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Mechatronic Systems: Solution of Exercise 1

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Problem 1: Floating mass

- Differential equation

$$m\ddot{x} = F$$

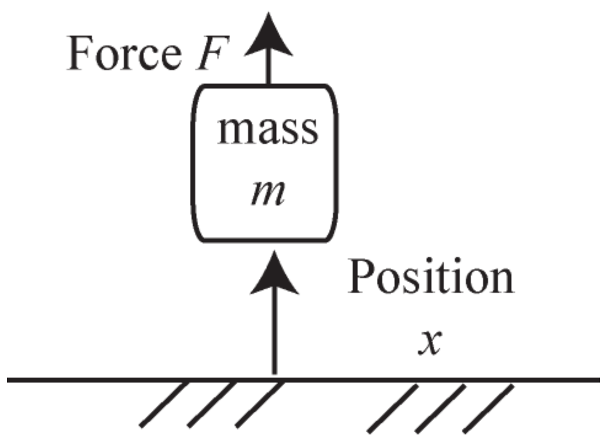
- Laplace transformation

$$ms^2x = F$$

- Transfer function

$$P(s) = \frac{x}{F} = \frac{1}{ms^2}$$

$$\frac{dx}{dt} = \dot{x}$$
$$\frac{d^2x}{dt^2} = \ddot{x}$$



Problem 2i: Damped mass-spring

- Differential equation

$$m\ddot{x} = F - c\dot{x} - kx$$

- Laplace transformation

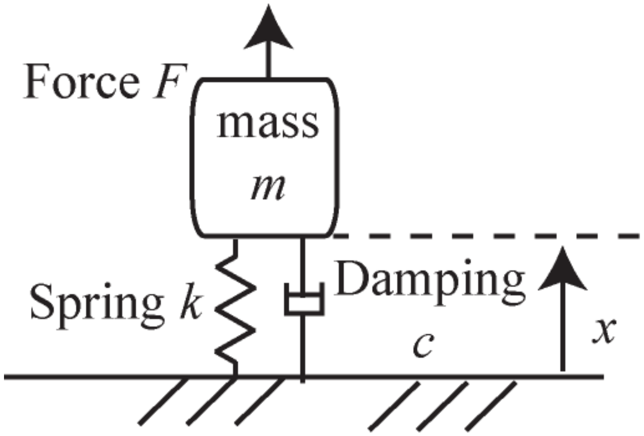
$$ms^2x = F - csx - kx$$

or

$$(ms^2 + cs + k)x = F$$

- Transfer function

$$P(s) = \frac{x}{F} = \frac{1}{ms^2 + cs + k}$$



Problem 2i: (Undamped) natural frequency

- Undamped mass spring system (i.e. *c=0*)

$$P(s) = \frac{x}{F} = \frac{1}{ms^2 + k}$$

- Laplace variable in the case of a steady state

$$s = \sigma + j\omega \quad \Rightarrow \quad s = j\omega$$

- At the natural frequency, the response becomes infinite, which corresponds to that the denominator of P(s) is zero.

$$\mathrm{k} - m{\omega_0}^2 = 0$$

- Natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{or} \quad f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Problem 2ii: Effects of low damping (1)

- Magnitude of un-damped system at ω_0

$$\left|P_{No\_c}\right| = \infty$$

- Magnitude of damped system at ω_0

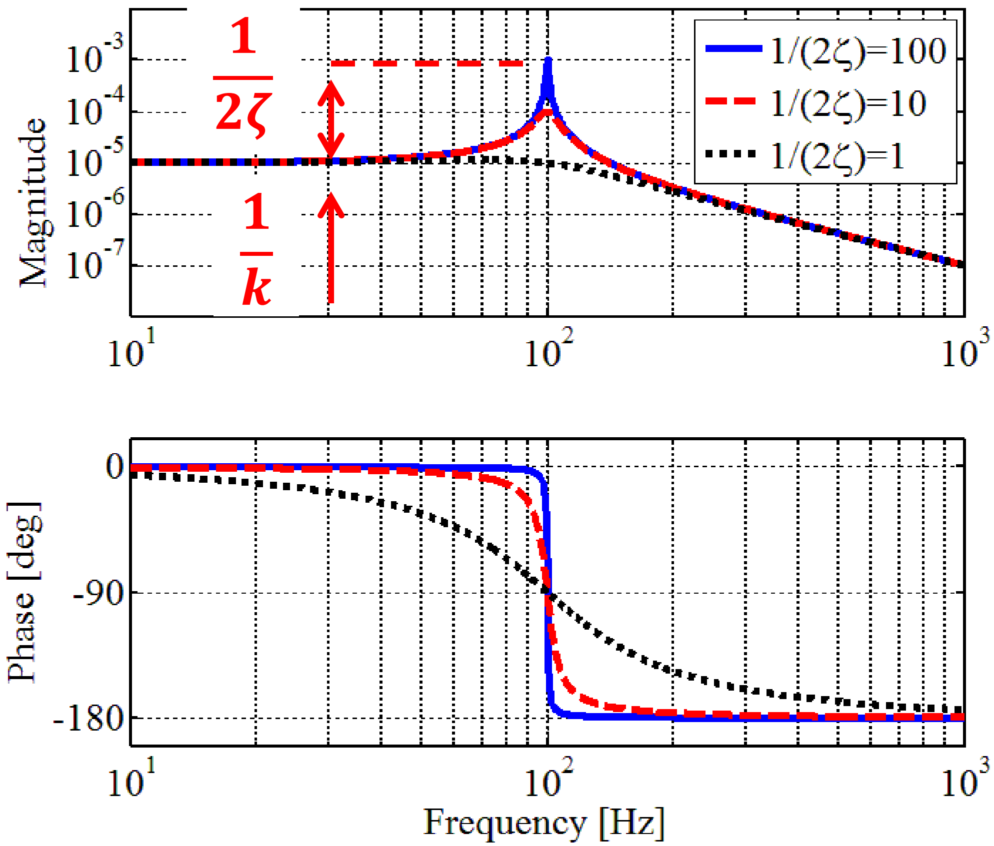
$$\left|P_{Low\_c}\right| = \left|\frac{1}{jc\omega_0}\right|$$

$$= \frac{1}{c}\sqrt{\frac{m}{k}}$$

Magnitude of spring line

$$= \frac{1}{k}\frac{1}{2\zeta} = \frac{Q}{k}$$

where $\zeta = c/(2\sqrt{km})$

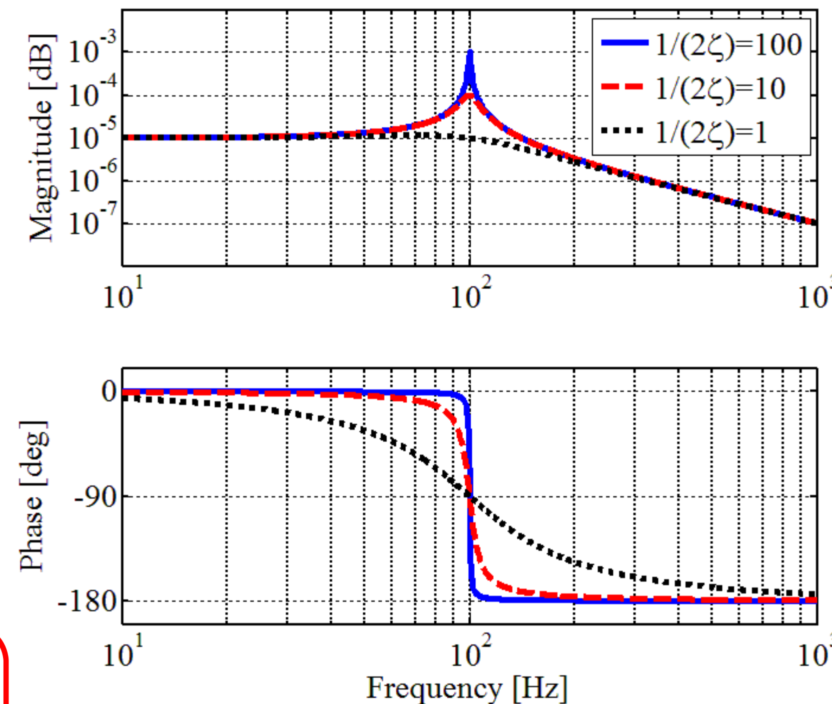


Problem 2ii: Effects of low damping (1)

- Damped resonant frequency ω_r
  - The frequency where the gain show a peak.

$$\omega_r = \arg\max_{\omega} |P(j\omega)|$$

$$= \omega_0 \sqrt{1 - 2\zeta^2}$$



- Without damping,

  $\omega_r = \omega_0$.

- For a slight damping ($\zeta < \sqrt{2}$),

  ω_r decreases.

Typical case of practice precision actuators

- For a large damping ($\zeta \geq \sqrt{2}$), what happens to ω_r?

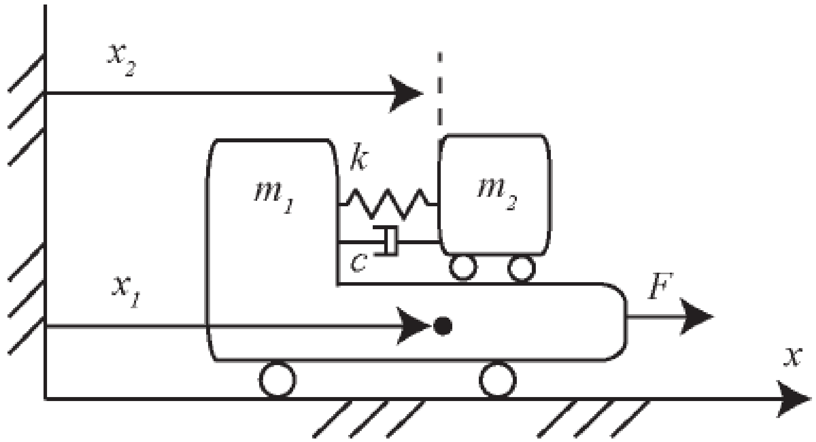
Problem (a)3i: Decoupling mass

- Equation of motion about m_1

$$m_1\ddot{x}_1 = -c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 + F$$

- Equation of motion about m_2

$$m_2\ddot{x}_2 = +c\dot{x}_1 - c\dot{x}_2 + kx_1 - kx_2$$



Problem (a)3ii: Transfer function (1)

- By rearrange the equations we get:

$$\ddot{x}_1 = -\frac{k}{m_1}x_1 + \frac{k}{m_1}x_2 - \frac{c}{m_1}\dot{x}_1 + \frac{c}{m_1}\dot{x}_2 + \frac{F}{m_1}$$
$$\ddot{x}_2 = +\frac{k}{m_2}x_1 - \frac{k}{m_2}x_2 + \frac{c}{m_2}\dot{x}_1 - \frac{c}{m_2}\dot{x}_2$$

- State space model: $\dot{x} = Ax + BF, \qquad y = Cx$

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & -c/m_1 & c/m_1\\ k/m_2 & -k/m_2 & c/m_2 & -c/m_2\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}x_1\\ x_2\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix}0\\ 0\\ 1/m_1\\ 0\end{bmatrix}}_{\mathbf{B}}F$$

Problem (a)3ii: Transfer function (2)

- Transfer function from F to x_1 is given with C=[1 0 0 0]

$$P_1(s) = X_1(s)/F(s) = C(sI - A)^{-1}B$$

$$= \frac{m_2 s^2 + cs + k}{s^2\{m_1 m_2 s^2 + (m_1 + m_2)(cs + k)\}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

- Transfer function from F to x_2 is given with C=[0 1 0 0]

$$P_2(s) = X_2(s)/F(s) = C(sI - A)^{-1}B$$

$$= \frac{cs + k}{s^2\{m_1 m_2 s^2 + (m_1 + m_2)(cs + k)\}}$$

Problem (a)3iii: Bode plot

$$P_1(s) = \frac{X_1(s)}{F} = \frac{m_2 s^2 + cs + k}{s^2\{m_1 m_2 s^2 + (m_1 + m_2)(cs + k)\}}$$

- Assuming that damping is sufficiently low (i.e. c=0),
  - Anti-resonance occurs at:

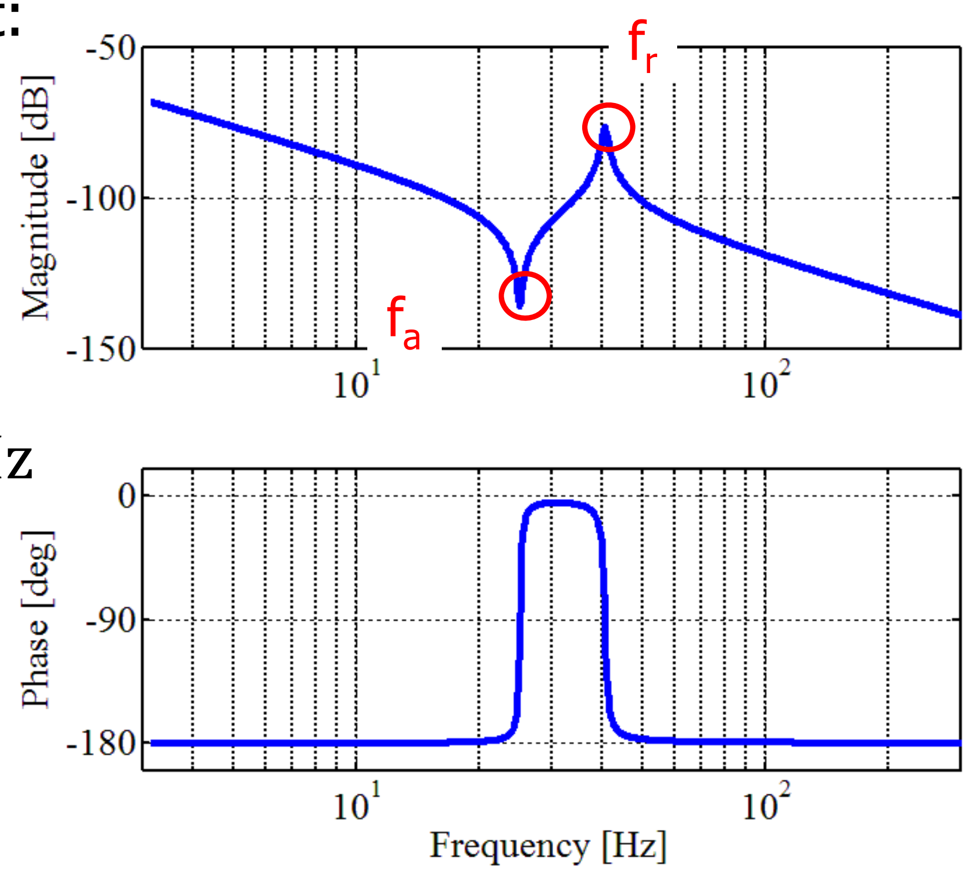
$$f_a = \frac{1}{2\pi}\sqrt{\frac{k}{m_2}} = 25\ Hz$$

  - Resonance occurs at:

$$f_r = \frac{1}{2\pi}\sqrt{\frac{(m_1 + m_2)k}{m_1 m_2}} = 41\ \text{Hz}$$

  - Ratio

$$\frac{f_r}{f_a} = \sqrt{1 + \frac{m_2}{m_1}} = 1.6$$



Problem (a)3iii: Bode plot

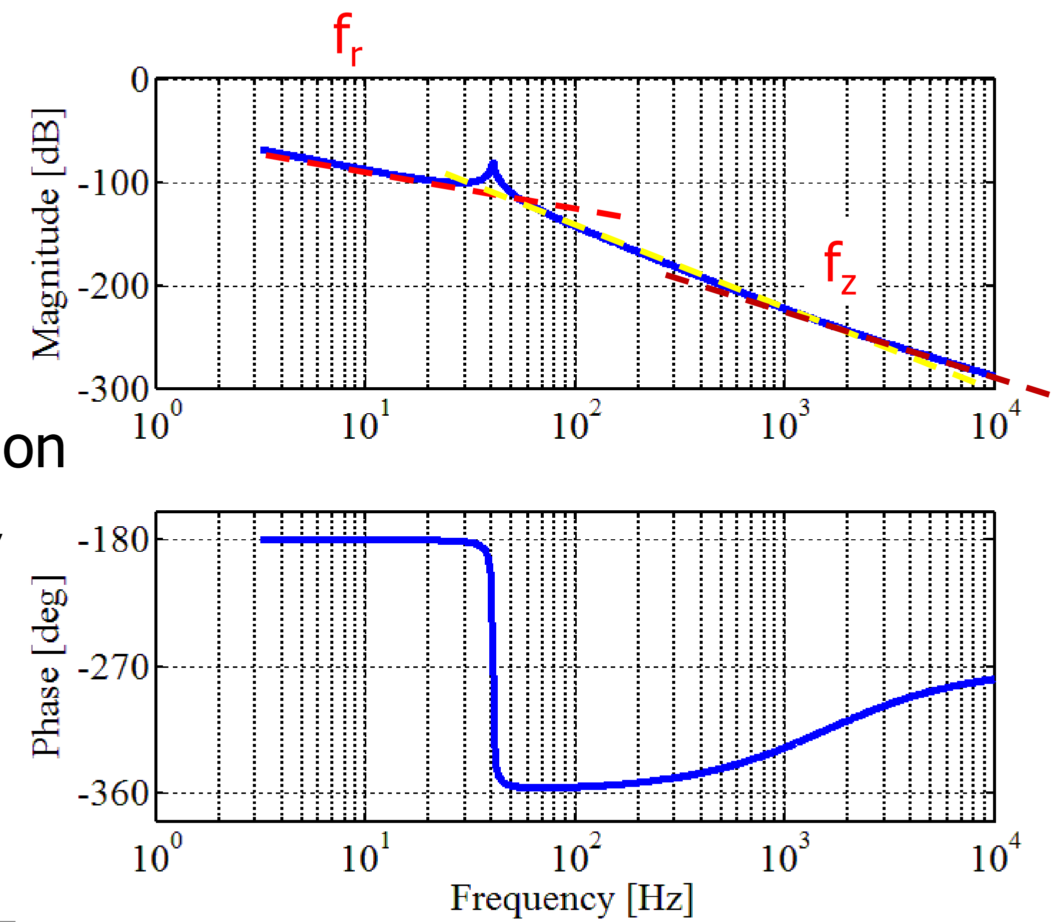
$$P_2(s) = \frac{X_2(s)}{F} = \frac{cs + k}{s^2\{m_1 m_2 s^2 + (m_1 + m_2)(cs + k)\}}$$

- Assuming that damping is sufficiently low (i.e. c=0),
  - Resonance occurs at:

$$f_r = \frac{1}{2\pi}\sqrt{\frac{(m_1 + m_2)k}{m_1 m_2}}$$

  - Straight line approximation gives a corner frequency

$$f_z = \frac{1}{2\pi}\frac{k}{c} = 1.6\ \text{kHz}$$



Problem (a)3iv: Bode plot