mathrm{T}}\left(\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \left(\frac{\partial V}{\partial \mathbf{q}}\right)^{\mathrm{T}}\right) . \end{aligned} \tag{5.98}$$

Substituting the equation of motion (5.93) gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E &= \dot{\mathbf{q}}^{\mathrm{T}}\left(-\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) + \frac{1}{2}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \left(\frac{\partial V}{\partial \mathbf{q}}\right)^{\mathrm{T}}\right) \\ &= \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} . \end{aligned} \tag{5.99}$$

This expression vanishes because $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$ is a skew-symmetric matrix.

To prove this, write the (j,k)th component of $\mathbf{N}(\mathbf{q},\dot{\mathbf{q}})$ in the form

$$\begin{aligned} N_{jk} &= \sum_i^n \left(\frac{\partial M_{jk}}{\partial q_i} - 2c_{ikj}\right)\dot{q}_i \\ &= \sum_i^n \left(\frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ji}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j}\right)\dot{q}_i \end{aligned} \tag{5.100}$$

which gives

$$N_{jk} = \sum_i^n \left(-\frac{\partial M_{ji}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j}\right)\dot{q}_i \ . \tag{5.101}$$

By interchanging the indices j and k one obtains analogously

$$N_{kj} = \sum_i^n \left(-\frac{\partial M_{ki}}{\partial q_j} + \frac{\partial M_{ij}}{\partial q_k}\right)\dot{q}_i \tag{5.102}$$

and considering the symmetry of the mass matrix $M_{ki} = M_{ik}$ this leads to $N_{jk} = -N_{kj}$. This shows that $\dot{E} = 0$ and thus the total energy E is constant in a conservative rigid body system.

Exercise 5.3 (Rotatory Two-Mass Oscillator). In this example, a rotatory two-mass oscillator as shown in Figure 5.8 is considered. It consists of two rigid bodies with moments of inertia $I_{xx,1}^{s1}$ and $I_{xx,2}^{s2}$, respectively, about the respective axis of rotation in x-direction of the coordinate system 0. It is assumed that the axes of rotation of bodies 1 and 2 simultaneously correspond to a principal axis of inertia, so that no products of inertia occur. The rigid bodies are coupled to each other or to the inertial system by linear torsional springs (stiffnesses c_1 or c_{12}, relaxed positions for $\varphi_1 = 0$ or $\varphi_1 - \varphi_2 = 0$) and viscous rotary dampers (damping constants d_1 or d_{12}). Furthermore, an external torque τ_1 or τ_2 acts on each of the bodies.

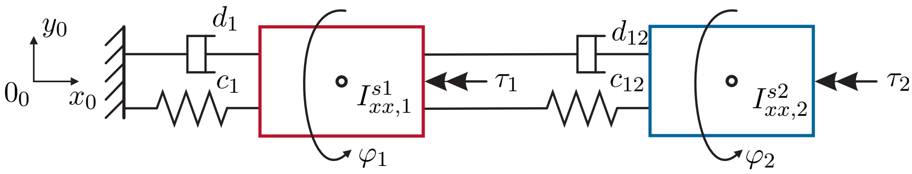


Figure 5.8: Rotatory two-mass oscillator.

Determine the equations of motion of this system in the representation (5.93).

Solution of exercise 5.3. The rotatory kinetic energies of the two rigid bodies result in

$$T_1 = \frac{1}{2} I^{s1}_{xx,1} \omega_1^2 \quad \text{and} \quad T_2 = \frac{1}{2} I^{s2}_{xx,2} \omega_2^2 \tag{5.103}$$

with $\omega_1 = \dot{\varphi}_1$ and $\omega_2 = \dot{\varphi}_2$. The potential energy of the torsional springs is given by

$$V_{c1} = \frac{1}{2} c_1 \varphi_1^2 \quad \text{and} \quad V_{c12} = \frac{1}{2} c_{12} (\varphi_2 - \varphi_1)^2 . \tag{5.104}$$

The torques on the rigid bodies result in

$$\tau_{e1} = -\tau_1 - d_1 \omega_1 - d_{12}(\omega_1 - \omega_2) \tag{5.105a}$$

$$\tau_{e2} = -\tau_2 + d_{12}(\omega_1 - \omega_2) \tag{5.105b}$$

The vector of generalized forces $\mathbf{f}_q^{np}$ due to these torques without potential can then be calculated directly to

$$\mathbf{f}_q^{np} = \begin{bmatrix} \tau_{e1} \\ \tau_{e2} \end{bmatrix} . \tag{5.106}$$

Thus the equations of motion can be specified in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_1 = \omega_1 \tag{5.107a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega_1 = \frac{1}{I^{s1}_{xx,1}} (-c_1 \varphi_1 - d_1 \omega_1 + c_{12}(\varphi_2 - \varphi_1) + d_{12}(\omega_2 - \omega_1) - \tau_1) \tag{5.107b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_2 = \omega_2 \tag{5.107c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega_2 = \frac{1}{I^{s2}_{xx,2}} (-c_{12}(\varphi_2 - \varphi_1) - d_{12}(\omega_2 - \omega_1) - \tau_2) \tag{5.107d}$$

If a representation according to (5.93) is used to calculate the equations of motion, then the mass matrix $\mathbf{M}$, the Coriolis matrix $\mathbf{C}$ and the vector of potential forces $\mathbf{g}$ are calculated to be

$$\mathbf{M} = \begin{bmatrix} I^{s1}_{xx,1} & 0 \\ 0 & I^{s2}_{xx,2} \end{bmatrix}, \quad \mathbf{C} = \mathbf{0}, \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} c_1 \varphi_1 + c_{12}(\varphi_1 - \varphi_2) \\ -c_{12}(\varphi_1 - \varphi_2) \end{bmatrix} . \tag{5.108}$$

Solution in MAPLE: `RotatorischerZweimassenschwinger.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Example 5.5 (Continuation Planar Manipulator). This example continues the planar manipulator from Example 5.1. Starting from the mass matrix and the potential energy of the system, the equations of motion of the system are now determined. It is now assumed that motors are installed in both joints, which introduce the torques τ_1 and τ_2 about the z-axis of the joints. Furthermore, it is assumed that viscous friction with the friction coefficient d_2 occurs in the second joint.

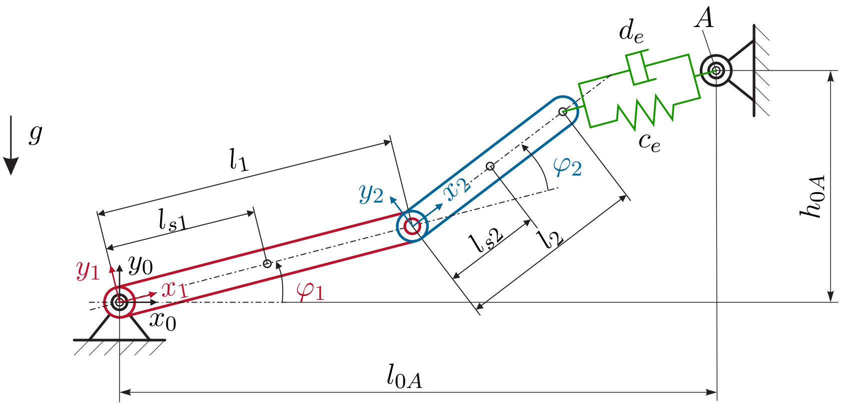


Figure 5.9: Sketch of the planar manipulator.

In the first step, the vector of generalized forces without potential is determined. The force of the damper at the end of the arm can be specified using $\mathbf{r}_e$ from (5.29) in the form

$$\mathbf{f}_{de} = d_e \dot{\mathbf{r}}_e = d_e \begin{bmatrix} l_2 \sin(\varphi_1 + \varphi_2)(\omega_1 + \omega_2) + l_1 \sin(\varphi_1)\omega_1 \\ -l_2 \cos(\varphi_1 + \varphi_2)(\omega_1 + \omega_2) - l_1 \cos(\varphi_1)\omega_1 \\ 0 \end{bmatrix} . \tag{5.109}$$

With the manipulator Jacobian matrix $(\mathbf{J_v})_0^{e2}$ the corresponding vector of generalized forces is calculated to

$$\mathbf{f}_{q,de} = \left((\mathbf{J_v})_0^{e2}\right)^{\mathrm{T}} \mathbf{f}_{de} = d_e \begin{bmatrix} -l_1^2\omega_1 - l_1 l_2 (2\omega_1 + \omega_2)\cos(\varphi_2) - l_2^2(\omega_1 + \omega_2) \\ -l_1 l_2 \omega_1 \cos(\varphi_2) - l_2^2(\omega_1 + \omega_2) \end{bmatrix} . \tag{5.110}$$

The vector of generalized forces due to the torques of the motors $\boldsymbol{\tau}_1^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & \tau_1 \end{bmatrix}$, $\boldsymbol{\tau}_2^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & \tau_2 \end{bmatrix}$ is obtained using the manipulator Jacobian matrices from Example 4.3 to

$$\mathbf{f}_{q,1} = \left((\mathbf{J}_{\boldsymbol{\omega}})_0^1\right)^{\mathrm{T}} \boldsymbol{\tau}_1 = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}, \quad \mathbf{f}_{q,2} = -\left((\mathbf{J}_{\boldsymbol{\omega}})_0^1\right)^{\mathrm{T}} \boldsymbol{\tau}_2 + \left((\mathbf{J}_{\boldsymbol{\omega}})_0^2\right)^{\mathrm{T}} \boldsymbol{\tau}_2 = \begin{bmatrix} 0 \\ \tau_2 \end{bmatrix} . \tag{5.111}$$

It must be taken into account in the calculation of $\mathbf{f}_{q,2}$ that the torque $\boldsymbol{\tau}_2$ acts on rod 1 and rod 2 according to the cutting principle. In the same way, the vector of the generalized force due to the viscous friction $\boldsymbol{\tau}_{d2}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & -d_2\omega_2 \end{bmatrix}$ is calculated to be $\mathbf{f}_{q,d2}^{\mathrm{T}} = \begin{bmatrix} 0 & -d_2\omega_2 \end{bmatrix}$.

Thus, all intermediate quantities for the determination of the equations of motion with the help of (5.93) are available. The results are not given due to the relatively extensive expressions. For checking, the Coriolis matrix $\mathbf{C}$ is given

$$\mathbf{C} = \begin{bmatrix} -m_2 l_1 l_{s2} \sin(\varphi_2)\omega_2 & -m_2 l_1 l_{s2} \sin(\varphi_2)(\omega_1 + \omega_2) \\ -m_2 l_1 l_{s2} \sin(\varphi_2)(\omega_1) & 0 \end{bmatrix} . \tag{5.112}$$

Solution in MAPLE: `Planarer_Manipulator.mw`
In this Maple file all calculation steps and the intermediate and final results are shown. Furthermore, you will find here a representation of the numerical solution of the equations of motion as well as a calculation of the equilibrium points of the system.
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise 5.4 (Continuation Tower Crane 2). This Exercise again considers the tower crane from Exercises 4.2, 4.4 and Example 5.2. It is now assumed that the tower (angle φ_1) is driven by a motor with the torque τ_1. Furthermore, the trolley (position s_2) is actuated by a motor with the force f_{LK} and the rope (position s_5) by a motor with the force f_S. For all degrees of freedom, viscous friction is assumed with the damping coefficients d_T of the tower, d_{LK} of the trolley, d_{St} of the translational motion of the rope (s_5) and d_{Sr} of the rotational motion of the rope (φ_3 and φ_4).

Calculate the equations of motion of the system! Determine all equilibrium points of the system and analyze the dynamic behavior of the tower crane by numerical simulation in MAPLE!

Solution of exercise 5.4.

Solution in MAPLE: `Turmdrehkran.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

5.3 References

[5.0] M. Calkin, *Lagrangian and Hamiltonian Mechanics*. Singapore: World Scientific, 1996.

[5.0] H. Goldstein, *Klassische Mechanik*. Wiesbaden: AULA-Verlag, 1991.

[5.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 1: Theoretical Basis*. Berlin: Springer, 2002.

[5.0] H. Hahn, *Rigid Body Dynamics of Mechanisms: Part 2: Applications*. Berlin: Springer, 2002.

[5.0] B. Heimann, W. Gerth, and K. Popp, *Mechatronik: Komponenten - Methoden - Beispiele*. Leipzig: Fachbuchverlag, 2003.

[5.0] R. Isermann, *Mechatronische Systeme: Grundlagen*. Berlin Heidelberg: Springer, 2008.

[5.0] A. Shabana, *Dynamics of Multibody Systems*. Cambridge: Cambridge University Press, 1998.

[5.0] B. Siciliano, L. Sciavicco, L. Villani, and G. Oriolo, *Robotics: Modelling, Planning and Control*. London, UK: Springer, 2009.

[5.0] M. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: John Wiley & Sons, 1989.

[5.0] P. Wellstead, *Physical System Modelling*. London: Academic Press, 1979.

A Exercises

This appendix contains several exercises and their solutions to support self-study.

Exercise A.1 (Drawbridge). To enable crossing the channel as well as the passage of ships, the bridge shown in Figure A.1 can be raised by pulling on rope 2. The bridge with mass m_B and center of gravity S_B is rotatably mounted at point B. The girder with mass m_T is rotatably mounted at point A, outside its center of gravity S_T. The length of rope 1 is chosen so that the girder and bridge are always oriented parallel to each other. To reduce the tensile force in rope 2, a counterweight m is mounted at the left end of the girder. Rope 2 wraps around the cylindrical part of the girder with radius R and is fixed at point C. For the following considerations, both ropes are assumed to be massless and all bearings are assumed to be ideally frictionless. The acceleration due to gravity g acts as shown in Figure A.1.

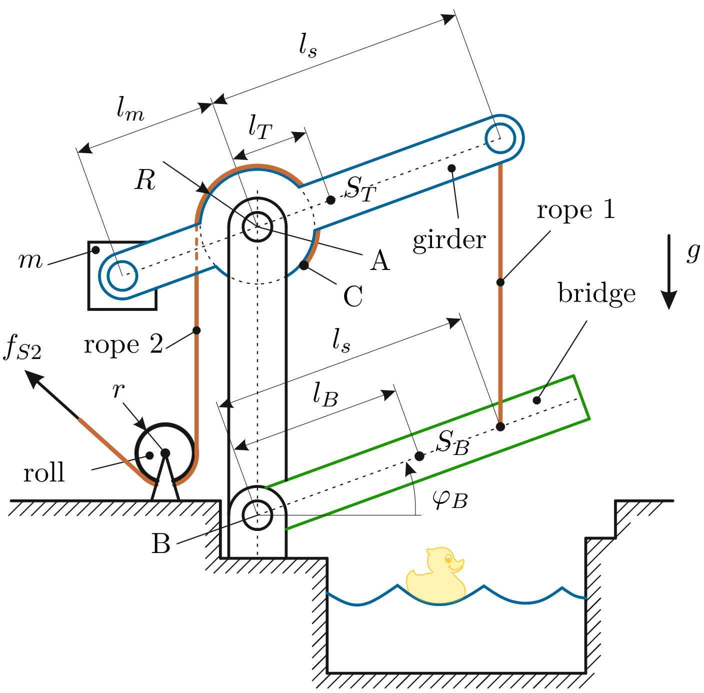


Figure A.1: Drawbridge.

The aim is to determine the bearing forces at A and B as well as the tensile forces in ropes 1 and 2 such that the system is in equilibrium. Based on these results, the mass m should be determined so that the system is in equilibrium for $f_{S2} = 0$.

Solution of exercise A.1. From Figure A.1 it can be seen that the bridge and the girder only influence each other via rope 1. If the bridge is cut free as shown in Figure A.2, the only force acting on it, besides the bearing forces at B and the gravitational force $m_B g$, is the rope force f_{S1}.

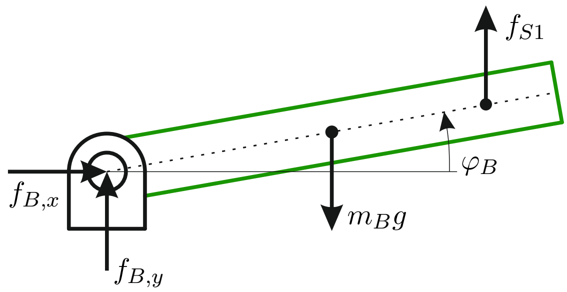


Figure A.2: Free-body diagram of the bridge.

With Figure A.2, the force balances in the x- and y-directions are obtained as

$$\mathbf{e}_x : 0 = f_{B,x} \tag{A.1a}$$

$$\mathbf{e}_y : 0 = f_{B,y} - m_B g + f_{S1}. \tag{A.1b}$$

From the force balance, two equations are obtained for the three unknowns $f_{B,x}$, $f_{B,y}$ and f_{S1}. To uniquely determine these three quantities, a third equation in the form of the torque balance is therefore necessary. A reference point must be chosen for the formulation of the torque balance. In order to minimize the number of unknowns and thus the effort for solving the resulting system of equations, it is advisable to choose bearing B as the reference point (analogously, the point of application of the rope force f_{S1} could have been chosen). The torque balance about the z-axis is obtained as

$$\mathbf{e}_z : 0 = -m_B g l_B \cos(\varphi_B) + f_{S1} l_s \cos(\varphi_B). \tag{A.2}$$

Simple rearrangement of the force and torque balance leads to

$$f_{S1} = m_B g \frac{l_B}{l_s} \tag{A.3}$$

for the tensile force in the rope and

$$\mathbf{f}_B = \begin{bmatrix} f_{B,x} \\ f_{B,y} \end{bmatrix} = \begin{bmatrix} 0 \\ m_B g\left(1 - \frac{l_B}{l_s}\right) \end{bmatrix} \tag{A.4}$$

for the bearing force at point B.

To calculate the bearing force $\mathbf{f}_A$ at point A and the tensile force f_{S2} in rope 2, the girder is now cut free according to Figure A.3.

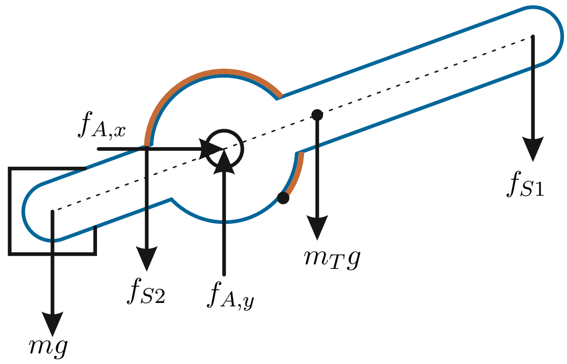


Figure A.3: Free-body diagram of the girder.

From the sketch, the force balances can be immediately read as

$$\mathbf{e}_x : 0 = f_{A,x} \tag{A.5a}$$

$$\mathbf{e}_y : 0 = -mg - f_{S2} + f_{A,y} - m_T g - f_{S1} \tag{A.5b}$$

Since f_{S1} is already known from the previous calculations, two equations are obtained for three unknowns, as was the case for the bridge. To calculate $f_{A,x}$, $f_{A,y}$ and f_{S2}, it is therefore again necessary to formulate the torque balance. Again, it is advantageous to choose the bearing as the reference point for the torque balance. The torque balance about point A in the z-direction is obtained as

$$\mathbf{e}_z : 0 = mgl_m \cos(\varphi_B) + f_{S2}R - m_T g l_T \cos(\varphi_B) - f_{S1} l_s \cos(\varphi_B). \tag{A.6}$$

Rearranging the force and torque balance yields

$$f_{S2} = \frac{m_B g l_B + m_T g l_T - m g l_m}{R} \cos(\varphi_B) \tag{A.7}$$

and

$$\mathbf{f}_A = \begin{bmatrix} 0 \\ m_B g\left(\frac{l_B}{l_s} + \frac{l_B}{R}\cos(\varphi_B)\right) + m_T g\left(1 + \frac{l_T}{R}\cos(\varphi_B)\right) - mg\left(1 + \frac{l_m}{R}\cos(\varphi_B)\right) \end{bmatrix}. \tag{A.8}$$

If the girder is now to be in equilibrium for $f_{S2} = 0$, the torque balance gives

$$m = \frac{m_B l_B + m_T l_T}{l_m}. \tag{A.9}$$

Solution in MAPLE: `Klappbruecke.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.2 (Rotating Rigid Body). In Fig. A.4 a rigid body system is shown, consisting of a frame (homogeneous mass density ρ, thickness d, width b) that is rotatably mounted at point A and a mass m_m (point mass). The frame is connected to the ground at its left end by a linear spring with stiffness c. The system is subjected to gravity g in the negative $\mathbf{e}_y$ direction.

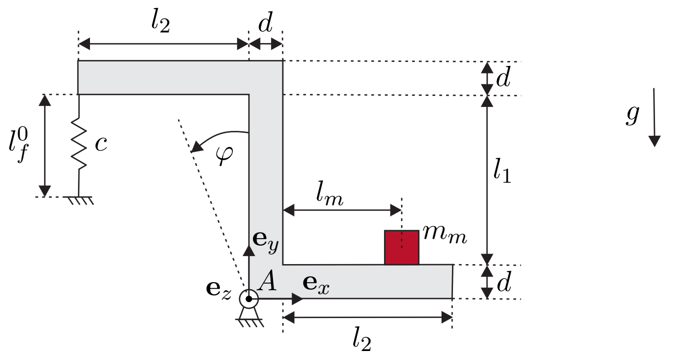


Figure A.4: Sketch of a rotatably mounted rigid body system.

For this system, the support forces and the preload of the spring are to be determined for the case $\varphi = 0$ such that the system is in equilibrium. Furthermore, the equations of motion for the rotational motion of the rigid body system are to be determined using the principle of conservation of angular momentum.

Solution of exercise A.2. In the first step, the support forces and the necessary spring preload are determined for $\varphi = 0$. For this purpose, the support (at point A) must be removed and replaced conceptually by the forces and toruqes occurring at the support. The support shown in Fig. A.4 does not allow movement in the x- and y-directions, but only a rotation about the z-axis. Therefore, this support can be equivalently replaced by the forces f_x and f_y, see Fig. A.5. The effect of the spring is replaced by the spring force f_c.

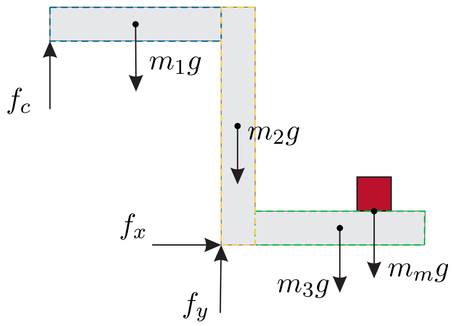


Figure A.5: Sketch of a rotatably mounted rigid body system.

To take into account the effect of gravity, the gravitational force $m_g g$, with the total mass m_g, must be applied at the center of gravity of the rigid body system. For efficient calculation, it proves useful to divide the entire rigid body system into four sub-bodies, see Fig. A.5. For each of these sub-bodies, the position of the center of gravity, the mass, and the gravitational force can then be easily determined.

If we first consider the sub-body framed in blue, its mass is calculated from (3.24) to be

$$m_1 = \int_{\mathcal{V}} \rho \, \mathrm{d}\mathcal{V} = l_2 d b \rho. \tag{A.10}$$

The center of gravity of the body framed in blue is calculated according to (3.28). For the position of the center of gravity in the x direction, one obtains according to (3.29), under the initially assumed condition $\varphi = 0$,

$$r_{Sm1,x} = \frac{1}{m_1} \int_{z=-b/2}^{b/2} \int_{y=l_1+d}^{l_1+2d} \int_{x=-l_2}^{0} x\rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = -\frac{l_2}{2}. \tag{A.11}$$

Remark: Of course, the position of the center of gravity of the homogeneous cuboid of mass m_1 can be read directly from the sketch without evaluating these integrals!

In an analogous manner, one obtains the position of the center of gravity in the y- and z-directions. Thus, we have

$$\mathbf{r}^0_{Sm1} = \begin{bmatrix} -\frac{1}{2} l_2 \\ l_1 + \frac{3}{2} d \\ 0 \end{bmatrix}, \tag{A.12}$$

where the index 0 has been used to denote the case $\varphi = 0$.

The masses of sub-bodies 2 and 3 are $m_2 = d(l_1 + 2d)b\rho$ and $m_3 = l_2 d b \rho$ and the positions of their centers of gravity can be determined to be

$$\mathbf{r}_{Sm2}^0 = \begin{bmatrix} \frac{1}{2}d \\ d+\frac{1}{2}l_1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_{Sm3}^0 = \begin{bmatrix} d+\frac{1}{2}l_2 \\ \frac{1}{2}d \\ 0 \end{bmatrix}, \quad \mathbf{r}_{Smm}^0 = \begin{bmatrix} d+l_m \\ d \\ 0 \end{bmatrix}. \tag{A.13}$$

Note that the mass m_m is modeled as a point mass acting directly on the rotating rigid body, as described in the problem statement.

For the further calculations, we still need the point of application $\mathbf{r}_{fc}^0$ of the spring force f_c. This is given by

$$\mathbf{r}_{fc}^0 = \begin{bmatrix} -l_2 \\ d+l_1 \\ 0 \end{bmatrix}. \tag{A.14}$$

For the system to be in equilibrium according to Fig. A.5, the force equilibrium and the torque equilibrium according to (3.12) must be satisfied. The force equilibrium in the x-direction yields

$$\mathbf{e}_x : \quad f_x = 0, \tag{A.15}$$

i.e., the support force in the x-direction must vanish. For the y-direction we obtain

$$\mathbf{e}_y : \quad f_c + f_y - m_1 g - m_2 g - m_3 g - m_m g = 0. \tag{A.16}$$

To establish the torque equilibrium for a rotation about the z-axis, a possible pivot point must be chosen. In the system under consideration, it is natural to choose point A, i.e., the actual pivot point of the system. Note, however, that the torque equilibrium must hold for any freely selectable pivot point of the free-body diagram shown in Fig. A.5. By appropriately choosing the pivot point, it can be achieved that certain unknown forces do not appear in the torque equilibrium, which can significantly simplify the calculation.

If point A is chosen as the pivot point in the considered system, it is recognized that the forces f_x and f_y do not contribute to the torque about this point. If, on the other hand, the point of application of the spring force were chosen as the pivot point, then f_c would not appear in the torque equilibrium.

For the chosen pivot point A, the torque $\tau_{m1}^{(A)}$ is obtained by applying (3.10) to

$$\boldsymbol{\tau}_{m1}^{(A),0} = \mathbf{r}_{Sm1}^0 \times \mathbf{f}_{Sm1} = \begin{bmatrix} -\frac{1}{2}l_2 \\ l_1+\frac{3}{2}d \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m_1 g \frac{1}{2} l_2 \end{bmatrix}, \tag{A.17}$$

and thus $\tau_{m1,z}^{(A),0} = m_1 g \frac{1}{2} l_2$. Again, the index 0 denotes the case $\varphi = 0$. The corresponding torques of the other partial masses or the spring force are $\tau_{m2,z}^{(A),0} = -m_2 g d/2$, $\tau_{m3,z}^{(A),0} = -m_3 g(d + l_2/2)$, $\tau_{mm,z}^{(A),0} = -m_m g(d + l_m)$ and $\tau_{fc,z}^{(A),0} = -f_c l_2$.

Remark: Of course, for the considered case, where all forces are directed either in the x- or y-direction, an evaluation of the cross product is not absolutely necessary. Instead, one can simply use the rule force times lever arm to determine the resulting torque. However, one must pay attention to the correct sign of the torque when using this procedure.

If we now apply the torque equilibrium, we immediately obtain

$$f_c = \frac{1}{l_2}\left(m_1 g \frac{l_2}{2} - m_2 g \frac{d}{2} - m_3 g\left(d + \frac{l_2}{2}\right) - m_m g(d + l_m)\right). \tag{A.18}$$

The necessary spring preload l_{f0} can be easily calculated with the length l_f^0 of the spring for the case $\varphi = 0$ from the equation

$$f_c = -c\left(l_f^0 - l_{f0}\right), \tag{A.19}$$

with f_c from (A.18). Substituting the solution for f_c into the force equilibrium in the y-direction yields the support force f_y.

To determine the equations of motion of the system, the degrees of freedom of the system must be identified. From Fig. A.4 it can be seen that the bearing at point A only allows rotation about the z-axis, so that a rotation by the angle φ represents the only degree of freedom of the system. The motion of the system can thus be described directly in terms of the conservation of angular momentum according to (3.119). This requires the effective torque of inertia of the system about the pivot point A and the torques acting about this point.

To determine the moment of inertia $I_{zz}^{(A)}$ about the pivot point, the moments of inertia of the sub-bodies according to Fig. A.5 are first determined about their respective centers of gravity. According to (3.124), one obtains for sub-body 1

$$I_{zz,m1}^{(S)} = \rho \int_{\tilde{z}=-b/2}^{b/2} \int_{\tilde{y}=-d/2}^{d/2} \int_{\tilde{x}=-l_2/2}^{l_2/2} \left(\tilde{x}^2 + \tilde{y}^2\right) \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{y}\, \mathrm{d}\tilde{z} = \frac{m_1}{12}\left(l_2^2 + d^2\right), \tag{A.20}$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ denote the distances from the center of gravity of sub-body 1. Using Steiner's theorem (3.126), the moment of inertia of sub-body 1 about the pivot point A can be determined as

$$I_{zz,m1}^{(A)} = I_{zz,m1}^{(S)} + m_1\left(r_{Sm1,x}^2 + r_{Sm1,y}^2\right). \tag{A.21}$$

The moments of inertia of the other sub-bodies can be obtained in an analogous manner. Since the mass m_m is modeled as a point mass, we have $I_{zz,mm}^{(S)} = 0$. Note, however, that due to Steiner's theorem, this does not imply $I_{zz,mm}^{(A)} = 0$, but rather $I_{zz,mm}^{(A)} = m_m(r_{Smm,x}^2 + r_{Smm,y}^2)$!

Remark: It would be possible to directly determine the moment of inertia $I_{zz}^{(A)}$ by using the distances to the pivot point instead of the distances $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. However, in general, this approach leads to significantly more complicated expressions for the integrals. Furthermore, the moments of inertia about the center of gravity are available in tables for many geometric bodies and can be adopted directly.

As the second part of the angular momentum balance, the sum of the torques is required. To determine this, the position of the centers of gravity and the point of application of the spring force must be described as a function of the angle φ. Starting from the position of the center of gravity $\mathbf{r}_{Sm1}^0$ for $\varphi = 0$, one obtains from geometrical considerations (or by using the rotation matrix of a rotation about the z-axis)

$$\mathbf{r}_{Sm1}(\varphi) = \begin{bmatrix} r_{Sm1,x}^0 \cos(\varphi) - r_{Sm1,y}^0 \sin(\varphi) \\ r_{Sm1,x}^0 \sin(\varphi) + r_{Sm1,y}^0 \cos(\varphi) \\ 0 \end{bmatrix}. \tag{A.22}$$

With this result, the torque $\tau_{m1,z}^{(A)}(\varphi)$ results from

$$\boldsymbol{\tau}_{m1}^{(A)}(\varphi) = \mathbf{r}_{Sm1}(\varphi) \times \mathbf{f}_{Sm1} = \begin{bmatrix} 0 \\ 0 \\ -\Big(r_{Sm1,x}^0 \cos(\varphi) - r_{Sm1,y}^0 \sin(\varphi)\Big) m_1 g \end{bmatrix}. \tag{A.23}$$

The torques $\tau_{m2,z}^{(A)}(\varphi)$, $\tau_{m3,z}^{(A)}(\varphi)$ and $\tau_{mm,z}^{(A)}(\varphi)$ can be determined in the same way.

The determination of the torque due to the spring force f_c, which changes as a function of the angle φ due to the change in length of the spring, is somewhat more difficult. The point of application of the spring results from the above considerations as

$$\mathbf{r}_{fc}(\varphi) = \begin{bmatrix} r_{fc,x}^0 \cos(\varphi) - r_{fc,y}^0 \sin(\varphi) \\ r_{fc,x}^0 \sin(\varphi) + r_{fc,y}^0 \cos(\varphi) \\ 0 \end{bmatrix}. \tag{A.24}$$

To determine the current length of the spring, the base point $\mathbf{r}_{fcfp}$ is determined in the form

$$\mathbf{r}_{fcfp} = \begin{bmatrix} -l_2 \\ d + l_1 - l_f^0 \\ 0 \end{bmatrix} \tag{A.25}$$

and the length of the spring can be determined from

$$l_f(\varphi) = \|\mathbf{r}_{fc}(\varphi) - \mathbf{r}_{fcfp}\|_2. \tag{A.26}$$

The spring force is of course directed along the connecting line between the base point and the point of application, so that one immediately obtains

$$\mathbf{f}_c(\varphi) = c(l_f(\varphi) - l_{f0})\frac{\mathbf{r}_{fc}(\varphi) - \mathbf{r}_{fcfp}}{l_f(\varphi)}. \tag{A.27}$$

The associated torque about the pivot point A is then given by $\boldsymbol{\tau}_{fc}^{(A)}(\varphi) = \mathbf{r}_{fc}(\varphi) \times \mathbf{f}_c(\varphi)$.

With these intermediate results, the angular momentum balance for the rotation about point A can now be stated:

$$I_{zz}^{(A)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi = \tau_z^{(A)}. \tag{A.28}$$

Here, $I_{zz}^{(A)}$ denotes the sum of all moments of inertia and $\tau_z^{(A)}$ is the sum of all torques about point A, cf. (3.119).

Solution in Maple: `DrehbarerStarrkoerper.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.3 (Holding Device). In a holding device according to Fig. A.6, a plate is clamped in a housing with the help of a small roller. The plate has the mass m and the width b. The roller with the diameter d has a negligible mass. At the contact point between the roller and the housing A as well as at the contact point between the roller and the plate B, the coefficient of static friction μ_H occurs. Between the plate and the housing wall, there is an ideally smooth (frictionless) contact. The angle α is known.

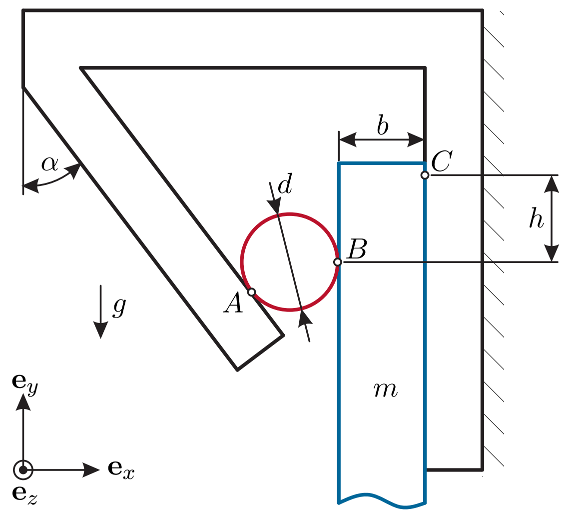


Figure A.6: Structure of a holding device.

The minimum required coefficient of static friction μ_H and the minimum clamping length h of the plate are sought, so that it is fixed in the holding device. In addition, the maximum permissible mass of the plate, which can be fixed by the friction in the device, is sought.

Solution of exercise A.3. Under the assumption that the plate is fixed in the device, the considered system is in static equilibrium. Based on the force and torque balances for the roller and the plate, the required coefficient of friction and the required clamping length can be calculated.

The first step is to free the roller, the plate, and the housing. Fig. A.7 shows the free body diagrams and the forces acting on them. The force at points A and B is composed of a normal component $f_{A,n}$ and $f_{B,n}$, respectively, as well as a tangential component due to static friction $f_{A,t}$ and $f_{B,t}$, respectively. Due to the ideally smooth contact of the plate and the wall, there is no friction between these two bodies. As a result, only the normal force $f_{C,n}$ acts at point C.

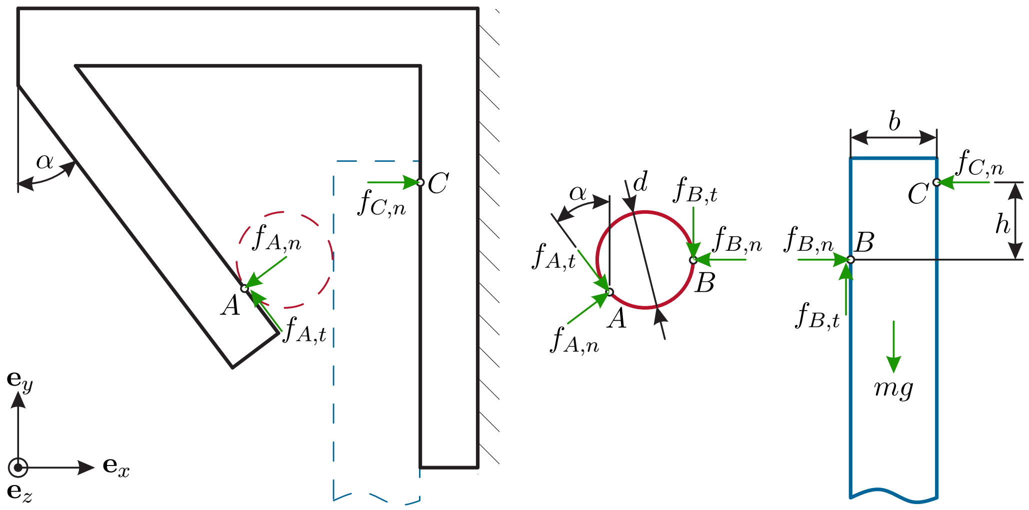


Figure A.7: Free body diagram of the housing, roller and plate of the holding device and the forces acting on them.

The force balance on the roller results in

$$\mathbf{e}_x : f_{A,n}\cos(\alpha) + f_{A,t}\sin(\alpha) - f_{B,n} = 0 \tag{A.29a}$$

$$\mathbf{e}_y : f_{A,n}\sin(\alpha) - f_{A,t}\cos(\alpha) - f_{B,t} = 0 \tag{A.29b}$$

for the x-direction and for the y-direction, respectively, under the angle α. With the center of the roller as the reference point, the torque balance in the z-direction results in

$$\mathbf{e}_z : f_{A,t}\frac{d}{2} - f_{B,t}\frac{d}{2} = 0 \tag{A.30}$$

directly $f_{A,t} = f_{B,t}$. Substituting this relationship into (A.29b) and then rearranging yields the relationship between the normal and frictional force at point A to

$$f_{A,t} = \frac{\sin(\alpha)}{1+\cos(\alpha)} f_{A,n}. \tag{A.31}$$

A comparison with (3.77) shows that the fraction in (A.31) can be interpreted as the coefficient of friction. In order for the roller and also the plate to adhere, the static friction coefficient μ_H at points A and B must be greater than the coefficient of friction from (A.31). This results in the condition

$$\mu_H \geq \frac{\sin(\alpha)}{1+\cos(\alpha)} \tag{A.32}$$

for the static friction coefficient.

Analogous to the previous approach, the determination of the minimum clamping length h is carried out via the force and torque balance of the plate. The force balance in the x-direction and in the y-direction is

$$\mathbf{e}_x : f_{B,n} - f_{C,n} = 0 \tag{A.33a}$$
$$\mathbf{e}_y : f_{B,t} - mg = 0. \tag{A.33b}$$

The torque balance in the z-direction with the reference point B results in

$$\mathbf{e}_z : f_{C,n}h - mg\frac{b}{2} = 0. \tag{A.34}$$

From equation (A.33b) it can be seen that only the friction between the roller and the plate counteracts the weight of the plate. Furthermore, it follows from (A.30)

$$f_{B,t} = f_{A,t} = mg. \tag{A.35}$$

An analogous relationship results for the normal forces. Substituting equations (A.29a) and (A.31) into (A.33a) yields

$$f_{C,n} = f_{B,n} = f_{A,n} = \frac{1+\cos(\alpha)}{\sin(\alpha)}mg. \tag{A.36}$$

By substituting this normal force into the torque balance (A.34), the minimum clamping length of the plate follows after rearranging to

$$h = \frac{b\sin(\alpha)}{2(1+\cos(\alpha))}. \tag{A.37}$$

From equations (A.32) and (A.37) it can be seen that neither the static friction coefficient nor the clamping length depends on the mass of the plate. This property of the holding device is also known as self-locking. Theoretically, the mass of the plate can be arbitrarily large. This consideration is valid as long as there is no mechanical deformation of the individual rigid bodies due to the acting forces.

Solution in MAPLE: `Haltevorrichtung.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.4 (Quarter-car vertical model). Figure A.8 shows a quarter-car vertical model. The wheel is modeled by a spring-damper system consisting of a spring (stiffness c_R) and a damper (viscous damping d_R). The wheel mass is given by m_R, and the vehicle body is modeled as a lumped mass m_A. The suspension is described by a spring (stiffness c_A) and a damper (velocity-dependent damping coefficient $d_A(v_{RA})$ with $v_{RA} = v_R - v_A$). For $x_A = x_R = x_U = 0$, all springs are unstretched. To describe road unevenness, the ground is parameterized by the displacement $x_U(t)$. The vertical coordinates of the wheel and the body are defined by x_R and x_A, respectively.

Determine the equations of motion and the static deflections for $x_U(t) = 0$ for the masses m_A and m_R!

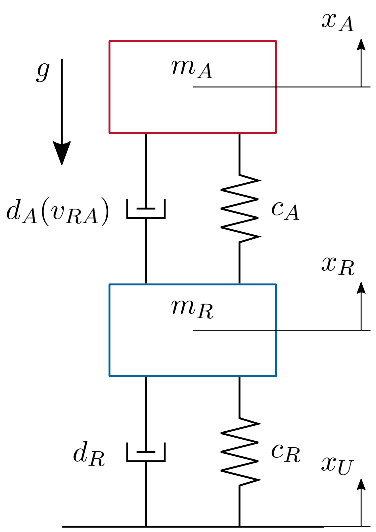


Figure A.8: Vertical model of a suspension.

Solution of exercise A.4. To determine the equations of motion of the substitute masses of the wheel and the vehicle body, the principle of conservation of linear momentum is applied. For this purpose, the substitute masses are cut free, and the corresponding spring and damper forces are applied to them. For the representation according to Fig. A.9, it is assumed that $x_U > x_R$ and $x_R > x_A$. For $x_U > x_R$, the wheel spring is compressed, and the spring force acts against the compression. Therefore, an upward force f_{cR} acts on the wheel. For $\dot{x}_U > \dot{x}_R$, the ground moves upwards faster than the substitute mass of the wheel. The damping force f_{dR} acts against the relative displacement and thus upwards on the wheel. The directions of the spring and damper force between the body and the wheel are defined analogously.

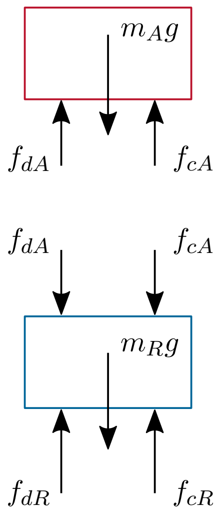


Figure A.9: Free body diagrams of the suspension masses.

The spring and damper forces are calculated as

$$f_{cA} = c_A(x_R - x_A) \qquad f_{dA} = d_A(v_{RA})(\dot{x}_R - \dot{x}_A) \tag{A.38a}$$

$$f_{cR} = c_R(x_U - x_R) \qquad f_{dR} = d_R(\dot{x}_U - \dot{x}_R). \tag{A.38b}$$

After determining the forces acting on the substitute masses, the conservation of linear momentum can be applied. This results in the equations of motion of the substitute masses according to

$$m_A \frac{\mathrm{d}}{\mathrm{d}t} v_A = f_{cA} + f_{dA} - m_A g \tag{A.39a}$$

$$m_R \frac{\mathrm{d}}{\mathrm{d}t} v_R = f_{cR} + f_{dR} - f_{cA} - f_{dA} - m_R g, \tag{A.39b}$$

with the velocities $v_A = \dot{x}_A$ and $v_R = \dot{x}_R$.

The static deflections of the masses m_A and m_R can be determined for $x_U = 0$ from the conservation of linear momentum for $\dot{x}_A = \dot{x}_R = \dot{x}_U = \ddot{x}_A = \ddot{x}_R = 0$. Then, it holds

$$0 = f_{cA} - m_A g \tag{A.40a}$$

$$0 = f_{cR} - f_{cA} - m_R g. \tag{A.40b}$$

Substituting the spring forces and solving for the unknown positions x_R and x_A yields

$$x_R = -\frac{(m_A + m_R)g}{c_R}. \tag{A.41}$$

and

$$x_A = -\frac{(m_A + m_R)g}{c_R} - \frac{m_A g}{c_A}. \tag{A.42}$$

Solution in MAPLE: `VertikalmodellFahrzeug.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.5 (Suspension). The arrangement shown in Fig. A.10 represents a suspension as used in racing. This suspension consists of the control arms Q1 and Q2, as well as the pushrod D. These are hinged to the vehicle. The normal force f_N and the lateral force f_S act on the wheel. It is assumed that these two forces act as point loads on the wheel, as shown in the figure. The forces in the control arms Q1 and Q2 as well as in the pushrod D are to be determined. Note that the control arms and the pushrod can only absorb forces in the respective axis of the rod due to their bearings.

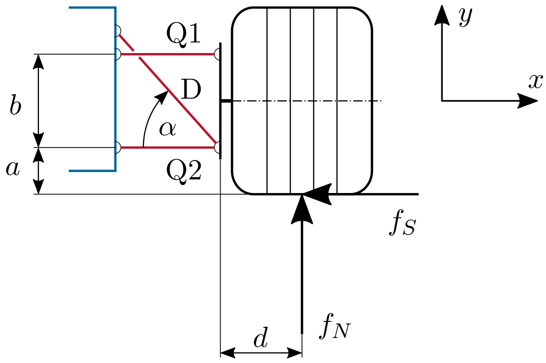


Figure A.10: Suspension of a race car.

Solution of exercise A.5. The control arms Q1 and Q2 as well as the pushrod D are considered as rods in this exercise. In order to be able to calculate the forces in the rods Q1, Q2 and D, the suspension must be cut free from the vehicle. It is recommended to make the cut directly through the rods, as shown in Fig. A.11.

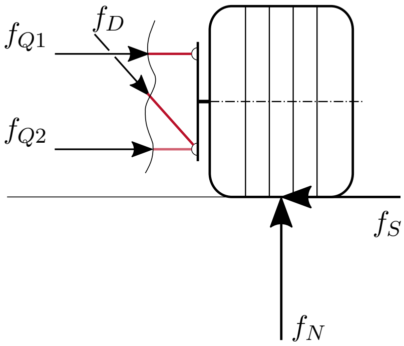


Figure A.11: Free body diagram of the suspension.

After cutting free the suspension, the equilibrium conditions can be formulated. The force balances in x- and y-direction read

$$\mathbf{e}_x : \quad 0 = f_{Q1} + f_{Q2} - f_S + f_D \cos(\alpha) \tag{A.43a}$$

$$\mathbf{e}_y : \quad 0 = -f_D \sin(\alpha) + f_N . \tag{A.43b}$$

The torque balance can be written around any point. To simplify further calculations, it is advantageous to write the torque balance around a point through which the largest number of unknown forces pass. In this example, it is the intersection of f_{Q2} and f_D, whereby the torque balance only contains f_{Q1} as an unknown. The corresponding torque balance reads

$$\mathbf{e}_z : \quad 0 = -f_{Q1} b - f_S a + f_N d \tag{A.44}$$

Solving the three equilibrium conditions for the forces in the rods yields

$$f_D = \frac{f_N}{\sin(\alpha)} \tag{A.45a}$$

$$f_{Q1} = \frac{f_N d - f_S a}{b} \tag{A.45b}$$

$$f_{Q2} = f_S \left(1 + \frac{a}{b}\right) - f_N \left(\cot(\alpha) + \frac{d}{b}\right). \tag{A.45c}$$

Solution in MAPLE: `Radaufhaengung.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.6 (Rotating Plate with Mass). Consider the rotating plate shown in Figure A.12, which rotates with a constant angular velocity ω. The mass is denoted by m, the coefficient of static friction between the surface and the mass by μ_H, the inclination angle of the rotating plate by β, the distance of the mass to the spring suspension point by s_m, the unstretched length of the spring by s_0, the distance between the spring suspension point and the axis of rotation with respect to the surface by l, and the acceleration due to gravity by g.

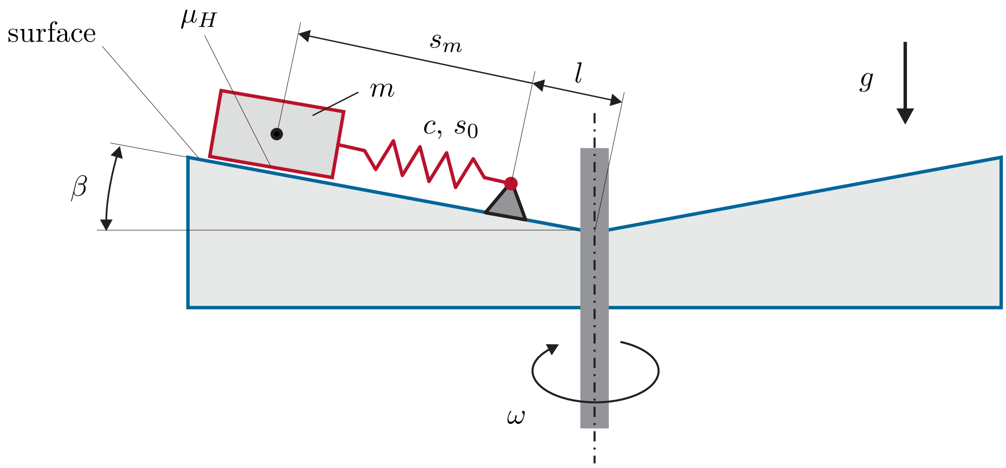


Figure A.12: Sketch of the rotating plate with mass.

In the following investigations, a stationary point is considered, i.e., the velocity $\dot{s}_m$ and the acceleration $\ddot{s}_m$ of the mass m are zero. For this system, all acting forces should be sketched and named, whereby the forces should be expressed as functions of the given quantities. Furthermore, the static friction conditions that apply to the mass m should be determined. Finally, the critical angular velocity ω_{krit} is to be determined, at which the mass m would start to move.

Solution of exercise A.6. In the first step, the centrifugal force acting on the mass m is calculated. In general, a centrifugal force can be expressed as $f_f = mr\omega^2$, where m represents the mass, r the distance with respect to the axis of rotation, and ω the angular velocity about the axis of rotation (see Figure 2.5). Applied to the rotating plate, the distance is $r = (s_m + l)\cos(\beta)$. Thus, the centrifugal force of the mass m can be expressed in the form

$$f_f = m(s_m + l)\cos(\beta)\omega^2. \tag{A.46}$$

In addition, due to the gravitational field, a gravitational force f_g acts on the mass m,

$$f_g = mg. \tag{A.47}$$

Since the mass m is also coupled to the rotating plate via a spring, the spring force

f_c acts,

$$f_c = c(s_m - s_0), \tag{A.48}$$

with the unstretched length s_0 of the spring.

Furthermore, a frictional force f_r acts on the mass. To describe this force, it is advantageous to separate the previous forces into their normal and tangential components with respect to the surface. The components of the gravitational force f_g can be described by

$$f_{g,n} = mg\cos(\beta) \tag{A.49a}$$
$$f_{g,t} = mg\sin(\beta). \tag{A.49b}$$

The centrifugal force f_f is equivalently composed of

$$f_{f,n} = m(s_m + l)\cos(\beta)\sin(\beta)\omega^2 \tag{A.50a}$$
$$f_{f,t} = m(s_m + l)\cos^2(\beta)\omega^2. \tag{A.50b}$$

The spring force is already aligned tangentially to the surface.

Thus, the frictional force f_r can be expressed in the form

$$f_r = \mu_H(f_{g,n} + f_{f,n}) = \mu_H m\Big(g\cos(\beta) + (s_m + l)\cos(\beta)\sin(\beta)\omega^2\Big). \tag{A.51}$$

Figure A.13 shows the forces and their normal and tangential components.

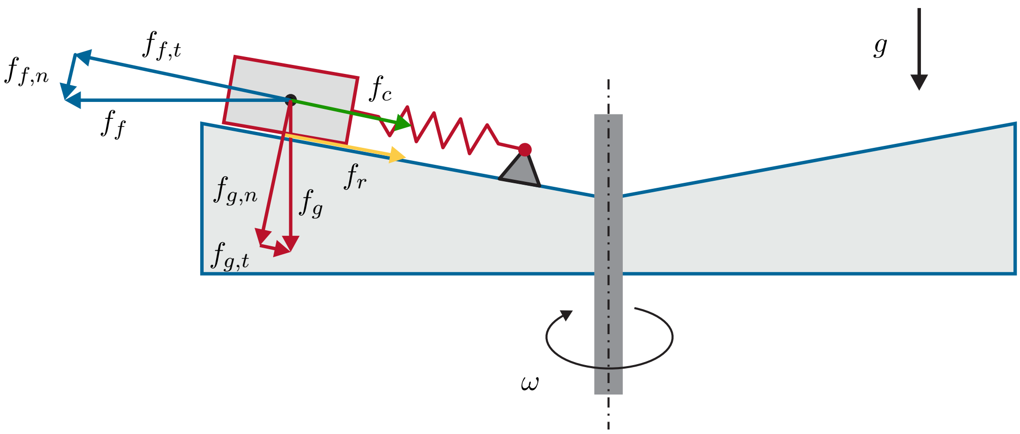


Figure A.13: Forces acting on the mass.

This makes it possible to establish the static friction conditions. Due to the angle β and the coupling of the mass m with the spring, the mass can move inwards even with a rotational movement despite the centrifugal forces. Therefore, when considering the static friction conditions, a distinction is made between outward and inward movement. These conditions can be expressed for outward movement in the form

$$f_{f,t} - f_{g,t} - f_c < \underbrace{\mu_H(f_{g,n} + f_{f,n})}_{f_r} \tag{A.52}$$

and for inward movement by

$$f_{f,t} - f_{g,t} - f_c > -\underbrace{\mu_H(f_{g,n} + f_{f,n})}_{f_r}. \tag{A.53}$$

In the next step, the critical angular velocities ω_{krit} are calculated. The basis for this are the static friction conditions from (A.52) and (A.53). Starting with (A.52), substituting the normal and tangential components yields the inequality

$$\begin{aligned} m(s_m + l)\cos^2(\beta)\omega^2 - mg\sin(\beta) - c(s_m - s_0) > \\ \mu_H m\Big(g\cos(\beta) + (x_m + l)\cos(\beta)\sin(\beta)\omega^2\Big). \end{aligned} \tag{A.54}$$

It is important to note that in this case the static frictional force f_r must be smaller than the resultant forces from $f_{f,t}$, $f_{g,t}$ and f_c. From this, the condition for ω_{krit}^2 can be easily determined according to

$$\omega_{krit}^2 = \frac{mg\sin(\beta) + \mu_H mg\cos(\beta) + c(s_m - s_0)}{m(s_m + l)\cos^2(\beta) - \mu_H m(s_m + l)\cos(\beta)\sin(\beta)}, \tag{A.55}$$

whereby immediately

$$\omega_{krit} = \pm\sqrt{\frac{mg\sin(\beta) + \mu_H mg\cos(\beta) + c(s_m - s_0)}{m(s_m + l)\cos^2(\beta) - \mu_H m(s_m + l)\cos(\beta)\sin(\beta)}} \tag{A.56}$$

follows. The procedure for determining the critical angular velocity ω_{krit} for inward movement is completely analogous. The basis is the static friction condition according to (A.53). Substituting the normal and tangential components accordingly leads to the inequality

$$\begin{aligned} m(s_m + l)\cos^2(\beta)\omega^2 - mg\sin(\beta) - c(s_m - s_0) > \\ -\mu_H m\Big(g\cos(\beta) + (s_m + l)\cos(\beta)\sin(\beta)\omega^2\Big), \end{aligned} \tag{A.57}$$

and thus

$$\omega_{krit} = \pm\sqrt{\frac{mg\sin(\beta) - \mu_H mg\cos(\beta) + c(s_m - s_0)}{m(s_m + l)\cos^2(\beta) + \mu_H m(s_m + l)\cos(\beta)\sin(\beta)}}. \tag{A.58}$$

Solution in MAPLE: `DrehtellermitMasse.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.7 (Centrifugal Governor). This exercise deals with a so-called centrifugal governor, which was first used in 1788 by James Watt to regulate a steam engine. The schematic representation in Figure A.14 shows a massless linkage that rotates about the $\mathbf{e}_z$-axis with the angular velocity $\dot{\varphi} = \omega_\varphi$ and is driven by the external torque τ. At the ends of the rods with length l, two point masses m are fixed, which influence the angle α due to the centrifugal force. Through the mechanism, point A slides up and down along the $\mathbf{e}_z$-axis. The height h_A of point A represents the output of the governor.

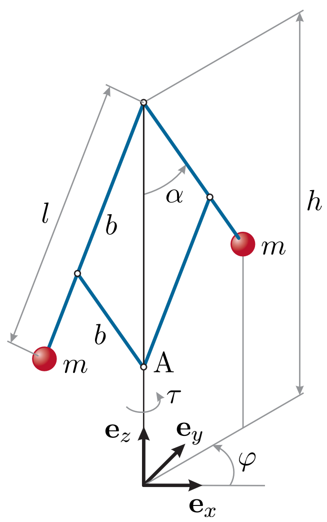


Figure A.14: Structure of the centrifugal governor.

Find the equations of motion of the centrifugal governor as well as the steady-state angle α_s, which is established for a constant angular velocity $\dot{\varphi}_s = \omega_{\varphi,s}$.

Solution of exercise A.7. A systematic derivation of the equation of motion is possible via the Euler-Lagrange equations. For this purpose, in the first step, the total kinetic and potential energy of the system is calculated as a function of the generalized coordinates. For the centrifugal governor shown, the generalized coordinates correspond to the two degrees of freedom φ and α, which are combined in the vector $\mathbf{q}(t) = \begin{bmatrix}\varphi(t) & \alpha(t)\end{bmatrix}^{\mathrm{T}}$. From this, the vector of generalized velocities is derived as $\dot{\mathbf{q}}(t) = \begin{bmatrix}\dot{\varphi}(t) & \dot{\alpha}(t)\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\omega_\varphi(t) & \omega_\alpha(t)\end{bmatrix}^{\mathrm{T}}$.

Due to the assumption of a massless linkage, the kinetic and potential energy result only from the energies of the point masses. Because of the symmetrical structure with respect to the z-axis, the two point masses each have the same kinetic and potential energy. Therefore, it is sufficient to consider only a single mass for the calculation of the energies. For the point mass, which is rotated by the angle φ with respect to the x-axis, the position vector from the origin of the coordinate system is

$$\mathbf{p}_0^m = \begin{bmatrix} l\sin(\alpha)\cos(\varphi) \\ l\sin(\alpha)\sin(\varphi) \\ h - l\cos(\alpha) \end{bmatrix} \tag{A.59}$$

and the derived velocity vector follows to

$$\dot{\mathbf{p}}_0^m = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_0^m = l\begin{bmatrix} \omega_\alpha\cos(\alpha)\cos(\varphi) - \omega_\varphi\sin(\alpha)\sin(\varphi) \\ \omega_\alpha\cos(\alpha)\sin(\varphi) + \omega_\varphi\sin(\alpha)\cos(\varphi) \\ \omega_\alpha\sin(\alpha) \end{bmatrix}. \tag{A.60}$$

The total kinetic energy of both point masses is calculated using

$$T = 2\frac{1}{2}m(\dot{\mathbf{p}}_0^m)^{\mathrm{T}}\dot{\mathbf{p}}_m \tag{A.61}$$

with the velocity (A.60) as

$$T = ml^2\Big(\omega_\alpha^2 + \omega_\varphi^2\sin^2(\alpha)\Big). \tag{A.62}$$

Assuming that the potential energy is zero for $p_{0,z}^m = 0$ (deflection of the point masses in z-direction), then with the acceleration due to gravity g, the total potential energy is

$$V = 2mg\Big(h - l\cos(\alpha)\Big). \tag{A.63}$$

The Lagrange function is obtained from the difference between the kinetic energy (A.62) and the potential energy (A.63) as

$$L = ml^2\Big(\omega_\alpha^2 + \omega_\varphi^2\sin^2(\alpha)\Big) - 2mg\Big(h - l\cos(\alpha)\Big). \tag{A.64}$$

The external torque is $\boldsymbol{\tau}_e = \begin{bmatrix} 0 & 0 & \tau \end{bmatrix}^{\mathrm{T}}$ and therefore the generalized forces follow as

$$f_\varphi = \tau \quad \text{and} \quad f_\alpha = 0. \tag{A.65}$$

Substituting (A.64) and (A.65) into the Euler-Lagrange equation (5.69) yields

$$
\begin{aligned}
\underbrace{2ml^2\left(\ddot{\varphi}\sin^2(\alpha) + 2\dot{\varphi}\dot{\alpha}\sin(\alpha)\cos(\alpha)\right)}_{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\varphi}}L} &- \underbrace{0}_{\frac{\partial}{\partial\varphi}L} &= \tau \\
\underbrace{2ml^2\ddot{\alpha}}_{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\alpha}}L} &- \underbrace{2ml\sin(\alpha)\left(l\dot{\varphi}^2\cos(\alpha) - g\right)}_{\frac{\partial}{\partial\alpha}L} &= 0,
\end{aligned}
\tag{A.66}
$$

from which the equations of motion of the centrifugal governor directly follow as

$$
\ddot{\varphi} = \frac{\tau - 4ml^2\dot{\varphi}\dot{\alpha}\sin(\alpha)\cos(\alpha)}{2ml^2\sin^2(\alpha)} \tag{A.67a}
$$

$$
\ddot{\alpha} = \frac{\sin(\alpha)\left(l\dot{\varphi}^2\cos(\alpha) - g\right)}{l} \tag{A.67b}
$$

The steady-state deflection for a constant angular velocity $\dot{\varphi}_s = \omega_{\varphi,s}$ follows by substituting $\dot{\alpha}_s = 0$ and $\ddot{\alpha}_s = \ddot{\varphi}_s = 0$ into (A.67). From the first equation (A.67a), setting the time derivatives to zero directly gives the external torque as $\tau_s = 0$. This result follows from the assumption that there is no friction in the system, which would have to be compensated by the external torque at the steady-state point. In the steady-state case, from (A.67b), the condition for the steady-state angle α_s follows as

$$
\sin(\alpha_s)\left(l\dot{\varphi}_s^2\cos(\alpha_s) - g\right) = 0 \tag{A.68}
$$

In addition to the trivial solution $\alpha_s = 0$, another equilibrium position results in

$$
\alpha_s = \arccos\left(\frac{g}{l\dot{\varphi}_s^2}\right) \tag{A.69}
$$

under the condition that $\dot{\varphi}_s = \omega_{\varphi,s} \geq \sqrt{\frac{g}{l}}$ holds for the angular velocity. If this inequality is not satisfied, then no other equilibrium position exists. The steady-state angle (A.69) can alternatively be determined by freeing the point mass and setting up the force balance, with the centrifugal force $f_f = m\omega_\varphi^2 l\cos(\alpha)$.

Solution in MAPLE: `Fliehkraftregler.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.8 (Drum hoist). A block of mass m_B hangs from a massless rope according to Fig. A.15. The rope runs frictionlessly over a massless pulley and is wound onto a drum (mass m_T, moment of inertia I_T). The drum rolls over the contact surface without slipping. In addition, a spring with the unstretched length s_{f0} and the constant spring stiffness c counteracts the motion of the drum. The entire system is located in the earth's gravitational field.

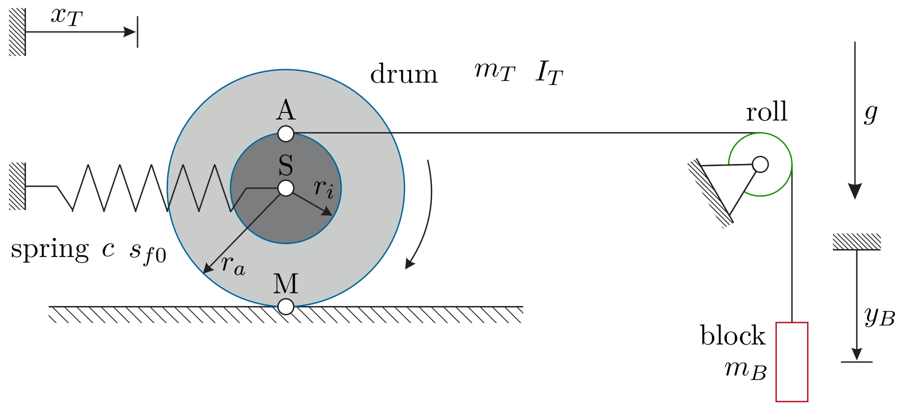


Figure A.15: Sketch of a drum hoist with load.

In the following, the following tasks should be solved:

a) In the first step, the bearing forces, the rope force and the spring force in the stationary state should be determined.

b) Subsequently, the equations of motion of the system should be specified using the conservation of linear and angular momentum. The position x_T of the drum should be used as the degree of freedom.

Solution of exercise A.8. To determine the bearing forces, the rope force, and the spring force, the drum, the pulley, and the block are cut free and the acting forces are drawn. In Fig. A.16 these cutting forces are shown for the given system.

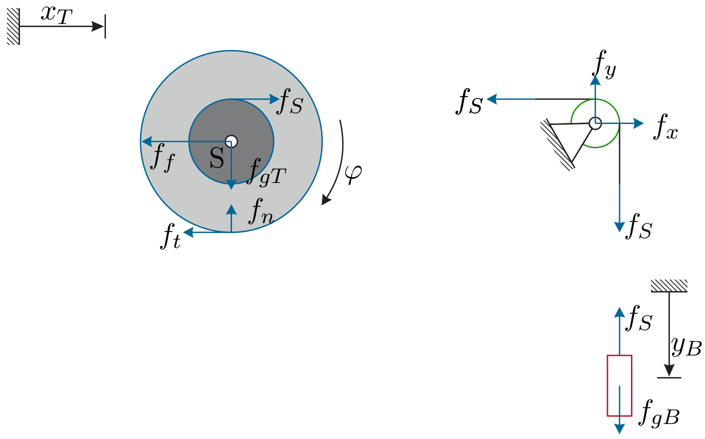


Figure A.16: Cut-free bodies and cutting forces.

The weight force $f_{gB} = m_B g$ and the rope force f_S act on the block of mass m_B in the y-direction. At the deflection pulley, the bearing forces f_x and f_y as well as the forces of the cut-free rope occur. The forces acting on the drum are the normal force f_n, the tangential force f_t, the weight force $f_{gT} = m_T g$, the spring force $f_f = c(x_T - s_{f0})$ and the rope force f_S.

The force balance in the y-direction for the block yields

$$f_S = f_{gB} = m_B g, \tag{A.70}$$

and the force balance for the pulley results in

$$f_x = f_S = m_B g \tag{A.71a}$$
$$f_y = f_S = m_B g\ . \tag{A.71b}$$

The normal force f_n can be determined directly from the force balance in the y-direction at the drum in the form

$$f_n = f_{gT} = m_T g \tag{A.72}$$

The force balance in the x-direction reads

$$f_S - f_f - f_t = 0. \tag{A.73}$$

To determine the unknown forces f_f and f_t, one uses the torque balance, written down e.g. around point S

$$f_S r_i + f_t r_a = 0, \tag{A.74}$$

wherewith immediately

$$f_t = -\frac{r_i}{r_a} F_S = -\frac{r_i}{r_a} m_B g \tag{A.75}$$

follows. If this result is inserted into the force balance in the x-direction, the unknown spring force is obtained

$$f_f = f_S - f_t = \left(1 + \frac{r_i}{r_a}\right) m_B g = \frac{r_i + r_a}{r_a} m_B g. \tag{A.76}$$

Remark: Note that the same result would have been obtained if, instead of the force balance in the x-direction, the torque balance around point M had been additionally written down at the drum. This reads

$$f_s(r_i + r_a) - f_f r_a = 0, \tag{A.77}$$

which obviously implies $f_f = (r_i + r_a)/r_a m_B g$.

To determine the equations of motion of the system, the number of degrees of freedom must first be determined. It can be seen that when the block moves in the y-direction, the rope passing over the deflection pulley leads to a rotation φ of the drum. Since it was further assumed that the drum rolls without slipping, a rotation of the drum simultaneously leads to a displacement x_T in the x-direction. Thus, the system has one degree of freedom, and this should be chosen to be x_T according to the specification.

In the first step, therefore, the angle φ and the position of the block y_B must be expressed as a function of the degree of freedom. Considering the rolling cylinder, it is obvious that the unrolled length (and thus the displacement of the drum in the x-direction) follows φr_a. Thus, we have

$$\varphi = \frac{x_T}{r_a}. \tag{A.78}$$

The change in rope length (and thus the change in position y_B of the block) results from the sum of the displacement of the drum x_T at point S and the unrolled rope due to the rotation of the drum, i.e.

$$y_B = x_T + r_i \varphi = x_T + \frac{r_i}{r_a} x_T = \frac{r_i + r_a}{r_a} x_T. \tag{A.79}$$

The determination of the equations of motion of the system can now be done either by using the conservation of linear and angular momentum or by applying the Euler-Lagrange formalism. In this example, the conservation of linear and angular momentum will be used. For this purpose, one formulates the conservation of linear and angular momentum for the drum and the block. Since the pulley and the rope were assumed to be massless, the momentum for these parts vanishes.

With the velocity $v_B = \dot{y}_B$ of the block, the conservation of linear momentum follows

$$m_B \frac{\mathrm{d}}{\mathrm{d}t} v_B = m_B g - f_S \tag{A.80}$$

and thus follows

$$f_S = -m_B \frac{\mathrm{d}}{\mathrm{d}t} v_B + m_B g = m_B \left(g - \frac{r_i + r_a}{r_a} \frac{\mathrm{d}}{\mathrm{d}t} v_T \right), \tag{A.81}$$

with the velocity of the drum v_T. Note that the rope force f_S from the above equation differs from the rope force in the static case!

The conservation of angular momentum for the drum written around point S results in

$$I_T \frac{\mathrm{d}}{\mathrm{d}t} \omega = f_S r_i + f_t r_a \tag{A.82}$$

and thus

$$f_t = \frac{1}{r_a} \left(I_T \frac{\mathrm{d}}{\mathrm{d}t} \omega - f_S r_i \right) \tag{A.83}$$

holds, where $\omega = \dot{\varphi}$ denotes the angular velocity of the drum. Substituting the rope force and the relationship between φ and x_T gives

$$f_t = \frac{1}{r_a^2} (r_i(r_i + r_a) m_B + I_T) \dot{v}_T - \frac{r_i}{r_a} m_B g. \tag{A.84}$$

In the last step, one formulates the conservation of linear momentum for the drum in the x-direction in the form

$$m_T \frac{\mathrm{d}}{\mathrm{d}t} v_T = f_S - f_f - f_t = f_S - c(x_T - s_{f0}) - f_t. \tag{A.85}$$

The desired equation of motion of the system is obtained by substituting the intermediate results for f_S and f_t and solving for $\dot{v}_T$. This yields

$$\dot{v}_T = \frac{(r_i + r_a) m_B g r_a - c(x_T - s_{f0}) r_a^2}{(r_i + r_a)^2 m_B + I_T + m_T r_a^2}. \tag{A.86}$$

Solution in MAPLE: `Seiltrommel.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.9 (Cable Pulley System). Given is the cable pulley system shown in Fig. A.17, consisting of a load L (mass m_L), a frictionless mounted pulley R_1 (mass m_1, moment of inertia I_1 about the axis of rotation, outer radius r_1, inner radius r_2) and a movable pulley R_2 (mass m_2, moment of inertia I_2 about the axis of rotation, radius r_2) on which an external vertical force f acts. The load is positioned on an inclined plane (angle α), whereby dry friction with a coefficient of friction μ_c occurs between the load and the plane. At time t_0 the load possesses the velocity v_0 (uphill). The load shall be accelerated to the velocity v_1 within the distance s_1 by the force f. The radii r_1 and r_2 of the pulley R_1 can be assumed to be constant and the ropes massless. Consider the following quantities as given: $s_1, v_0, v_1, m_L, m_1, m_2, I_1, I_2, r_1, r_2, \alpha, \mu_c$.

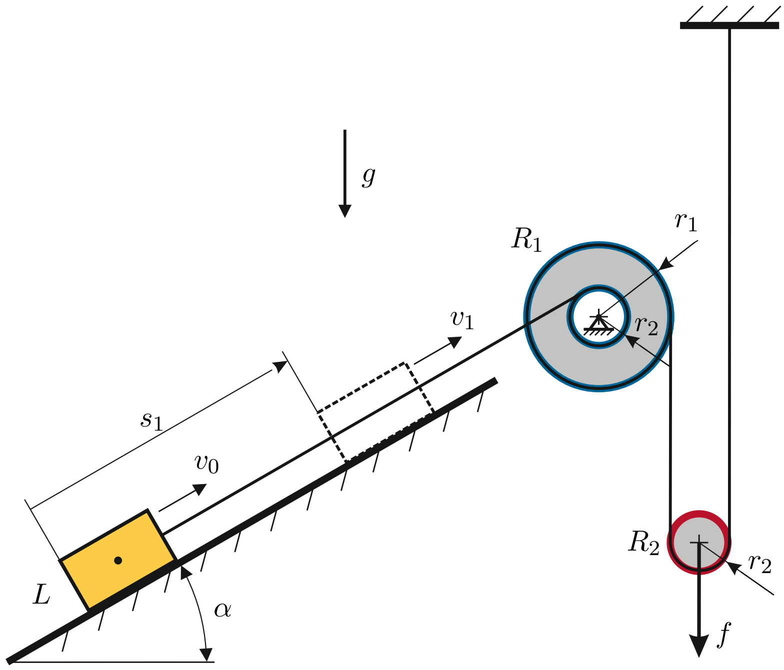


Figure A.17: Sketch of a cable pulley system.

We are looking for the time-constant force f which accelerates the mass m_L with the initial velocity v_0 within the distance $s_1 - s_0$ to the velocity v_1. Assume that the position s at the beginning is $s(t_0) = s_0 = 0$.

Solution of exercise A.9. An elegant way to calculate the required force is by applying the principle of conservation of energy in the form of comparing the energy at time t_0 and at time t_1, when the load reaches the velocity v_1. Due to the principle of conservation of energy, the following applies

$$T(t_0) + V(t_0) + W_F - W_R = T(t_1) + V(t_1) \ , \tag{A.87}$$

where T is the kinetic energy and V the potential energy of the system. W_F denotes the work done by the force f and W_R the dissipated energy due to the occurring friction.

As a first step to solve the problem, a consideration of the kinematics, especially the determination of the degrees of freedom of the system, is necessary. If there were no tensioned rope, then the following 4 degrees of freedom would result: the position s of the load, the angle φ_{R1} of the pulley R_1 as well as the position s_{R2} and the angle φ_{R2} of the pulley R_2. For each rope connection one degree of freedom is lost and thus for 3 connections between the objects 1 degree of freedom results. Thus, it is possible to express the entire kinematics of the system by one independent variable (degree of freedom). For solving the problem it is advantageous to choose the position s of the load.

The kinetic energy T of the system results in

$$T = \frac{m_L v^2}{2} + \frac{I_1 \omega_{R1}^2}{2} + \frac{I_2 \omega_{R2}^2}{2} + \frac{m_2 v_{R2}^2}{2} \ . \tag{A.88}$$

For the angular velocity ω_{R1} of the pulley R_1 the following relationship applies

$$\omega_{R1} = \frac{v}{r_2} \tag{A.89}$$

and for the exit velocity v_2 of the rope from the pulley R_1 one obtains

$$v_2 = r_1 \omega_{R1} = v \frac{r_1}{r_2} \ . \tag{A.90}$$

At the movable pulley R_2 the relations for a pulley system apply (see Fig. 3.17), i.e.

$$v_{R2} = \frac{v_2}{2} = v \frac{r_1}{2r_2}, \quad \omega_{R2} = \frac{v_{R2}}{r_2} = v \frac{r_1}{2r_2^2} \ . \tag{A.91}$$

The potential energy of the system at time t_1

$$V(t_1) = m_L g s_1 \sin(\alpha) - m_2 g s_1 \frac{r_1}{2r_2} + V(t_0) \ , \tag{A.92}$$

is composed of the rise of the load and the lowering of the pulley R_2. Since $V(t_0)$ is canceled out in the energy balance, $V(t_0) = 0$ can be chosen. The work done by the force f is calculated as

$$W_F = \int_{\tilde{x}=0}^{s_1 r_1/(2r_2)} f \, \mathrm{d}\tilde{x} = f s_1 \frac{r_1}{2r_2}, \tag{A.93}$$

where a constant force f was assumed.

To calculate the dissipated energy W_R due to the occurring friction, one needs the normal force f_n which the load exerts on the inclined plane:

$$f_n = m_L g \cos(\alpha) \ . \tag{A.94}$$

The work done results with the tangential force $f_t = f_n \mu_c$ to

$$W_R = s_1 f_t = s_1 \mu_c m_L g \cos(\alpha) \ . \tag{A.95}$$

Substituting the expressions for the energy balance and rearranging yields the sought expression for the force f:

$$\begin{aligned} f = \frac{2r_2}{s_1 r_1}\Bigg(& m_L \frac{v_1^2 - v_0^2}{2} + I_1 \frac{\omega_{R1}(t_1)^2 - \omega_{R1}(t_0)^2}{2} + I_2 \frac{\omega_{R2}(t_1)^2 - \omega_{R2}(t_0)^2}{2} \\ & + m_2 \frac{v_{R2}(t_1)^2 - v_{R2}(t_0)^2}{2} + m_L g s_1 (\sin(\alpha) + \mu_c \cos(\alpha)) - m_2 g s_1 \frac{r_1}{2r_2}\Bigg) \end{aligned} \tag{A.96}$$

with the previously defined quantities ω_{R1}, ω_{R2} and v_{R2}.

Remark: This problem can also be solved with the help of the equations of motion of the system. This approach, however, is much more complex and requires the solution of the differential equation of motion. This solution path is also shown in the sample solution in MAPLE.

Solution in MAPLE: `Seilzug.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.10 (Toggle Press). Fig. A.18 shows a sketch of a toggle press as it is used, for example, in the massive forming of steel. By applying a force f to the knee of the press, the slide can be moved horizontally. The advantage of this construction is that very large forces can be built up in the horizontal direction when the knee is almost fully extended. The two legs have masses m_1 and m_2 and moments of inertia I_1 and I_2 about the z-axis. The lengths L_1 and L_2 between the joints are also known. It can be assumed that the centers of gravity of the legs are located in their middle. The return spring has the spring constant c_F and a relaxed length $s_{F,0}$. Gravity with the acceleration due to gravity g acts in the negative y-direction.

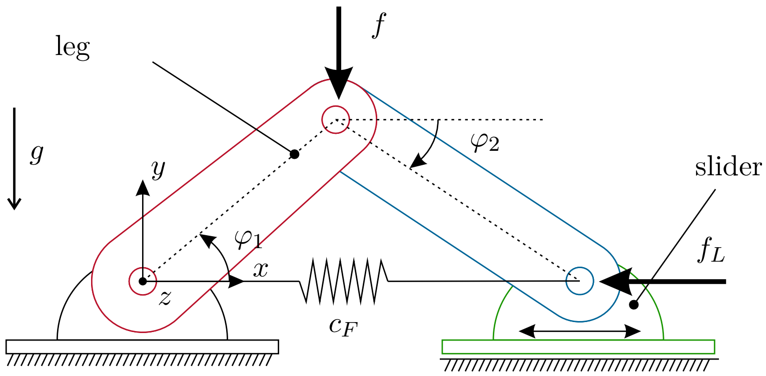


Figure A.18: Sketch of a toggle press.

The task is to determine the potential and kinetic energy of the system, the vector of generalized forces, and the equations of motion.

Solution of exercise A.10. Before beginning with the mathematical description of the toggle press, some considerations must be made regarding the degrees of freedom of the system. The first leg is rotatably mounted at the origin of the coordinate system but cannot be moved in either the x- or y-direction. Thus, it only has one degree of freedom. The second leg is rotatably connected to the first leg at the knee. Without the slide, this leg would also have one degree of freedom, but its movement is restricted by the slide. This constraint makes the rotation of the second leg dependent on the movement of the first. Consequently, one degree of freedom is sufficient to completely describe the movement of the system.

The choice of coordinates to describe the system is not unique. So far, it has only been determined that one degree of freedom is sufficient to describe the system completely. For example, the angle φ_1 or the x-coordinate of the slide can be used to describe the system. However, it turns out that a suitable choice of the degree of freedom can greatly simplify the derivation of the equations of motion.

In the further steps, the angle

$$q = \varphi_1(t) \tag{A.97}$$

of the first leg will be used to describe the movement of the toggle press. Since only one generalized coordinate is needed, the angle φ_2 must be dependent on it. The constraint of the slide now states that the endpoint of the second leg

$$\mathbf{p}_0^e = L_1 \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1) \\ 0 \end{bmatrix} + L_2 \begin{bmatrix} \cos(\varphi_2) \\ -\sin(\varphi_2) \\ 0 \end{bmatrix} \tag{A.98}$$

remains on the x-axis for all times, i.e., that the equation

$$L_1 \sin(\varphi_1) - L_2 \sin(\varphi_2) = 0 \tag{A.99}$$

must be satisfied. The angle φ_2 can thus be expressed by

$$\varphi_2(q) = \arcsin\left(\frac{L_1}{L_2}\sin(q)\right) \tag{A.100}$$

To derive the equations of motion using the Euler-Lagrange equations, the Lagrange function of the system and the generalized forces must be determined. First, the position vectors to the centers of gravity of the two legs are determined. Since the centers of gravity are located in the middle of the legs (i.e., at $L_1/2$ and $L_2/2$, respectively), the expressions

$$\mathbf{p}_0^{s1} = \frac{L_1}{2}\begin{bmatrix}\cos(q)\\ \sin(q)\\ 0\end{bmatrix} \tag{A.101a}$$

$$\mathbf{p}_0^{s2} = L_1\begin{bmatrix}\cos(q)\\ \sin(q)\\ 0\end{bmatrix} + \frac{L_2}{2}\begin{bmatrix}\cos(\varphi_2(q))\\ -\sin(\varphi_2(q))\\ 0\end{bmatrix} \tag{A.101b}$$

can be given for the center of gravity vectors. The translational velocities of the centers of gravity are obtained by taking the time derivative of the position vectors:

$$\mathbf{v}_0^{s1} = \frac{L_1}{2}\begin{bmatrix}-\sin(q)\\ \cos(q)\\ 0\end{bmatrix}\dot{q} \tag{A.102a}$$

$$\mathbf{v}_0^{s2} = L_1\begin{bmatrix}-\sin(q)\\ \cos(q)\\ 0\end{bmatrix}\dot{q} + \frac{L_2}{2}\begin{bmatrix}-\sin(\varphi_2(q))\\ -\cos(\varphi_2(q))\\ 0\end{bmatrix}\frac{\partial\varphi_2(q)}{\partial q}\dot{q} \tag{A.102b}$$

and the angular velocities of the legs are

$$\omega_1 = \dot{q} \tag{A.103a}$$

$$\omega_2 = -\frac{\partial\varphi_2(q)}{\partial q}\dot{q}\ . \tag{A.103b}$$

For the sake of clarity, the derivative of $\varphi_2(q)$ was not evaluated here.

The kinetic energy is then given by

$$
\begin{aligned}
T =& \frac{1}{8}m_1 L_1^2 \dot{q}^2 + \\
& \frac{1}{2}m_2 L_1^2 \dot{q}^2 + \frac{1}{8}m_2 L_2^2 \left(\frac{\partial \varphi_2(q)}{\partial q}\right)^2 \dot{q}^2 - \frac{1}{2}m_2 L_1 L_2 \cos(q + \varphi_2(q)) \frac{\partial \varphi_2(q)}{\partial q} \dot{q}^2 + \\
& \frac{1}{2} I_1 \dot{q}^2 + \frac{1}{2} I_2 \left(\frac{\partial \varphi_2(q)}{\partial q} \dot{q}\right)^2
\end{aligned}
\tag{A.104}
$$

where the trigonometric functions that occur can be greatly simplified using $\sin(\varphi_1)^2 + \cos(\varphi_1)^2 = 1$ and $\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2) = \cos(\varphi_1 + \varphi_2)$.

The potential energy due to the weights of the legs is given by

$$
V_g = m_1 g \frac{L_1}{2} \sin(q) + m_2 g \left(L_1 \sin(q) - \frac{L_2}{2} \sin(\varphi_2(q))\right) + V_{g,0} \tag{A.105}
$$

and the potential energy of the spring follows from (3.70) and the x-coordinate of the endpoint $\mathbf{p}_0^e$:

$$
V_F = \frac{1}{2} c_F (L_1 \cos(q) + L_2 \cos(\varphi_2(q)) - s_{F,0})^2 \ . \tag{A.106}
$$

The generalized forces due to the knee force f and the load force f_L are calculated as

$$
f_{q,f} = \begin{bmatrix} 0 & -F & 0 \end{bmatrix} \frac{\partial}{\partial q} \begin{bmatrix} L_1 \cos(q) \\ L_1 \sin(q) \\ 0 \end{bmatrix} = -f L_1 \cos(q) \tag{A.107a}
$$

$$
f_{q,f_L} = \begin{bmatrix} -f_L & 0 & 0 \end{bmatrix} \frac{\partial \mathbf{p}_0^e}{\partial q} = f_L \left(L_1 \sin(q) + L_2 \sin(\varphi_2(q)) \frac{\partial \varphi_2(q)}{\partial q}\right) \ . \tag{A.107b}
$$

With this, the Euler-Lagrange equations can be written using the Lagrange function $L = T - V_g - V_F$ and the generalized forces $f_{q,f}$ and f_{q,f_L}.

Solution in MAPLE: `Kniehebelpresse.mw`
`https://www.acin.tuwien.ac.at/bachelor/modellbildung/`

Exercise A.11 (Planar Robot). In this exercise, the mechanical system (planar robot) from Figure A.19 is considered. The setup consists of three segments $i = \{1, 2, 3\}$ with the lengths l_i, the masses m_i, and the moments of inertia I_i about the z-axis with respect to the respective center of gravity S_i. Segment 1 is rotatably mounted at the constant height h. The bearing can be assumed to be ideally frictionless. At both ends of segment 1, segments 2 and 3 are again rotatably mounted without friction.

For all segments, a homogeneous density can be assumed both over the cross-section and over the length. As can be seen from Figure A.19, the acceleration due to gravity g acts in the negative $\mathbf{e}_y$ direction. In addition, an external force $\mathbf{f}_e = \begin{bmatrix} -f_{e,x} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ acts on the free end of segment 2.

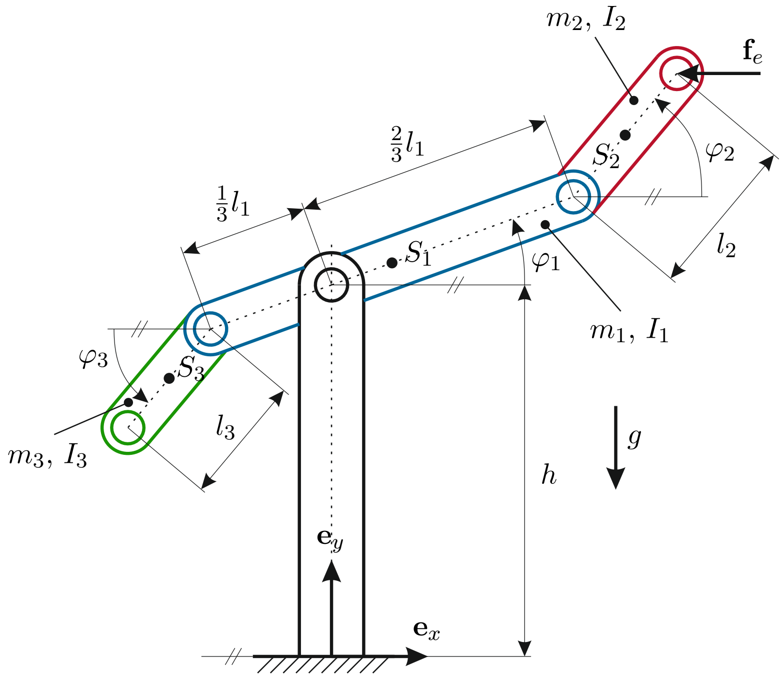


Figure A.19: Simple planar manipulator.

For the given mechanical system, the equations of motion are to be derived using the Euler-Lagrange formalism.

Solution of exercise A.11. The system under consideration has three degrees of freedom (the rotations of the three segments). In the first step, the vector of generalized coordinates is defined. With regard to a simple calculation of the kinetic and potential energy, the choice of

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix}^{\mathrm{T}} \tag{A.108}$$

proves to be useful.

Remark: The choice of generalized coordinates is not unique. In the present example, the choice of

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \varphi_1 & \varphi_2 - \varphi_1 & \varphi_3 - \varphi_1 \end{bmatrix}^{\mathrm{T}} \tag{A.109}$$

would also be conceivable.

The central quantity for the application of the Euler-Lagrange equations is the Lagrangian $L = T - V$. The kinetic energy in the system is calculated from the sum of the kinetic energies of the individual bodies as

$$T = \sum_{i=1}^{3} (T_{t,i} + T_{r,i}), \tag{A.110}$$

with the translational energies

$$T_{t,i} = \frac{1}{2} m_i \left(\dot{\mathbf{p}}_0^{S_i}\right)^{\mathrm{T}} \dot{\mathbf{p}}_0^{S_i} \tag{A.111}$$

and the rotational energies

$$T_{r,i} = \frac{1}{2} I_i \dot{\varphi}_i^2 \tag{A.112}$$

The position vectors to the centers of gravity of the segments result from Figure A.19 as

$$\mathbf{p}_0^{S_1} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} + \frac{l_1}{6} \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \\ 0 \end{bmatrix} \tag{A.113a}$$

$$\mathbf{p}_0^{S_2} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} + \frac{2l_1}{3} \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \\ 0 \end{bmatrix} + \frac{l_2}{2} \begin{bmatrix} \cos(q_2) \\ \sin(q_2) \\ 0 \end{bmatrix} \tag{A.113b}$$

$$\mathbf{p}_0^{S_3} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} + \frac{l_1}{3} \begin{bmatrix} -\cos(q_1) \\ -\sin(q_1) \\ 0 \end{bmatrix} + \frac{l_3}{2} \begin{bmatrix} -\cos(q_3) \\ -\sin(q_3) \\ 0 \end{bmatrix} \tag{A.113c}$$

wherewith the velocities of the centers of gravity are obtained as

$$\dot{\mathbf{p}}_0^{S_1} = \frac{l_1}{6}\begin{bmatrix} -\sin(q_1) \\ \cos(q_1) \\ 0 \end{bmatrix}\dot{q}_1 \tag{A.114a}$$

$$\dot{\mathbf{p}}_0^{S_2} = \frac{2l_1}{3}\begin{bmatrix} -\sin(q_1) \\ \cos(q_1) \\ 0 \end{bmatrix}\dot{q}_1 + \frac{l_2}{2}\begin{bmatrix} -\sin(q_2) \\ \cos(q_2) \\ 0 \end{bmatrix}\dot{q}_2 \tag{A.114b}$$

$$\dot{\mathbf{p}}_0^{S_3} = \frac{l_1}{3}\begin{bmatrix} \sin(q_1) \\ -\cos(q_1) \\ 0 \end{bmatrix}\dot{q}_1 + \frac{l_3}{2}\begin{bmatrix} \sin(q_3) \\ -\cos(q_3) \\ 0 \end{bmatrix}\dot{q}_3 \tag{A.114c}$$

For the squares of the velocity vectors, one obtains after a short calculation

$$\left(\dot{\mathbf{p}}_0^{S_1}\right)^{\mathrm{T}}\dot{\mathbf{p}}_0^{S_1} = \left(\frac{l_1}{6}\right)^2\dot{q}_1^2 \tag{A.115a}$$

$$\left(\dot{\mathbf{p}}_0^{S_2}\right)^{\mathrm{T}}\dot{\mathbf{p}}_0^{S_2} = \left(\frac{2l_1}{3}\right)^2\dot{q}_1^2 + \left(\frac{l_2}{2}\right)^2\dot{q}_2^2 + \frac{2}{3}l_1 l_2\cos(q_1 - q_2)\dot{q}_1\dot{q}_2 \tag{A.115b}$$

$$\left(\dot{\mathbf{p}}_0^{S_3}\right)^{\mathrm{T}}\dot{\mathbf{p}}_0^{S_3} = \left(\frac{l_1}{3}\right)^2\dot{q}_1^2 + \left(\frac{l_3}{2}\right)^2\dot{q}_3^2 + \frac{1}{3}l_1 l_3\cos(q_1 - q_3)\dot{q}_1\dot{q}_3, \tag{A.115c}$$

where the trigonometric identity $\cos(q_1 - q_i) = \cos(q_1)\cos(q_i) + \sin(q_1)\sin(q_i)$ for $i = \{2, 3\}$ was used to simplify the expressions.

With these preparations, the kinetic energy in the system is now calculated as

$$\begin{aligned} T = \frac{1}{2}&\left(m_1\left(\frac{l_1}{6}\right)^2 + m_2\left(\frac{2l_1}{3}\right)^2 + m_3\left(\frac{l_1}{3}\right)^2 + I_1\right)\dot{q}_1^2 \\ &+ \frac{1}{2}\left(m_2\left(\frac{l_2}{2}\right)^2 + I_2\right)\dot{q}_2^2 + \frac{1}{2}\left(m_3\left(\frac{l_3}{2}\right)^2 + I_3\right)\dot{q}_3^2 \\ &+ \frac{1}{3}m_2 l_1 l_2\cos(q_1 - q_2)\dot{q}_1\dot{q}_2 + \frac{1}{6}m_3 l_1 l_3\cos(q_1 - q_3)\dot{q}_1\dot{q}_3 \end{aligned} \tag{A.116}$$

The potential energy of the system results only from gravitation. If the reference potential is chosen at $y = 0$, the potential energy is obtained with the acceleration due to gravity g according to Figure A.19 as

$$\begin{aligned} V &= m_1 g\mathbf{e}_y^{\mathrm{T}}\mathbf{p}_0^{S1} + m_2 g\mathbf{e}_y^{\mathrm{T}}\mathbf{p}_0^{S2} + m_3 g\mathbf{e}_y^{\mathrm{T}}\mathbf{p}_0^{S3} \\ &= m_1 g\left(h + \frac{l_1}{6}\sin(q_1)\right) + m_2 g\left(h + \frac{2l_1}{3}\sin(q_1) + \frac{l_2}{2}\sin(q_2)\right) \\ &\quad + m_3 g\left(h - \frac{l_1}{3}\sin(q_1) - \frac{l_3}{2}\sin(q_3)\right) \end{aligned} \tag{A.117}$$

To describe the equation of motion, the generalized forces are still needed. According to (5.60) one obtains the generalized forces from

$$\mathbf{f}_q = \left(\frac{\partial \mathbf{p}_0^e}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{f}_e \tag{A.118}$$

with the position vector of the point of application of the force

$$\mathbf{p}_0^e = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} + \frac{2l_1}{3}\begin{bmatrix} \cos(q_1) \\ \sin(q_1) \\ 0 \end{bmatrix} + l_2 \begin{bmatrix} \cos(q_2) \\ \sin(q_2) \\ 0 \end{bmatrix} \tag{A.119}$$

to

$$\mathbf{f}_q = \begin{bmatrix} \frac{2l_1}{3}\sin(q_1) \\ l_2 \sin(q_2) \\ 0 \end{bmatrix} f_{e,x} \tag{A.120}$$

With these preparations, the equations of motion of the system can be calculated from

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = f_{q,i}\,, \qquad i \in \{1,2,3\} \tag{A.121}$$

These can be compactly written as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{f}_q \tag{A.122}$$

with

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} \frac{l_1^2}{36}(m_1 + 16m_2 + 4m_3) + I_1 & \frac{l_1 l_2}{3} m_2 \cos(q_1 - q_2) & \frac{l_1 l_3}{6} m_3 \cos(q_1 - q_3) \\ \frac{l_1 l_2}{3} m_2 \cos(q_1 - q_2) & \frac{l_2^2}{4} m_2 + I_2 & 0 \\ \frac{l_1 l_3}{6} m_3 \cos(q_1 - q_3) & 0 & \frac{l_3^2}{4} m_3 + I_3 \end{bmatrix} \tag{A.123a}$$

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} 0 & \frac{l_1 l_2}{3} m_2 \sin(q_1 - q_2)\dot{q}_2 & \frac{l_1 l_3}{6} m_3 \sin(q_1 - q_3)\dot{q}_3 \\ -\frac{l_1 l_2}{3} m_2 \sin(q_1 - q_2)\dot{q}_2 & 0 & 0 \\ \frac{l_1 l_3}{6} m_3 \sin(q_1 - q_3)\dot{q}_3 & 0 & 0 \end{bmatrix} \tag{A.123b}$$

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} \frac{l_1}{6} m_1 g \cos(q_1) + \frac{2l_1}{3} m_2 g \cos(q_1) - \frac{l_1}{3} m_3 g \cos(q_1) \\ \frac{l_2}{2} m_2 g \cos(q_2) \\ -\frac{l_3}{2} m_3 g \cos(q_3) \end{bmatrix} \tag{A.123c}$$

Solution in Maple: `PlanarerRoboter.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.12 (Rotationally Constrained Hollow Cylinder). A hollow cylinder (mass m_1, length l_1, outer diameter d_1, inner diameter d_2) is mounted on a pivot at point K and subjected to an external torque τ_e, as shown in Fig. A.20. A second cylindrical rod (mass m_2, length l_2, diameter d_2) is connected to the hollow cylinder via a spring element (spring constant c, unstretched length s_0).

During relative motion of the two cylinders, a velocity-proportional friction force f_d acts, with the friction coefficient $d(s)$. The friction coefficient is proportional to the contact area between the cylinders, with $d(s) = d_0 A(s)$. Both cylinders are subjected to gravitational acceleration g. The moments of inertia $I_{1,zz}^{(S)}$ and $I_{2,zz}^{(S)}$ of the cylinders about their respective $\mathbf{e}_z^S$-axes are assumed to be known. For the determination of the center of gravity of the hollow cylinder, the base area can be neglected.

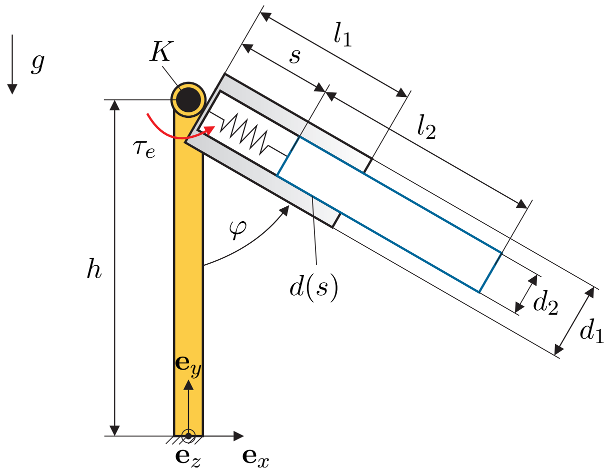


Figure A.20: Setup of the rotationally constrained hollow cylinder.

For this configuration, the equations of motion are to be derived using the Euler-Lagrange formalism. Furthermore, the stationary point (equilibrium position) of the system is to be determined for $\tau_e = 0$.

Solution of exercise A.12. As a first step, a suitable choice of generalized coordinates $\mathbf{q}$ (i.e., the degrees of freedom of the system) must be made. The present system consists of two rigid bodies, each of whose unrestricted planar motion has three degrees of freedom (displacements in the x- and y-directions and rotation about the z-axis). In the case under consideration, the system is subject to 4 constraints: (i) The x- and y-positions of cylinder 1 are fixed, leaving it with only the rotation about the angle φ as a degree of freedom. (2) Cylinder 2 is connected to cylinder 1, so it must undergo the same rotation. The only remaining degree of freedom for this cylinder is movement in the direction of the degree of freedom s.

From these considerations, the following possible choice of generalized coordinates (degrees of freedom) results

$$\mathbf{q} = \begin{bmatrix} \varphi \\ s \end{bmatrix}. \tag{A.124}$$

In the next step, the position vectors to the centers of mass of the respective cylinders are formulated. The position vector $\mathbf{p}_0^{s1}$ to the center of gravity of the hollow cylinder results from geometrical considerations to

$$\mathbf{p}_0^{s1} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} + \frac{l_1}{2}\begin{bmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{bmatrix}, \tag{A.125}$$

and the position vector $\mathbf{p}_0^{s2}$ to the second cylinder is calculated as

$$\mathbf{p}_0^{s2} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} + \left(s + \frac{l_2}{2}\right)\begin{bmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{bmatrix}. \tag{A.126}$$

The translational velocities $\mathbf{v}_0^{s1}$ and $\mathbf{v}_0^{s2}$ of the centers of gravity of the two cylinders, which are necessary for the calculation of the kinetic energy, are obtained as

$$\mathbf{v}_0^{s1} = \dot{\mathbf{p}}_0^{s1} = \frac{l_1}{2}\begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}\dot{\varphi} \tag{A.127a}$$

$$\mathbf{v}_0^{s2} = \dot{\mathbf{p}}_0^{s2} = \begin{bmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{bmatrix}\dot{s} + \left(s + \frac{l_2}{2}\right)\begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}\dot{\varphi}. \tag{A.127b}$$

The rotational part of the kinetic energy is obtained directly as

$$T_r = \frac{1}{2}\left(I_{1,zz}^{(S)} + I_{2,zz}^{(S)}\right)\dot{\varphi}^2, \tag{A.128}$$

where it should be noted that the moments of inertia $I_{1,zz}^{(S)}$ and $I_{2,zz}^{(S)}$ are defined about the respective center of gravity of the cylinders.

The translational part results with the velocities $\mathbf{v}_0^{s1}$ and $\mathbf{v}_0^{s2}$ to $T_t = T_{t1} + T_{t2}$, with

$$T_{t1} = \frac{1}{2}m_1\left(\mathbf{v}_0^{s1}\right)^{\mathrm{T}}\mathbf{v}_0^{s1} = \frac{1}{8}m_1 l_1^2\dot{\varphi}^2 \tag{A.129a}$$

$$T_{t2} = \frac{1}{2}m_2\left(\mathbf{v}_0^{s2}\right)^{\mathrm{T}}\mathbf{v}_0^{s2} = \frac{1}{2}m_2\left(\dot{s}^2 + \dot{\varphi}^2\left(s + \frac{l_2}{2}\right)^2\right). \tag{A.129b}$$

The potential energy V_f of the linear spring results in

$$V_f = \frac{1}{2}c(s - s_0)^2, \tag{A.130}$$

where s_0 describes the unstretched length of the spring. The potential energies due to gravity can be written in the form

$$V_1 = m_1 g\left(h - \frac{l_1}{2}\cos(\varphi)\right) \tag{A.131a}$$

$$V_2 = m_2 g\left(h - \left(s + \frac{l_2}{2}\right)\cos(\varphi)\right). \tag{A.131b}$$

This expression is obtained directly by using the y component of the center of gravity vectors. The total potential energy is thus calculated to be $V = V_f + V_1 + V_2$.

Using the Lagrange function

$$L = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) \tag{A.132}$$

the equations of motion of the mechanical system can be calculated. The essential intermediate results are calculated as follows:

$$\frac{\partial}{\partial\dot{\varphi}}L = \frac{1}{4}m_1 l_1^2\dot{\varphi} + m_2\left(s + \frac{l_2}{2}\right)^2\dot{\varphi} + \left(I_{1,zz}^{(S)} + I_{2,zz}^{(S)}\right)\dot{\varphi}, \tag{A.133a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial\dot{\varphi}}L\right) = \frac{1}{4}m_1 l_1^2\ddot{\varphi} + m_2\left(s + \frac{l_2}{2}\right)^2\ddot{\varphi} + 2m_2\left(s + \frac{l_2}{2}\right)\dot{\varphi}\dot{s} + \left(I_{1,zz}^{(S)} + I_{2,zz}^{(S)}\right)\ddot{\varphi}, \tag{A.133b}$$

$$\frac{\partial}{\partial\varphi}L = -m_1 g\frac{l_1}{2}\sin(\varphi) - m_2 g\left(s + \frac{l_2}{2}\right)\sin(\varphi), \tag{A.133c}$$

$$\frac{\partial}{\partial\dot{s}}L = m_2\dot{s}, \tag{A.133d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial\dot{s}}L\right) = m_2\ddot{s}, \tag{A.133e}$$

$$\frac{\partial}{\partial s}L = m_2\dot{\varphi}^2\left(s + \frac{l_2}{2}\right) - c(s - s_0) + m_2 g\cos(\varphi) \tag{A.133f}$$

To account for the effect of the friction force f_d, the principle of virtual work is used. The friction force was assumed to be proportional to the relative velocity of the surfaces of cylinders 1 and 2, i.e. proportional to $\dot{s}$. Furthermore, it was assumed that the friction is proportional to the contact area $A(s)$. This results in $A(s) = (l_1 - s)d_2\pi$, where d_2 is the diameter of cylinder 2. This gives

$$\mathbf{f}_d = d_0(l_1 - s)d_2\pi\dot{s}\begin{bmatrix} -\sin(\varphi) \\ \cos\varphi \\ 0 \end{bmatrix} = f_d\begin{bmatrix} -\sin(\varphi) \\ \cos\varphi \\ 0 \end{bmatrix}. \tag{A.134}$$

The point of application $\mathbf{p}_0^d$ of the friction force $\mathbf{f}_d$ can be assumed to be at the beginning of cylinder 2 (forces can be shifted along their line of action). Thus we have

$$\mathbf{p}_0^d = \begin{bmatrix} s\sin(\varphi) \\ h - s\cos(\varphi) \\ 0 \end{bmatrix}. \tag{A.135}$$

If we now use D'Alembert's principle, the generalized forces with respect to the degrees of freedom are calculated as

$$f_{f_d,s} = -f_d \tag{A.136a}$$
$$f_{f_d,\varphi} = 0 \tag{A.136b}$$

This result could also have been derived directly (with some practice in dealing with the calculation of generalized forces) from the fact that a change in the degree of freedom φ does not produce a displacement of the friction force and thus no (virtual) work.

By analogous considerations, one obtains the generalized forces due to the external torque τ_e. Here, a displacement with respect to the degree of freedom s does no virtual work, while a rotation with respect to the degree of freedom φ directly results in work with the external torque τ_e. Thus one obtains

$$f_{\tau_e,s} = 0 \tag{A.137a}$$
$$f_{\tau_e,\varphi} = \tau_e \tag{A.137b}$$

and finally

$$f_s = f_{f_d,s} + f_{\tau_e,s} = -f_d \tag{A.138a}$$
$$f_\varphi = f_{f_d,\varphi} + f_{\tau_e,\varphi} = \tau_e \tag{A.138b}$$

The equations of motion are obtained by assembling the intermediate results to

$$\begin{aligned}\left(\frac{1}{4}m_1 l_1^2 + m_2\left(s+\frac{l_2}{2}\right)^2 + I_{1,zz}^{(S)} + I_{2,zz}^{(S)}\right)\ddot{\varphi} + 2m_2\left(s+\frac{l_2}{2}\right)\dot{\varphi}\dot{s} \\ +g\left(m_1\frac{l_1}{2} + m_2\left(s+\frac{l_2}{2}\right)\right)\sin(\varphi) = \tau_e\end{aligned} \tag{A.139a}$$

$$m_2\ddot{s} + c(s-s_0) - m_2 g\cos(\varphi) - m_2\dot{\varphi}^2\left(s+\frac{l_2}{2}\right) = -f_d \tag{A.139b}$$

The equilibrium positions (stationary points) of a system are characterized by $\ddot{\mathbf{q}} = \dot{\mathbf{q}} = \mathbf{0}$. Thus, the equilibrium positions of the system for $\tau_e = 0$ follow as

$$\varphi_R = k\pi \quad \text{with} \quad k \in \mathbb{Z}, \tag{A.140}$$

$$s_R = \frac{m_2 g\cos(\varphi_R)}{c} + s_0 = s_0 \pm \frac{m_2 g}{c}. \tag{A.141}$$

Obviously the spring is stretched in the lower equilibrium position $\varphi_R = 0$ ($s_R > s_0$) and compressed in the upper equilibrium position $\varphi = \pi$ ($s_R < s_0$). Furthermore, it is immediately clear that a rotation of the mechanism by $k2\pi$, $k \in \mathbb{Z}$, does not change the stationary conditions. Thus, although the system has an infinite number of equilibrium positions, the stationary behavior can be completely characterized by the two essential equilibrium positions (upper and lower equilibrium position).

Solution in MAPLE: `DrehgelagerterHohlzylinder.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.13 (Rigid Body with Torsional Spring). Given is the mechanical system shown in Figure A.21. A beam is mounted at the origin, rotatable by an angle φ, and consists of two rigidly connected rods. The rods have the lengths L_s and $L_s/2$ and the masses m and $m/2$, respectively. The mass moment of inertia of a rod about its center of gravity can be approximated by $I_{zz,s}^{(S)} = \frac{m_s L_s^2}{12}$. The gravitational acceleration g acts in the negative $\mathbf{e}_y$ direction, and the spring is relaxed at $\varphi = 0$. The force $\mathbf{f}$ with magnitude f acts in the direction of the second rod's axis.

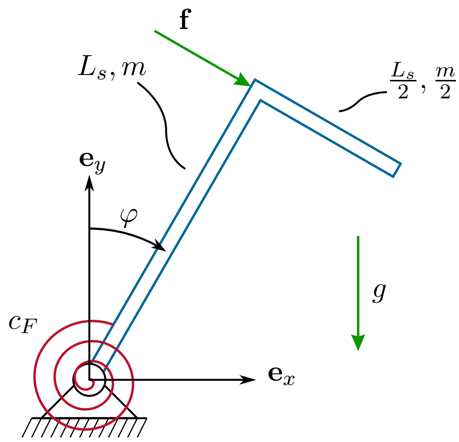


Figure A.21: Rigid body system with torsional spring.

For this system, the first step is to calculate the location of the center of gravity and the total moment of inertia about this center of gravity. Subsequently, the equations of motion of the system are to be determined using the Euler-Lagrange equations.

Solution of exercise A.13. In the first step, the vector $\mathbf{p}_0^s$ of the entire beam for $\varphi = 0$ and the mass moment of inertia of the beam about its center of gravity are determined. First, the center of gravity vectors and the mass moments of inertia of the individual rods are specified. The center of gravity vectors of the two rods of the beam for $\varphi = 0$ can be read directly from the sketch and are

$$\mathbf{p}_0^{s1} = \begin{bmatrix} 0 \\ \frac{L_s}{2} \\ 0 \end{bmatrix} \tag{A.142}$$

and

$$\mathbf{p}_0^{s2} = \begin{bmatrix} \frac{L_s}{4} \\ L_s \\ 0 \end{bmatrix}, \tag{A.143}$$

respectively. Applying formula (3.31), the resulting center of gravity of the beam can be determined as

$$\mathbf{p}_0^s = \frac{m\mathbf{p}_0^{s1} + \frac{m}{2}\mathbf{p}_0^{s2}}{m + \frac{m}{2}} = \begin{bmatrix} \frac{L_s}{12} \\ \frac{2L_s}{3} \\ 0 \end{bmatrix}. \tag{A.144}$$

The mass moments of inertia of the two rods about their respective centers of gravity are given by

$$I_{zz}^{(s1)} = \frac{mL_s^2}{12} \tag{A.145a}$$

$$I_{zz}^{(s2)} = \frac{mL_s^2}{96}. \tag{A.145b}$$

The total mass moment of inertia of the beam about the total center of gravity is calculated by applying formula (3.126) as

$$I_{zz}^{(s)} = I_{zz}^{(s1)} + m\left(x_{s,s1}^2 + y_{s,s1}^2\right) + I_{zz}^{(s2)} + \frac{m}{2}\left(x_{s,s2}^2 + y_{s,s2}^2\right) = \frac{19mL_s^2}{96}, \tag{A.146}$$

where the variables $x_{s,s1}$, $y_{s,s1}$ and $x_{s,s2}$, $y_{s,s2}$ denote the x and y coordinates of the vectors

$$\mathbf{p}_{s,s1} = \mathbf{p}_0^s - \mathbf{p}_0^{s1} = \begin{bmatrix} \frac{L_s}{12} \\ \frac{L_s}{6} \\ 0 \end{bmatrix} \tag{A.147a}$$

$$\mathbf{p}_{s,s2} = \mathbf{p}_0^s - \mathbf{p}_0^{s2} = \begin{bmatrix} -\frac{L_s}{6} \\ -\frac{L_s}{3} \\ 0 \end{bmatrix}, \tag{A.147b}$$

respectively.

In the next step, the center of gravity vector and the center of gravity velocity are determined as a function of the generalized coordinate φ. The center of gravity vector can be expressed, starting from $\mathbf{p}_0^s$ for $\varphi = 0$ via geometric considerations or using the rotation matrix, as

$$\mathbf{p}_0^s = \begin{bmatrix} \frac{L_s}{12}\cos(\varphi) + \frac{2L_s}{3}\sin(\varphi) \\ -\frac{L_s}{12}\sin(\varphi) + \frac{2L_s}{3}\cos(\varphi) \\ 0 \end{bmatrix}. \tag{A.148}$$

The translational velocity of the center of gravity is then obtained by taking the time derivative of the center of gravity vector:

$$\mathbf{v}_0^s = \begin{bmatrix} -\frac{L_s}{12}\sin(\varphi)\dot{\varphi} + \frac{2L_s}{3}\cos(\varphi)\dot{\varphi} \\ -\frac{L_s}{12}\cos(\varphi)\dot{\varphi} - \frac{2L_s}{3}\sin(\varphi)\dot{\varphi} \\ 0 \end{bmatrix}. \tag{A.149}$$

The translational kinetic energy is

$$T_t = \frac{1}{2}\frac{3m}{2}(\mathbf{v}_0^s)^{\mathrm{T}}\mathbf{v}_0^s = \frac{65}{192}m\dot{\varphi}^2 L_s^2, \tag{A.150}$$

and the rotational energy is

$$T_r = \frac{1}{2} I_{zz}^{(s)} \dot{\varphi}^2 = \frac{19}{192} m \dot{\varphi}^2 L_s^2. \tag{A.151}$$

The potential energy is composed of the potential energy in the gravitational field and the potential spring energy and can be calculated as

$$V = V_g + V_F = -\frac{L_s}{8} mg \sin(\varphi) + mgL_s \cos(\varphi) + \frac{1}{2} c_F \varphi^2. \tag{A.152}$$

The generalized force due to the force $\mathbf{f}$ is $f_{q,f} = fL_s$. This is obtained directly from the fact that the force $\mathbf{f}$ always acts in the direction of rod 2 (and thus orthogonal to rod 1). The generalized force $f_{q,f}$ thus corresponds to the moment acting about the pivot point. This result would also be obtained by applying D'Alembert's principle (5.57). The point of application $\mathbf{p}_0^f$ of the force $\mathbf{f}$ is calculated as

$$\mathbf{p}_0^f = \begin{bmatrix} L_s \sin(\varphi) \\ L_s \cos(\varphi) \\ 0 \end{bmatrix}, \tag{A.153}$$

and the force can be calculated as a function of the angle φ in the form

$$\mathbf{f} = \begin{bmatrix} f \cos(\varphi) \\ -f \sin(\varphi) \\ 0 \end{bmatrix}. \tag{A.154}$$

The generalized force can thus be determined via

$$f_{q,f} = \mathbf{f}^{\mathrm{T}} \frac{\partial \mathbf{p}_0^f}{\partial \varphi} = fL_s. \tag{A.155}$$

In the last step, the equation of motion of the system is calculated using the Euler-Lagrange equations. For this purpose, the Lagrangian

$$L = T_r + T_t - V = \frac{7}{16} m \dot{\varphi}^2 L_s^2 + \frac{L_s}{8} mg \sin(\varphi) - mgL_s \cos(\varphi) - \frac{1}{2} c_F \varphi^2 \tag{A.156}$$

is used and inserted into the Euler-Lagrange equations (5.69)

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{\varphi}} L - \frac{\partial}{\partial \varphi} L = f_{q,f}. \tag{A.157}$$

Evaluating the Euler-Lagrange equations yields the equation of motion of the system

$$\frac{7}{8} m L_s^2 \ddot{\varphi} - \frac{1}{8} mgL_s \cos(\varphi) - mgL_s \sin(\varphi) + c_F \varphi = fL_s. \tag{A.158}$$

Solution in MAPLE: `StarrkoerpermitDrehfeder.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.14 (Elastically Mounted Cantilever Beam). A beam B (piecewise rectangular, constant thickness d and homogeneous density ρ, moment of inertia $I_{B,zz}^{(S)}$ about the center of gravity) is, as shown in Figure A.22, rotatably mounted in a joint A on a carriage S (mass m_s). Viscous friction (proportional to the angular velocity $\dot{\varphi}$) occurs in the bearing A with the constant friction parameter $d_2 > 0$. A torsional spring acts between the beam B and the carriage S, whose torque increases linearly with the deflection φ of the beam (spring constant $c_2 > 0$). The carriage S is mounted on the carriage guide SF, which only allows a translational degree of freedom in the direction s. In the carriage bearing there is velocity-proportional friction with the constant friction parameter $d_1 > 0$. Between the carriage S and the ground there is a linear spring with the constant spring stiffness $c_1 > 0$. An external force $\mathbf{f}^e$ with $\|\mathbf{f}^e\| = f^e$ acts on the beam, which is always perpendicular to the beam. Figure A.22 shows the system with relaxed springs ($s = s_{10}, \varphi = 0$).

Consider the following quantities as given for the calculations: m_s, ρ, $I_{B,zz}^{(S)}$, b_1, b_2, b_3, d, l_1, l_2, l_3, c_1, s_{10}, c_2, d_1, d_2, f^e.

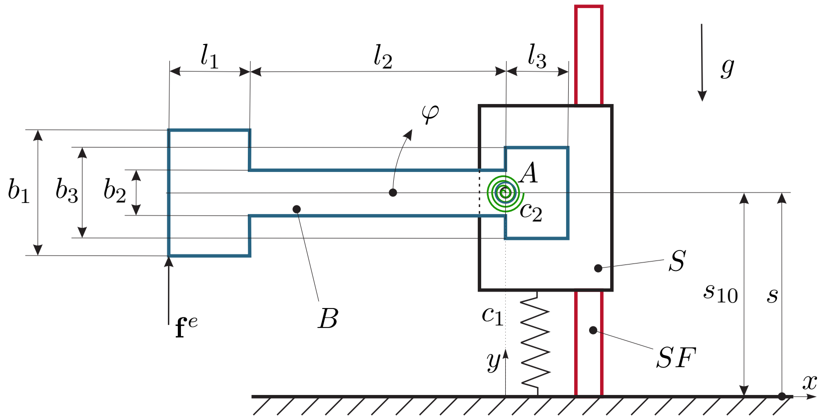


Figure A.22: Elastically mounted cantilever beam.

For this system, the equations of motion are to be determined using the Euler-Lagrange equations.

Solution of exercise A.14. To set up the equations of motion, the Lagrange function $L = T - V$ with the kinetic energy T and the potential energy V must be determined. The generalized coordinates (degrees of freedom) are given by $\mathbf{q}^{\mathrm{T}} = \begin{bmatrix} s & \varphi \end{bmatrix}$.

In the first step, the potential energy $V = V_g + V_f$ is determined. This is composed of the potential energy V_g due to gravity and the potential energy V_f stored in the two springs.

The mass m_B of the beam B required for calculating the energies is obtained from (3.24) to

$$m_B = \int_{\mathcal{V}} \rho \,\mathrm{d}\mathcal{V} = \rho d(b_1 l_1 + b_2 l_2 + b_3 l_3). \tag{A.159}$$

Due to the constant thickness d, the homogeneous density ρ and the vertical symmetry for $\varphi = 0$, the center of gravity of the beam B lies on the horizontal line passing through the bearing A. The distance l_s of the center of gravity of the beam B from the joint axis A (pivot point) is obtained from (3.29) to

$$l_s = \frac{b_1 l_1 \left(\frac{l_1}{2} + l_2\right) + \frac{b_2 l_2^2}{2} - \frac{b_3 l_3^2}{2}}{b_1 l_1 + b_2 l_2 + b_3 l_3}. \tag{A.160}$$

Thus, the vector $\mathbf{p}_0^{sB}$ to the center of gravity of the beam as a function of the generalized coordinates $\mathbf{q}^T = \begin{bmatrix} s & \varphi \end{bmatrix}$ is

$$\mathbf{p}_0^{sB} = \begin{bmatrix} -l_s \cos(\varphi) \\ s + l_s \sin(\varphi) \\ 0 \end{bmatrix}. \tag{A.161}$$

With these results, the potential energy due to gravity with the reference level $y = 0$ can be calculated for the beam to

$$V_{g,B} = g m_B (s + l_s \sin(\varphi)) \tag{A.162}$$

and for the carriage S to

$$V_{g,S} = g m_S s. \tag{A.163}$$

Since the springs were defined as linear in this example, i.e. $c_1 = \text{const.}$ and $c_2 = \text{const.}$, their potential energy is given by

$$V_{c1} = \frac{1}{2} c_1 (s - s_{10})^2 \tag{A.164a}$$

$$V_{c2} = \frac{1}{2} c_2 \varphi^2. \tag{A.164b}$$

The total potential energy is finally

$$V = V_{g,B} + V_{g,S} + V_{c1} + V_{c2}. \tag{A.165}$$

The kinetic energy $T = T_{t,S} + T_{t,B} + T_{r,B}$ is composed of the translational kinetic energy of the carriage $T_{t,S}$, the translational kinetic energy of the beam $T_{t,B}$, and the rotational kinetic energy of the beam $T_{r,B}$. The translational kinetic energy of the carriage follows to

$$T_{t,S} = \frac{1}{2} m_S \dot{s}^2 \tag{A.166}$$

and that of the beam to

$$T_{t,B} = \frac{1}{2} m_B \left(\dot{\mathbf{p}}_0^{sB}\right)^{\mathrm{T}} \dot{\mathbf{p}}_0^{sB} = \frac{1}{2} m_B \left(\dot{\varphi}^2 l_s^2 + \dot{s}^2 + 2 l_s \dot{\varphi} \dot{s} \cos(\varphi)\right) . \tag{A.167}$$

The moment of inertia $I_{B,zz}^{(S)}$ of the beam B is defined about an axis passing through the center of gravity. Thus, one obtains for the rotational part of the kinetic energy of the beam

$$T_{r,B} = \frac{1}{2} I_{B,zz}^{(S)} \dot{\varphi}^2 . \tag{A.168}$$

With these results, the Lagrange function $L = T - V$ is completely determined.

Now, the vector of generalized forces $\mathbf{f}_q^{\mathrm{T}} = \begin{bmatrix} f_{q,s} & f_{q,\varphi} \end{bmatrix}$ is still missing, which is composed of the effect of the external force $\mathbf{f}^e$ and the dissipative forces. The external force can be represented in the form

$$\mathbf{f}^e = \begin{bmatrix} f^e \sin(\varphi) \\ f^e \cos(\varphi) \\ 0 \end{bmatrix} \tag{A.169}$$

and the vector $\mathbf{p}_0^f$ to the point of application of the external force $\mathbf{f}^e$ results in

$$\mathbf{p}_0^f = \begin{bmatrix} -(l_1 + l_2)\cos(\varphi) - \frac{b_1}{2}\sin(\varphi) \\ s + (l_1 + l_2)\sin(\varphi) - \frac{b_1}{2}\cos(\varphi) \\ 0 \end{bmatrix} . \tag{A.170}$$

The contributions to the generalized force due to the external force $\mathbf{f}^e$ are calculated as

$$f_{q,s,f^e} = \left(\frac{\partial \mathbf{p}_0^f}{\partial s}\right)^{\mathrm{T}} \mathbf{f}^e = f^e \cos(\varphi) \tag{A.171a}$$

$$f_{q,\varphi,f^e} = \left(\frac{\partial \mathbf{p}_0^f}{\partial \varphi}\right)^{\mathrm{T}} \mathbf{f}^e = f^e (l_1 + l_2) . \tag{A.171b}$$

Since the friction is assumed to be proportional to velocity with constant friction parameters, the proportion of the generalized force due to the friction forces or torquess is obtained as

$$f_{q,s,d} = -d_1 \dot{s} \tag{A.172a}$$

$$f_{q,\varphi,d} = -d_2 \dot{\varphi} . \tag{A.172b}$$

In sum, the vector of the generalized force results in

$$f_{q,s} = f_{q,s,f^e} + f_{q,s,d} \tag{A.173a}$$
$$f_{q,\varphi} = f_{q,\varphi,f^e} + f_{q,\varphi,d} \ . \tag{A.173b}$$

The Euler-Lagrange equations for the rotatably mounted cantilever beam from Figure A.22 are therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = f_{q,s} \tag{A.174a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = f_{q,\varphi} \ . \tag{A.174b}$$

A presentation of the evaluation of (A.174) is omitted here, as it is very extensive. For control, these can be taken from the sample solution in MAPLE.

Solution in MAPLE: `ElastischgelagerterAusleger.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/

Exercise A.15 (Trebuchet). To win the pumpkin throwing competition, the dynamics of the catapult used for this purpose should be analyzed. It is known that a trebuchet, as sketched in Fig. A.23, has the best efficiency of all throwing machines. A heavy counterweight of mass M accelerates the projectile (pumpkin) with mass m. The throwing arm has the length $L+l$ and the counterweight is connected to the throwing arm by a pendulum mechanism of length h. A sling with length r provides additional range. For simplification, it can be assumed that the moment of inertia of the throwing arm, the counterweight and the projectile can be neglected. Furthermore, the ropes are assumed to be massless.

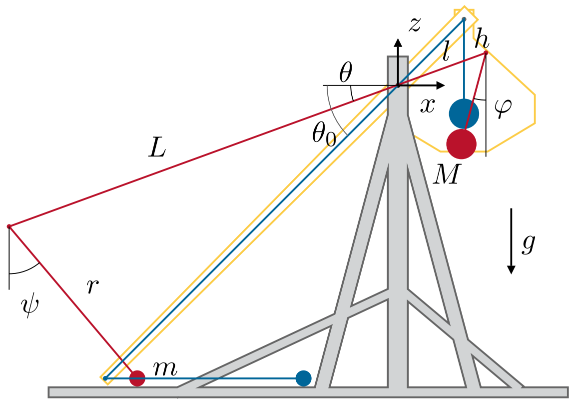


Figure A.23: Sketch of the trebuchet: Starting position blue, position during the acceleration phase red.

For this system, the equations of motion should be derived.

Remark: To better understand the dynamics of the catapult, a typical motion sequence should first be discussed:

In the starting position, which is shown in blue in the sketch, the tip of the throwing arm is on the ground and the counterweight M is accordingly at the highest possible point. If the release mechanism, which is not considered here, is now actuated, the counterweight moves downwards and the throwing arm begins to rotate around the pivot point (here the origin of the coordinate system). The rope attached to the end of the arm transfers this movement to the projectile m, which is now accelerated along a certain path until launch. Note that the projectile first slides along the ground and only leaves it at a certain point.

The motion sequence of the machine can thus be divided into three phases:

1. In the first phase after release, the projectile slides along the ground until the time it leaves the ground.
2. In the second phase, the projectile has left the ground and is accelerated until it is decoupled from the rope.
3. In the third phase, the projectile is separated from the throwing machine. The arm and the counterweight perform a pendulum motion until the catapult has come to rest again.

In the context of this example, only the first and second phases will be considered.

Solution of exercise A.15. In the first step, the necessary degrees of freedom (generalized coordinates) to describe the system are determined. The throwing arm rotates around the y-axis with the angle θ, whereby $\theta = \theta_0$ should apply in the initial state. The counterweight M is rotatably attached to the short end of the arm by the angle φ. This is measured here relative to the vertical. The projectile is rotatably attached to the end of the throwing arm by a rope at an angle ψ (again defined relative to the vertical). To describe the dynamics of the system, the following generalized coordinates are chosen:

$$\mathbf{q} = \begin{bmatrix} \theta \\ \varphi \\ \psi \end{bmatrix} . \tag{A.175}$$

In the next step, the vectors $\mathbf{p}_0^M$ and $\mathbf{p}_0^m$ from the origin of the coordinate system to the counterweight M and projectile m, respectively, are determined. As shown in the sketch A.23, the origin of the coordinate system is located at the pivot point of the throwing arm. The vectors $\mathbf{p}_0^M$ and $\mathbf{p}_0^m$ are thus obtained by

$$\mathbf{p}_0^M = \begin{bmatrix} l\cos(\theta) - h\sin(\varphi) \\ 0 \\ l\sin(\theta) - h\cos(\varphi) \end{bmatrix} \tag{A.176a}$$

$$\mathbf{p}_0^m = \begin{bmatrix} -L\cos(\theta) + r\sin(\psi) \\ 0 \\ -L\sin(\theta) - r\cos(\psi) \end{bmatrix} . \tag{A.176b}$$

In the first phase of the movement, the projectile m moves along the ground. Thus, there is a constraint in this phase of the motion. With the height $h_m = -L\sin(\theta_0)$ of the projectile at the beginning of the motion, this constraint $\mathbf{e}_z^{\mathrm{T}}\mathbf{p}_0^m = h_m$ can be formulated as an equation of the form

$$L\sin(\theta) + r\cos(\psi) = L\sin(\theta_0) \tag{A.177}$$

This constraint must be fulfilled throughout the first phase, whereby the angles ψ and θ are not independent of each other in this phase. If the angle θ is now used as an independent coordinate (degree of freedom) in this phase, then ψ can be determined in the form

$$\psi(\theta) = \arccos\left(\frac{L}{r}(\sin(\theta_0) - \sin(\theta))\right) \tag{A.178}$$

This relationship must be taken into account in the further derivation of the equations of motion for phase 1.

Remark: A much more systematic approach to describing this constraint is to use Lagrange multipliers. However, since these are not part of this course, their use will be dispensed with.

The velocities of the projectile and the counterweight, respectively, are obtained in phase 2 by taking the total time derivative of the position vectors:

$$\mathbf{v}_0^M = \begin{bmatrix} -l\sin(\theta)\dot{\theta} - h\cos(\varphi)\dot{\varphi} \\ 0 \\ l\cos(\theta)\dot{\theta} + h\sin(\varphi)\dot{\varphi} \end{bmatrix} \tag{A.179a}$$

$$\mathbf{v}_0^{m,2} = \begin{bmatrix} L\sin(\theta)\dot{\theta} + r\cos(\psi)\dot{\psi} \\ 0 \\ -L\cos(\theta)\dot{\theta} + r\sin(\psi)\dot{\psi} \end{bmatrix}. \tag{A.179b}$$

The velocity vector for phase 1 is calculated by substituting the constraint (A.178):

$$\mathbf{v}_0^{m,1} = \begin{bmatrix} L\sin(\theta)\dot{\theta} + r\cos(\psi)\dot{\psi} \\ 0 \\ 0 \end{bmatrix}. \tag{A.180}$$

The kinetic energies for phase 1 and 2 are calculated to be:

$$T_1 = \frac{1}{2}M\Big(l^2\dot{\theta}^2 + h^2\dot{\varphi}^2 + 2lh\dot{\theta}\dot{\varphi}\sin(\theta+\varphi)\Big) + \frac{1}{2}m\Big(L\sin(\theta)\dot{\theta} + r\cos(\psi)\dot{\psi}\Big)^2 \tag{A.181a}$$

$$T_2 = \frac{1}{2}M\Big(l^2\dot{\theta}^2 + h^2\dot{\varphi}^2 + 2lh\dot{\theta}\dot{\varphi}\sin(\theta+\varphi)\Big) + \frac{1}{2}m\Big(L^2\dot{\theta}^2 + r^2\dot{\psi}^2 + 2Lr\dot{\theta}\dot{\psi}\sin(\theta-\psi)\Big), \tag{A.181b}$$

where, for clarity, the substitution of the relationship

$$\dot{\psi} = \frac{\partial\psi}{\partial\theta}\dot{\theta} \tag{A.182}$$

in T_1 for phase 1 has been omitted.

The potential energies of the projectile and the counterweight result in:

$$V_M = Mg(l\sin(\theta) - h\cos(\varphi)) + V_{M,0} \tag{A.183}$$

$$V_m = mg(-L\sin(\theta) - r\cos(\psi)) + V_{m,0} \tag{A.184}$$

and the total potential energy of the system results, depending on the phase, in

$$V_1 = V_M \tag{A.185}$$

$$V_2 = V_M + V_m, \tag{A.186}$$

since in phase 1 only the potential energy of the counterweight has to be considered (the potential energy of the projectile is constant in this phase).

The equations of motion of the system can now be determined directly by applying the Euler-Lagrange formalism. A further detailed derivation of the equations of motion using the Lagrange formalism is omitted due to the rather unwieldy expressions. The equations of motion of the catapult for phase two are therefore

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0} \tag{A.187}$$

with the mass matrix

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} L^2 m + l^2 M & Mlh\sin(\theta+\varphi) & mLr\sin(\theta-\psi) \\ Mlh\sin(\theta+\varphi) & Mh^2 & 0 \\ mLr\sin(\theta-\psi) & 0 & mr^2 \end{bmatrix} \tag{A.188}$$

and the remaining terms

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -mLr\cos(\theta-\psi)\dot{\psi}^2 + Mlh\cos(\theta+\varphi)\dot{\varphi}^2 - g(mL - Ml)\cos(\theta) \\ Mlh\cos(\theta+\varphi)\dot{\theta}^2 + Mgh\sin(\varphi) \\ mLr\cos(\theta-\psi)\dot{\theta}^2 + mgr\sin(\psi) \end{bmatrix} . \tag{A.189}$$

The entire equation of motion is presented in MAPLE in the sample solution.

Solution in MAPLE: `Trebuchet.mw`
https://www.acin.tuwien.ac.at/bachelor/modellbildung/