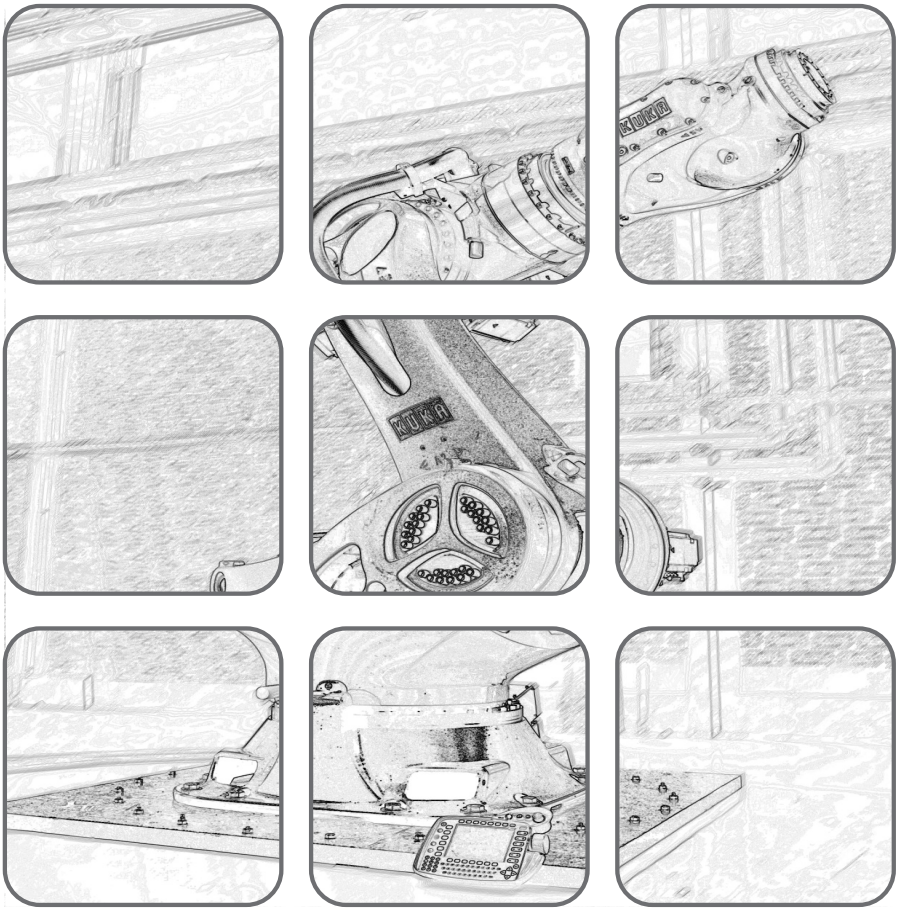
TECHNISCHE UNIVERSITÄT WIEN

AUTOMATION & CONTROL INSTITUTE
INSTITUT FÜR AUTOMATISIERUNGS- & REGELUNGSTECHNIK

Lecture
2026W

CONTROL SYSTEMS

Univ.-Prof. Dr.-Ing. Wolfgang Kemmetmüller
Univ.-Prof. Dr. techn. Andreas Kugi



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Univ.-Prof. Dr.-Ing. Wolfgang Kemmetmüller
Univ.-Prof. Dr. techn. Andreas Kugi

TU Wien
Automation and Control Institute
Complex Dynamical Systems Group

Gußhausstraße 27–29
1040 Wien
Phone: +43 1 58801 – 37615
Internet: https://www.acin.tuwien.ac.at

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1 Identification Methods

This chapter deals with the process identification of linear dynamic systems. After a brief introduction to the concept of identification, some essential representatives of non-parametric and parametric identification methods will be discussed. It should be noted at this point that the terminology used is strongly based on the book by L. Ljung [1.1], as this book also forms the basis of the MATLAB Identification Toolbox, in which all the algorithms discussed in this chapter and many more are implemented.

1.1 General Aspects

Figure 1.1 shows the basic task of process identification. The process is affected by the

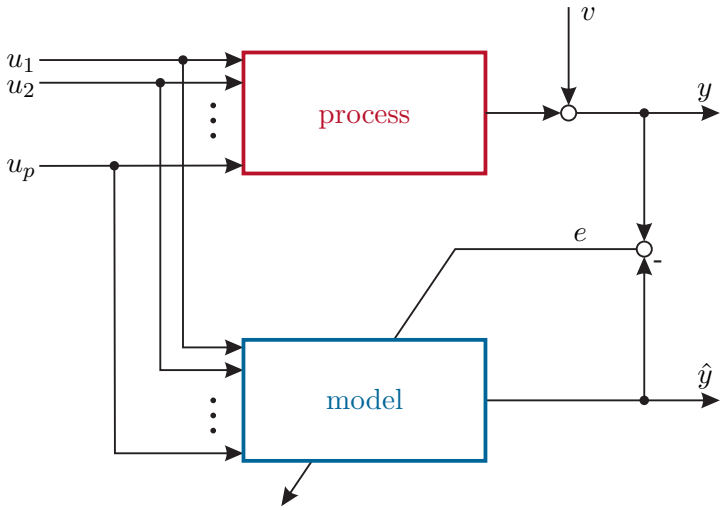


Figure 1.1: On the concept of process identification.

input variables $u_1, \ldots, u_p$, and it is assumed that only one output variable y is measured. This measured output variable y is disturbed by the noise signal v. The process model should now describe the process as accurately as possible, whereby as much a priori knowledge about the process as possible should be taken into account. The quality of the model is then evaluated, for example, using the error e between the disturbed measured output y and the model output $\hat{y}$. This error is then used to improve the model. The various steps to be carried out within the identification task are summarized in Figure 1.2 and must be iteratively repeated several times depending on the problem:

(A) **Choice of Input Variables:** In this step, it must be decided which input variables

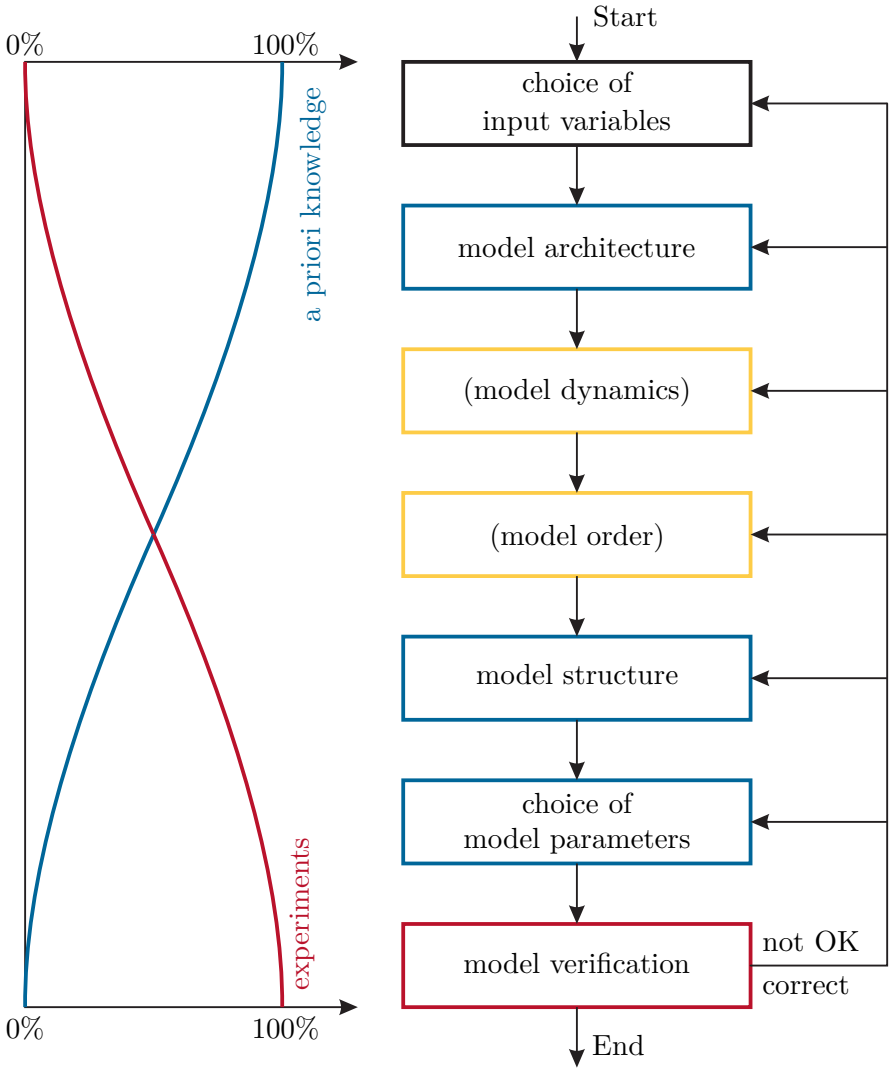


Figure 1.2: Identification procedure.

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are used for process identification and which signals are suitable for the individual input variables.

(B) **Model Architecture:** Here it should be determined whether it is a static or dynamic model, for what purpose the model is needed (simulation, controller design, error detection, etc.), the resulting dynamic range, whether the identification should be on- or off-line, etc.

(C,D) **Model Dynamics** and **Model Order:** In these steps, a description of the system dynamics is carried out, for example, in the form of a transfer function or a state-space model, and a system order should be determined, which, however, if not known from a priori knowledge, can only be sensibly estimated within the framework of targeted experiments.

(E) **Model Structure:** In this step, the model structure underlying the identification task is determined. Reference is made here to Section 1.3.1 for the different models of parametric identification methods for linear dynamic systems (ARMA, ARX, ARMAX, etc.).

(F) **Choice of Model Parameters:** Often, the model parameters are already fixed by the previous steps, in particular the choice of model structure and system order. However, there is still the possibility to specifically select parameters adapted to the identification task.

(G) **Model Verification:** Depending on the purpose of the model (step (B)), it must be checked whether the model has the corresponding quality. It is particularly important to ensure in this step that not the same data set is used for model verification as for solving the identification task.

In the literature, a distinction is also made between *white-box*, *black-box*, and *grey-box* models:

- In white-box models, all equations and parameters can be derived on the basis of physical considerations. White-box models are also referred to as such when the model is completely derived from physical laws and some so-called constitutive parameters (friction parameters, leakage parameters, stray inductances, etc.) are determined from experiments. The advantages of these models lie in the very good extrapolatability of the model beyond the data obtained through experiments, high reliability, good insight into the model, and the scalability of the model, which makes it applicable even for systems that have not yet been realized (prototyping). As a disadvantage of white-box models, it can be stated that the creation is generally relatively time-consuming and requires precise knowledge of the system.

- Black-box models are based solely on experimental results and have no (or very little) a priori knowledge of the system. Of course, it should be borne in mind that the model thus obtained is only valid in the data set covered by the identification. The main advantage is that relatively little knowledge about the system is required. All advantages of white-box models can be listed here as disadvantages.

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- Grey-box models combine model building based on physical considerations with principles of process identification.

1.2 Non-parametric Methods

Linear, time-invariant systems can be characterized by their transfer function (s-transfer function in continuous time and z-transfer function in discrete time) or their impulse response. The goal of *non-parametric identification methods* is now to determine the frequency response or impulse response directly from measurements of the input and output variables, without resorting to a specific model structure underlying the identification task. Since these methods do not rely on a finite number of parameters to be identified, they are called non-parametric. The following considerations are based on a system of the form shown in Figure 1.3 with the deterministic input sequence (u_k), the stochastic disturbance (noise) (v_k), the undisturbed output sequence $(\bar{y}_k)$, the output sequence (y_k), and the BIBO-stable z-transfer function $G(z)$ with the impulse sequence $(g_k) = \mathcal{Z}^{-1}\{G(z)\}$.

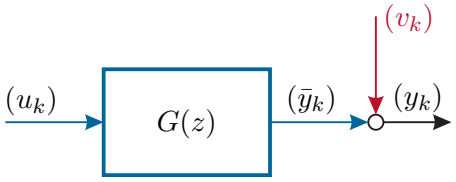


Figure 1.3: System considered for non-parametric identification.

For the output sequence, the following applies:

$$y_k = \sum_{i=0}^{k} g_{k-i} u_i + v_k \tag{1.1}$$

or in the z-domain

$$y_z(z) = G(z) u_z(z) + v_z(z) \tag{1.2}$$

with $y_z(z)$, $v_z(z)$, and $u_z(z)$ as the z-transforms of the sequences (y_k), (v_k), and (u_k).

1.2.1 Impulse Response Analysis

If the input variable (u_k) is chosen as the impulse sequence

$$u_k = \delta_k = \begin{cases} \alpha & \text{for } k = 0 \\ 0 & \text{for } k > 0, \end{cases} \tag{1.3}$$

then the output sequence is

$$y_k = \alpha g_k + v_k \ . \tag{1.4}$$

If the noise is small compared to the impulse response, i.e., $|v_k| \ll |\alpha g_k|$, then the impulse response can be determined from the measured sequence values y_k in the form

$$\hat{g}_k = \frac{y_k}{\alpha} \tag{1.5}$$

The problem with this method is that many technical systems do not allow impulse-like inputs.

1.2.2 Frequency Response Analysis - Excitation with a Harmonic Function

For the system in Figure 1.3, in the steady state, the response to the harmonic input sequence

$$(u_k) = (U \sin(\omega_0 k T_a)) \tag{1.6}$$

with the sampling time T_a is the harmonic output sequence (y_k)

$$y_k = Y \sin(\omega_0 k T_a + \varphi) + v_k \tag{1.7}$$

with

$$Y = U\left|G\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right| \quad \text{and} \quad \varphi = \arg\left(G\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right) \tag{1.8}$$

or

$$y_k = Y_c \cos(\omega_0 k T_a) + Y_s \sin(\omega_0 k T_a) + v_k \tag{1.9}$$

with

$$Y_c = Y \sin(\varphi) \quad \text{and} \quad Y_s = Y \cos(\varphi)\ . \tag{1.10}$$

In view of (1.7)-(1.10), it is reasonable to choose an approach of the form

$$\hat{y}_k = \hat{Y} \sin(\omega_0 k T_a + \hat{\varphi}) = \hat{Y}_c \cos(\omega_0 k T_a) + \hat{Y}_s \sin(\omega_0 k T_a) \tag{1.11}$$

with the parameters to be estimated $\hat{Y}_c = \hat{Y} \sin(\hat{\varphi})$ and $\hat{Y}_s = \hat{Y} \cos(\hat{\varphi})$ for the estimate $\hat{y}_k$ of the measured output variable y_k. Obviously, the parameters $\hat{Y}$ and $\hat{\varphi}$ can then be easily determined using the relationships

$$\hat{Y} = \sqrt{\hat{Y}_c^2 + \hat{Y}_s^2} \quad \text{and} \quad \hat{\varphi} = \arctan\left(\frac{\hat{Y}_c}{\hat{Y}_s}\right) \tag{1.12}$$

It must now be clarified by which choice of $\hat{Y}_c$ and $\hat{Y}_s$ the quadratic error

$$L = \frac{1}{N} \sum_{k=0}^{N-1} (y_k - \hat{y}_k)^2 \tag{1.13}$$

for N measurements of y_k is minimized. By substituting (1.11) into (1.13), differentiating with respect to $\hat{Y}_c$ and setting to zero, we obtain

$$\frac{\partial L}{\partial \hat{Y}_c} = -\frac{2}{N} \sum_{k=0}^{N-1} \Big(y_k - \hat{Y}_c \cos(\omega_0 k T_a) - \hat{Y}_s \sin(\omega_0 k T_a)\Big) \cos(\omega_0 k T_a) = 0 \tag{1.14}$$

or with $(\cos(\alpha))^2 = \frac{1}{2}(1+\cos(2\alpha))$ and $\cos(\alpha)\sin(\alpha) = \frac{1}{2}\sin(2\alpha)$

$$\hat{Y}_c - \frac{2}{N}\sum_{k=0}^{N-1} y_k \cos(\omega_0 k T_a) + \frac{\hat{Y}_c}{N}\sum_{k=0}^{N-1}\cos(2\omega_0 k T_a) + \frac{\hat{Y}_s}{N}\sum_{k=0}^{N-1}\sin(2\omega_0 k T_a) = 0\ . \tag{1.15}$$

Using the Euler formulas $\cos(\alpha) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{I}\alpha} + \mathrm{e}^{-\mathrm{I}\alpha}\right)$ and $\sin(\alpha) = \frac{1}{2\mathrm{I}}\left(\mathrm{e}^{\mathrm{I}\alpha} - \mathrm{e}^{-\mathrm{I}\alpha}\right)$, the second and third sums of (1.15) can be rewritten as follows

$$\sum_{k=0}^{N-1}\cos(2\omega_0 k T_a) = \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathrm{e}^{\mathrm{I}2\omega_0 k T_a} + \mathrm{e}^{-\mathrm{I}2\omega_0 k T_a}\right) \tag{1.16a}$$

$$\sum_{k=0}^{N-1}\sin(2\omega_0 k T_a) = \frac{1}{2\mathrm{I}}\sum_{k=0}^{N-1}\left(\mathrm{e}^{\mathrm{I}2\omega_0 k T_a} - \mathrm{e}^{-\mathrm{I}2\omega_0 k T_a}\right) \tag{1.16b}$$

If we now only use discrete values of the form

$$\omega_0 = \frac{2\pi l}{N T_a} \quad \text{with} \quad l = 1, 2, \ldots \tag{1.17}$$

for the angular frequency ω_0, then using the relationship

$$\sum_{k=0}^{N-1} z^{-k} = \frac{1 - z^{-N}}{1 - z^{-1}} \tag{1.18}$$

we obtain the following expression for the sum

$$\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{I}2\omega_0 k T_a} = \sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{I}\frac{4\pi l k}{N}} = \frac{1-\mathrm{e}^{-\mathrm{I}4\pi l}}{1-\mathrm{e}^{-\mathrm{I}\frac{4\pi l}{N}}} = \begin{cases} N & \text{for } l = \frac{r}{2}N,\ r = \pm1, \pm2, \ldots \\ 0 & \text{for } l = \pm1, \pm2, \ldots \end{cases} \tag{1.19}$$

The analogous result is obtained, of course, if I is replaced by $-\mathrm{I}$. For ω_0 according to (1.17), the expressions (1.16) are therefore calculated as

$$\sum_{k=0}^{N-1}\cos(2\omega_0 k T_a) = \begin{cases} N & \text{for } l = \frac{r}{2}N,\ r = \pm1, \pm2, \ldots \\ 0 & \text{for } l = \pm1, \pm2, \ldots \end{cases} \tag{1.20a}$$

$$\sum_{k=0}^{N-1}\sin(2\omega_0 k T_a) = 0 \tag{1.20b}$$

If the test frequency ω_0 is chosen according to (1.17) such that $l < N/2$, then the expression in (1.20a) is also zero, and the optimal solution $\hat{Y}_c$ is calculated from (1.15) in the form

$$\hat{Y}_c = \frac{2}{N}\sum_{k=0}^{N-1} y_k \cos\left(\frac{2\pi l}{N}k\right)\ . \tag{1.21}$$

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Note 1.1. Note that, for a given sampling time T_a, the maximum possible frequency of a uniquely representable harmonic function must be strictly less than the Nyquist frequency $\omega_{\max} = \pi/T_a$. It is immediately apparent that the value $l = N/2$ inserted into ω_0 from (1.17) corresponds to this Nyquist frequency.

Exercise 1.1. Show that the optimal $\hat{Y}_s$ is calculated as follows:

$$\hat{Y}_s = \frac{2}{N} \sum_{k=0}^{N-1} y_k \sin\left(\frac{2\pi l}{N} k\right) \tag{1.22}$$

This can now be summarized as follows: If the system from Figure 1.3 is excited with the harmonic sequence

$$(u_k) = (U \sin(\omega_0 k T_a)) \quad \text{with} \quad \omega_0 = \frac{2\pi l}{N T_a},\ l = 1, 2, \ldots, \frac{N}{2} - 1 \tag{1.23}$$

then, from the N measured values y_k, $k = 0, \ldots, N-1$, the discrete frequency response $G\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)$ at the frequencies $\omega_0 = \frac{2\pi l}{N T_a}$, $l = 1, 2, \ldots, \frac{N}{2} - 1$ can be approximated via the relationships (see (1.7),(1.8))

$$\hat{Y}_c = U \left|\hat{G}\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right| \sin\left(\arg\left(\hat{G}\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right)\right) \tag{1.24a}$$

$$\hat{Y}_s = U \left|\hat{G}\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right| \cos\left(\arg\left(\hat{G}\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right)\right) \tag{1.24b}$$

as follows

$$\left|\hat{G}\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right| = \frac{\sqrt{\hat{Y}_s^2 + \hat{Y}_c^2}}{U} \tag{1.25a}$$

$$\arg\left(\hat{G}\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)\right) = \arctan\left(\frac{\hat{Y}_c}{\hat{Y}_s}\right) \tag{1.25b}$$

with $\hat{Y}_c$ and $\hat{Y}_s$ from (1.21) and (1.22), respectively.

The above relationships (1.21) and (1.22) are closely related to the *discrete Fourier transform (DFT)* of the sequences (u_k) and (y_k). As a reminder, the discrete Fourier transform $F_n(\omega)$ of a sequence (f_k) is

$$F_n(\omega) = \sum_{k=0}^{N-1} f_k \mathrm{e}^{-\mathrm{I}\omega k T_a} \quad \text{with} \quad \omega = \frac{2\pi n}{N T_a}, \quad n = 0, 1, \ldots, N-1 \tag{1.26}$$

and the inverse discrete Fourier transform (IDFT) is

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n \mathrm{e}^{\mathrm{I}\omega k T_a} \quad \text{with} \quad \omega = \frac{2\pi n}{N T_a}, \quad k = 0, 1, \ldots, N-1 \tag{1.27}$$

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The DFT of the input sequence (u_k) according to (1.23) and the DFT of the measured output sequence (y_k) are, considering (1.21) and (1.22) with $\omega_0 = \frac{2\pi l}{NT_a}$,

$$\begin{aligned} U_n &= \sum_{k=0}^{N-1} u_k \mathrm{e}^{-\mathrm{I}\omega k T_a} = \frac{U}{2\mathrm{I}} \sum_{k=0}^{N-1} \left(\mathrm{e}^{\mathrm{I}\omega_0 k T_a} - \mathrm{e}^{-\mathrm{I}\omega_0 k T_a}\right) \mathrm{e}^{-\mathrm{I}\omega k T_a} \\ &= \frac{U}{2\mathrm{I}} \sum_{k=0}^{N-1} \left(\mathrm{e}^{\mathrm{I}\frac{2\pi k}{N}(l-n)} - \mathrm{e}^{-\mathrm{I}\frac{2\pi k}{N}(l+n)}\right) = \begin{cases} \frac{NU}{2\mathrm{I}} & \text{for } l = n \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1.28}$$

and

$$Y_n = \sum_{k=0}^{N-1} y_k \mathrm{e}^{-\mathrm{I}\omega k T_a} = \sum_{k=0}^{N-1} y_k \left(\cos\left(\frac{2\pi n}{N} k\right) - \mathrm{I}\sin\left(\frac{2\pi n}{N} k\right)\right) = \frac{N}{2}\left(\hat{Y}_c - \mathrm{I}\hat{Y}_s\right) . \tag{1.29}$$

Exercise 1.2. Prove the relationship in (1.28).

Thus, it can be seen that the estimation of the discrete frequency response $G\left(\mathrm{e}^{\mathrm{I}\omega_0 T_a}\right)$ at the frequencies $\omega_0 = \frac{2\pi l}{NT_a}$, $l = 1, 2, \ldots, \frac{N}{2} - 1$ according to (1.25) can be very easily calculated using the discrete Fourier transforms of the input and output sequences according to (1.28) and (1.29) – namely,

$$\frac{Y_n(\omega_0)}{U_n(\omega_0)} = \frac{\mathrm{DFT}((y_k))}{\mathrm{DFT}((u_k))} = \frac{\frac{N}{2}\left(\hat{Y}_c - \mathrm{I}\hat{Y}_s\right)}{\frac{NU}{2\mathrm{I}}} = \frac{\sqrt{\hat{Y}_c^2 + \hat{Y}_s^2}}{U} \mathrm{e}^{\mathrm{I}\arctan\frac{\hat{Y}_c}{\hat{Y}_s}} . \tag{1.30}$$

For practical application, it is of course advisable to perform the calculation of the discrete Fourier transforms using the *more efficient FFT (Fast Fourier Transform)* algorithm.

1.2.3 Frequency Response Analysis - ETFE

The relationship (1.30) now suggests choosing as the input sequence (u_k) not only a harmonic signal with constant frequency ω_0, but a signal containing several frequencies, thereby making use of the *superposition principle of linear systems*. The estimation of the frequency response according to relationship (1.30) is then called *empirical transfer function estimate (ETFE)* in the English-language literature – see also the Matlab command of the same name. The estimation is called empirical because, apart from the assumption of *linearity and time invariance of the system*, no further assumptions about the model structure are made. If, for certain frequencies ω, the Fourier transform $U(\omega) = 0$ due to the excitation (u_k), then the ETFE is not defined at these frequencies. Examples of suitable input sequences are impulse sequences, the 3-2-1 step, chirp signals, or PRBS signals.

A linear chirp signal with trapezoidal windowing is given by

$$u_k = U_0 + r_k \sin\left(\omega_{start} k T_a + \frac{(\omega_{end} - \omega_{start})}{NT_a} \frac{(kT_a)^2}{2}\right) \tag{1.31a}$$

$$r_k = U \mathrm{sat}\left(\frac{10k}{N}\right) \mathrm{sat}\left(\frac{10(N-k)}{N}\right) \tag{1.31b}$$

with

$$\mathrm{sat}(x) = \begin{cases} 1 & \text{for } x \geq 1 \\ x & \text{for } -1 < x < 1 \\ -1 & \text{for } x \leq -1 \end{cases} \tag{1.32}$$

and $k = 0, 1, \ldots, N-1$, for the sampling time T_a, the number of sampling points N, the constant U_0 to compensate for an offset of the amplitude U, and the lower and upper chirp frequencies ω_{start} and ω_{end}.

Exercise 1.3. Plot the time response of the chirp signal (1.31) and the response of the corresponding discrete Fourier transform for $N = 256$, $U = 1$, $\omega_{start} = 0.1$, and $\omega_{end} = 5/2$, $T_a = 0.5\,\mathrm{s}$ in Matlab.

So-called PRBS (Pseudo Random Binary Signal) sequences are frequently used in identification tasks because they exhibit similar properties to white noise. A PRBS signal of order O_p can be determined by the difference equation

$$p_k = \mathrm{mod}\Big(a_1 p_{k-1} + a_2 p_{k-2} + \ldots + a_{O_p} p_{k-O_p}, 2\Big) \tag{1.33}$$

with the coefficients $a_j \in \{0, 1\}$, $j = 1, \ldots, O_p$, which are shown for different orders in Table 1.1. Note that the PRBS signal repeats every $2^{O_p} - 1$ sampling steps.

Order O_p	$a_j \neq 0$ for the following j
2	1, 2
3	2, 3
4	1, 4
5	2, 5
6	1, 6
7	3, 7
8	1, 2, 7, 8
9	4, 9
10	7, 10
11	9, 11

Table 1.1: Coefficients of the PRBS signal.

To influence the frequency characteristics of the PRBS signal, oversampling of the signal p_k with a factor $P_p \geq 1$ is often useful. Given a sampling time T_a, the values of p_k are upsampled P_p-times, i.e.,

$$u((P_p k + j)T_a) = U p_k, \quad j = 0, \ldots, P_p - 1, \quad k = 0, \ldots, 2^{O_p} - 2 \tag{1.34}$$

with the amplitude U of the signal. It can be shown that this results in a low-pass filtering of the PRBS signal in the frequency domain. In the literature, an oversampling

of $P_p = 4$ is typically recommended. With an order O_p and an oversampling P_p, a number of $N = (2^{O_p} - 1)P_p$ sample values are available.

Exercise 1.4. Plot the time response of the PRBS signal (1.34) and the response of the corresponding discrete Fourier transform for $O_p = 10$, $U = 1$, and a sampling time $T_a = 0.5$s in Matlab. Investigate the influence of different values for the oversampling P_p and initialize the difference equation (1.34) with $p_0 = 1$.

Exercise 1.5. Assume that the sequences (u_k) and (y_k) consist of N equidistant sampling points with the sampling time T_a. Show that the minimum resolvable frequency is given by $\omega_{\min} = \frac{2\pi}{NT_a}$ and the maximum resolvable frequency by $\omega_{\max} = \frac{\pi}{T_a}$.

Exercise 1.6. Using the `fft` routine from Matlab, write a program to calculate the frequency response of a linear, time-invariant sampled-data system with the transfer function $G(z)$ according to (1.30). The input parameters should be the sampled sequences of the input and output variables (u_k) and (y_k), the sampling time T_a, and the number of measurement points N. The result should be the discrete frequencies $\omega = \frac{2\pi}{NT_a}n$, $\left|G\left(\mathrm{e}^{\mathrm{I}\omega T_a}\right)\right|$, and $\arg\left(G\left(\mathrm{e}^{\mathrm{I}\omega T_a}\right)\right)$ for $n = 1, \ldots, \frac{N}{2} - 1$. Test the program using the system

$$G(s) = \frac{1}{s^2 + 0.25s + 1}$$

using a chirp signal according to (1.31), (1.32).

The results of the ETFE are generally very good for deterministic input signals, especially for those frequencies that are sufficiently well excited by the input signal. For stochastic input signals, additional measures, such as smoothing the ETFE using suitable window functions (Hamming, Bartlett, Kaiser, etc.), must usually be taken to obtain usable results. These details will not be discussed in this lecture; interested students are referred to the literature listed at the end, in particular the book by L. Ljung [1.1].

In practical applications, it has proven very helpful to incorporate *a priori knowledge* about the process into the identification process. This can be done, for example, by choosing the input signal in such a way that it specifically excites the system in the frequency range of interest. Another possibility is to incorporate known parts of the transfer function to be identified. To this end, the transfer function $G(z)$ is factored into a known and an unknown part in the form

$$G(z) = \underbrace{\frac{z_b(z)}{n_b(z)}}_{\text{known}} \underbrace{G_1(z)}_{\text{unknown}} \tag{1.35}$$

and then the identification task for $G_1(z)$ is solved according to Figure 1.4.

This method, also known as *Clary's method*, has proven to be very effective in the identification of certain mechanical systems. In these systems, for example, it is known that the s-transfer function $G(s)$ has a double pole at $s_i = 0$. This means that the z-transfer function to be identified has a double pole at $z_i = 1$ according to the relationship

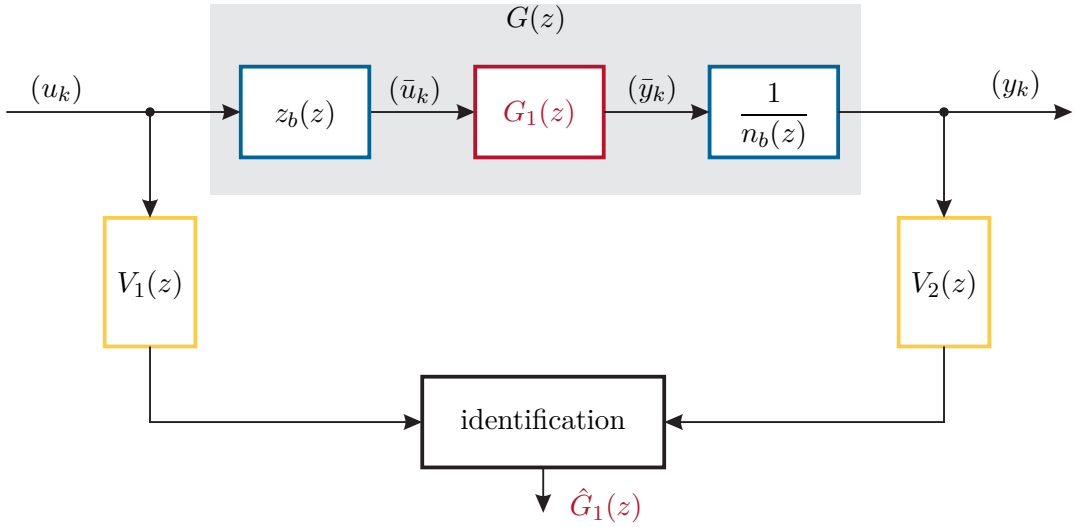


Figure 1.4: Identification considering known parts of the transfer function (Clary's method).

$z = \exp(sT_a)$ with the sampling time T_a. This direct assignment of the poles of s- and z-transfer functions is known not to be transferable to their zeros. In particular, the zeros outside the unit circle usually have only a small influence on the frequency response in the frequency range of interest, but are extremely difficult to identify. A non-exact but effective procedure is to assign the value $z_j = -1$ to these zeros z_j outside the unit circle. In order for the sequences (u_k) and (y_k) to be prepared for identification according to Figure 1.4, the individual prefilters $V_1(z)$ and $V_2(z)$ must be proper, i.e., the numerator degree must be less than or equal to the denominator degree. To ensure this, the known part of the transfer function from (1.35) is written in the form

$$\frac{z_b(z)}{n_b(z)} = \frac{z_b(z)}{z^n}\frac{z^n}{n_b(z)} \quad \text{with} \quad n = \max(\deg(z_b(z)), \deg(n_b(z))). \tag{1.36}$$

The prefilters are then

$$V_1(z) = \frac{z_b(z)}{z^n} \quad \text{and} \quad V_2(z) = \frac{n_b(z)}{z^n}\,. \tag{1.37}$$

Exercise 1.7. Identify the system transfer function

$$G(s) = \frac{1}{s^2}\frac{1}{s^2 + 0.25s + 1} \tag{1.38}$$

using Clary's method for the sampling time $T_a = 0.5\,\mathrm{s}$. Assume that you know from the system that it possesses a double integrator. Use the chirp signal from Exercise 1.3 as the input signal.

1.3 Parametric Methods

As the name implies, these methods use models with a finite number of parameters. These parameters are then determined within the framework of the identification task such that the model agrees with the system to be identified as well as possible in the sense of a quality criterion. The literature contains a vast number of different model structures, which is why a classification of these models is carried out in the first step. The following classification is essentially based on the Matlab System Identification Toolbox.

1.3.1 Model Structures

The starting point of the considerations is again the system from Figure 1.3. For the further steps, it is necessary to characterize the stochastic disturbance (noise) (v_k) in a suitable form. A relatively simple approach is to model (v_k) as the output sequence of a linear time-invariant system with the transfer function $H(z)$ and *white noise* (w_k) (sequence of independent random variables) with a given probability density function, i.e.,

$$v_k = \sum_{i=-\infty}^{k} h_{k-i} w_i = \sum_{i=0}^{\infty} h_i w_{k-i} \quad \text{with} \quad (h_k) = \mathcal{Z}^{-1}\{H(z)\} \; . \tag{1.39}$$

This approach is sufficient for most practical applications; however, not every possible stochastic disturbance (v_k) can be characterized in this way. Note that the character of the stochastic disturbance signal (v_k) can be specifically influenced by the choice of the probability density function of (w_k). For example, it is plausible that for one and the same system with the transfer function $H(z)$, a white noise $(w_{2,k})$ with the probability density function

$$\begin{aligned} w_{2,k} &= 0 \quad \text{with probability } 1-\mu \\ w_{2,k} &= r \quad \text{with probability } \mu \end{aligned} \tag{1.40}$$

for a very small μ and a uniformly distributed random variable $r \in (-1, 1)$ gives a completely different picture for $(v_{2,k})$ than a white noise $(w_{1,k})$ with the probability density function $w_{1,k} = r$.

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Note the time respoonse of $(v_{1,k})$ and $(v_{2,k})$ for $H(z) = \mathbf{Z}\{H(s)\}$, with

$$H(s) = \frac{1}{\frac{s^2}{(2\pi 10)^2} + 2\frac{0.2s}{2\pi 10} + 1}, \tag{1.41}$$

in Figure 1.5, where $T_a = 10\text{ms}$ and $\mu = 0.07$ were used.

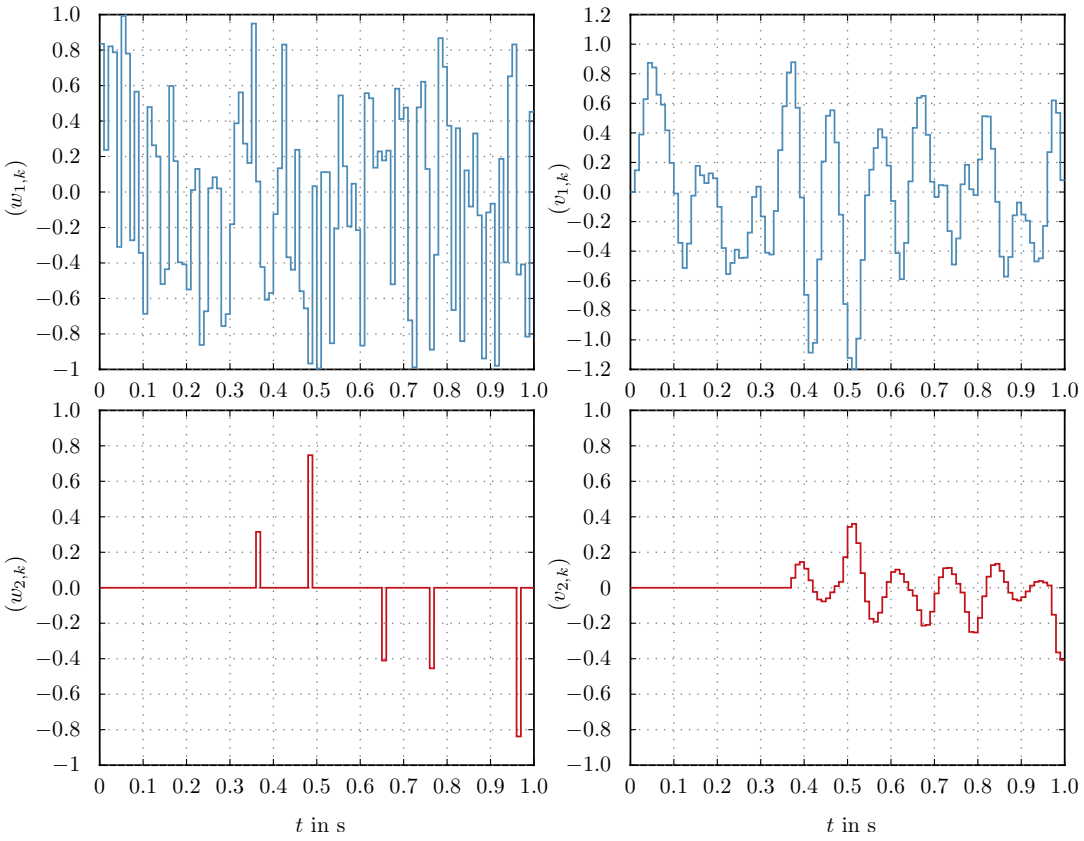


Figure 1.5: Responses $(v_{1,k})$ and $(v_{2,k})$ of the second-order linear time-invariant discrete-time system (1.41) to white noise $(w_{1,k})$ and $(w_{2,k})$ with different probability density functions.

Combining (1.1) with (1.39), the output variable y_k from Figure 1.3 is

$$y_k = \sum_{i=0}^{\infty} g_i u_{k-i} + \sum_{i=0}^{\infty} h_i w_{k-i} \tag{1.42}$$

or, introducing the shift operator δ with $u_{k+1} = \delta u_k$ or $u_{k-1} = \delta^{-1} u_k$,

$$y_k = \sum_{i=0}^{\infty} g_i \delta^{-i} u_k + \sum_{i=0}^{\infty} h_i \delta^{-i} w_k \ . \tag{1.43}$$

To save writing effort later on, we also write (1.43) as

$$y_k = G(\delta) u_k + H(\delta) w_k \tag{1.44}$$

with the transfer operators

$$G(\delta) = \sum_{i=0}^{\infty} g_i \delta^{-i} \quad \text{and} \quad H(\delta) = \sum_{i=0}^{\infty} h_i \delta^{-i} \ . \tag{1.45}$$

Note that the expressions for the transfer operators $G(\delta)$ and $H(\delta)$ are the same as the z-transfer functions of linear time-invariant sampled-data systems with the impulse responses (g_k) and (h_k). However, if $G(\delta)$ and $H(\delta)$ were interpreted as z-transfer functions, the notation of (1.44) would not be permissible.

By representing $G(\delta)$ and $H(\delta)$ in the form of rational transfer operators with their associated numerator and denominator polynomials and combining all common poles of $G(\delta)$ and $H(\delta)$ in the polynomial $A(\delta)$, (1.44) becomes (see Figure 1.6)

$$A(\delta) y_k = \frac{B(\delta)}{F(\delta)} u_k + \frac{C(\delta)}{D(\delta)} w_k \ . \tag{1.46}$$

Based on (1.46), a classification of the various model structures can now be carried out.

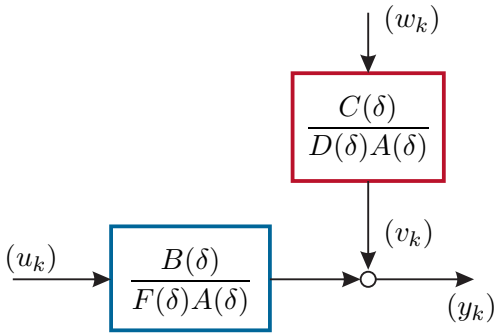


Figure 1.6: On the model structures of parametric identification.

ARMA Model

In econometrics, one often deals with models that do not have a deterministic input variable, i.e., $(u_k) = (0)$. In these so-called *time series models*, a distinction is made between the following special cases:

The model

$$y_k = \frac{1}{D(\delta)} w_k \tag{1.47}$$

or for $D(\delta) = d_0 + d_1\delta^{-1} + \ldots + d_n\delta^{-n}$ with $d_0 = 1$

$$y_k = -d_1 y_{k-1} - d_2 y_{k-2} - \ldots - d_n y_{k-n} + w_k \tag{1.48}$$

is called an *autoregressive model* or *AR model*. Equation (1.48) can also be written in the form

$$y_k = \mathbf{s}_k^{\mathrm{T}} \mathbf{p} \tag{1.49}$$

with the *parameter vector* $\mathbf{p}$ and the *data vector* $\mathbf{s}_k$ in the form

$$\mathbf{p}^{\mathrm{T}} = \begin{bmatrix} -d_1 & -d_2 & \ldots & -d_n & 1 \end{bmatrix} \tag{1.50a}$$

$$\mathbf{s}_k^{\mathrm{T}} = \begin{bmatrix} y_{k-1} & y_{k-2} & \ldots & y_{k-n} & w_k \end{bmatrix} \tag{1.50b}$$

In statistics, models that are linear in the parameters $\mathbf{p}$, such as (1.48), are also called *linear regression models*. If the data vector $\mathbf{s}_k$ contains only past values y_j, $j < k$, of the quantity y_k to be calculated, then this model is called *autoregressive*.

A model of the form

$$y_k = C(\delta) w_k \tag{1.51}$$

or for $C(\delta) = c_0 + c_1\delta^{-1} + \ldots + c_m\delta^{-m}$ with $c_0 = 1$

$$y_k = w_k + c_1 w_{k-1} + c_2 w_{k-2} + \ldots + c_m w_{k-m} \tag{1.52}$$

is called a *moving average model* or *MA model*.

Exercise 1.8. Show that the impulse response sequence (g_k) of the z-transfer function

$$G(z) = c_0 + c_1 z^{-1} + \ldots + c_m z^{-m} \tag{1.53}$$

is calculated as follows:

$$(g_k) = (c_0, c_1, \ldots, c_m, 0, 0, \ldots) \tag{1.54}$$

For this reason, a transfer function of the form (1.53) is also called a FIR (finite impulse response) model. Analogously, a z-transfer function of the form

$$G(z) = \frac{1}{d_0 + d_1 z^{-1} + \ldots + d_n z^{-n}} \tag{1.55}$$

is called an IIR (infinite impulse response) model.

Combining (1.47) and (1.51), we obtain the so-called *ARMA (autoregressive moving average) model*

$$y_k = \frac{C(\delta)}{D(\delta)} w_k \ . \tag{1.56}$$

For $C(\delta) = c_0 + c_1\delta^{-1} + \ldots + c_m\delta^{-m}$ and $D(\delta) = d_0 + d_1\delta^{-1} + \ldots + d_n\delta^{-n}$, $d_0 = 1$, the output variable y_k is

$$y_k = c_0 w_k + c_1 w_{k-1} + c_2 w_{k-2} + \ldots + c_m w_{k-m} - d_1 y_{k-1} - d_2 y_{k-2} - \ldots - d_n y_{k-n} \ . \tag{1.57}$$

ARX Model

If the AR model (1.47) is extended to include the influence of the deterministic input variable u_k (*exogenous input*), then the *ARX (autoregressive with exogenous input) model* is obtained:

$$y_k = \frac{B(\delta)}{A(\delta)} u_k + \frac{1}{A(\delta)} w_k \ . \tag{1.58}$$

As can be seen from (1.58), in this case the denominator polynomials of the transfer operators $G(\delta)$ and $H(\delta)$ according to (1.44) are identical. A mathematical justification for this choice of structure will be given later in the context of least-squares identification. As will be shown, the main advantage of this model structure lies in the fact that the parameters are linear in the estimation error and can therefore be very easily estimated using linear least-squares methods.

ARMAX Model

Analogously to the ARX model, the *ARMAX (autoregressive moving average with exogenous input) model* is given by

$$y_k = \frac{B(\delta)}{A(\delta)} u_k + \frac{C(\delta)}{A(\delta)} w_k, \tag{1.59}$$

where again the denominator polynomials of the transfer operators from the deterministic and stochastic inputs u_k and w_k to the output y_k are equal. It should be mentioned at this point that there are a large number of other models in the literature, but they are all composed in a similar way to those discussed so far and are special cases of (1.46). See Table 1.2 below.

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Model structure	Model equation
MA	$y_k = C(\delta) w_k$
AR	$y_k = \frac{1}{D(\delta)} w_k$
ARMA	$y_k = \frac{C(\delta)}{D(\delta)} w_k$
ARX	$y_k = \frac{B(\delta)}{A(\delta)} u_k + \frac{1}{A(\delta)} w_k$
ARMAX	$y_k = \frac{B(\delta)}{A(\delta)} u_k + \frac{C(\delta)}{A(\delta)} w_k$
ARARX	$y_k = \frac{B(\delta)}{A(\delta)} u_k + \frac{1}{D(\delta)A(\delta)} w_k$
ARARMAX	$y_k = \frac{B(\delta)}{A(\delta)} u_k + \frac{C(\delta)}{D(\delta)A(\delta)} w_k$
OE (output error)	$y_k = \frac{B(\delta)}{F(\delta)} u_k + w_k$
BJ (Box-Jenkins)	$y_k = \frac{B(\delta)}{F(\delta)} u_k + \frac{C(\delta)}{D(\delta)} w_k$

Table 1.2: Model structures and model equations.

1.3.2 Least Squares Method

Given is the *overdetermined* linear system of equations

$$\mathbf{y} = \mathbf{S}\mathbf{p} \tag{1.60}$$

with the $(m \times n)$-matrix $\mathbf{S} \in \mathbb{R}^{m \times n}$, the m-dimensional vector $\mathbf{y} \in \mathbb{R}^m$, and the n-dimensional vector of unknowns $\mathbf{p} \in \mathbb{R}^n$. For $m > n$ and $\operatorname{rank}(\mathbf{S}) = n \neq \operatorname{rank}([\mathbf{S}, \mathbf{y}])$, the system of equations (1.60) has no solution for $\mathbf{p}$. We now seek the solution $\mathbf{p}_0$ that minimizes the quadratic error

$$\min_{\mathbf{p}} \|\mathbf{e}\|_2^2 \quad \text{with} \quad \mathbf{e} = \mathbf{y} - \mathbf{S}\mathbf{p}. \tag{1.61}$$

Setting the derivative of $\|\mathbf{e}\|_2^2$ with respect to $\mathbf{p}$ equal to zero

$$\frac{\partial}{\partial \mathbf{p}} \mathbf{e}^{\mathrm{T}} \mathbf{e} = \frac{\partial}{\partial \mathbf{p}} \underbrace{(\mathbf{y} - \mathbf{S}\mathbf{p})^{\mathrm{T}} (\mathbf{y} - \mathbf{S}\mathbf{p})}_{\mathbf{y}^{\mathrm{T}}\mathbf{y} - 2\mathbf{y}^{\mathrm{T}}\mathbf{S}\mathbf{p} + \mathbf{p}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{p}} = -2\mathbf{y}^{\mathrm{T}}\mathbf{S} + 2\mathbf{p}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{S} = \mathbf{0}^{\mathrm{T}}, \tag{1.62}$$

the optimal solution $\mathbf{p} = \mathbf{p}_0$ in the sense of (1.61) is

$$\mathbf{p}_0 = \left(\mathbf{S}^{\mathrm{T}}\mathbf{S}\right)^{-1} \mathbf{S}^{\mathrm{T}} \mathbf{y} \ . \tag{1.63}$$

The expression $\mathbf{S}^{\dagger} = \left(\mathbf{S}^{\mathrm{T}}\mathbf{S}\right)^{-1} \mathbf{S}^{\mathrm{T}}$ is also called the *pseudoinverse* of the matrix $\mathbf{S}$ (see the Matlab commands `\` and `pinv`). It can be seen that for the regularity of $\mathbf{S}^{\mathrm{T}}\mathbf{S}$, the matrix $\mathbf{S}$ must have full column rank, i.e., $\operatorname{rank}(\mathbf{S}) = n$.

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Exercise 1.9 (Polynomial approximation in the least squares sense). Given are N measurement points that describe the relationship $y_j = f(x_j)$, $j = 1, \ldots, N$. Assume an n-th order polynomial with $n + 1 < N$ of the form

$$f(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_{n-1} x^{n-1} + p_n x^n$$

for the function $f(x)$ and determine the polynomial coefficients $p_0, p_1, \ldots, p_n$ using the least squares method. Test your approach using the function $g(x) = \tanh(x/10)$ for $x_j = -20 + j$, $j = 0, \ldots, 40$, and different orders n of the polynomial.

Exercise 1.10. The mathematical model of a externaly excited DC motor is

$$L_a \frac{d}{dt} i_a = u_a - R_a i_a - k_a \omega$$
$$J_r \frac{d}{dt} \omega = k_a i_a - d_v \omega - d_c \text{sign}(\omega)$$

with the armature current i_a, the angular velocity ω, the armature voltage u_a, the armature inductance L_a, the armature resistance R_a, the motor constant k_a, the moment of inertia J_r, and the friction parameters d_v and d_c. Determine the parameters k_a, R_a, d_v, and d_c from stationary measurements of u_a, i_a, and ω using the least squares method. Test your algorithm using the measurement data `data.mat` and `data_rausch.mat`, in which the measured values of u_a, i_a, and ω are stored with and without measurement noise. Compare your identification results with the nominal parameters $R_a = 1.373\,\Omega$, $k_a = 0.0652\,\text{V s/rad}$, $d_c = 0.0188\,\text{N m}$, and $d_v = 43.3 \cdot 10^{-6}\,\text{N m s/rad}$.



Data for Exercise 1.10:
https://www.acin.tuwien.ac.at/file/teaching/master/Regelungssysteme-1/Daten_Aufgabe_1_10.zip.

The optimal solution $\mathbf{p}_0$ according to (1.63) of the optimization problem (1.61) allows the following geometrical interpretation: The n linearly independent column vectors of the matrix $\mathbf{S}$ span an n-dimensional subspace $\mathcal{U}$ in $\mathbb{R}^m$. The solvability of the system of equations (1.60) is equivalent to the question of whether a linear combination (described by the entries in the vector $\mathbf{p}$) of the column vectors of $\mathbf{S}$ exists such that the vector $\mathbf{y}$ can be represented. Thus, the system of equations (1.60) is uniquely solvable if $\mathbf{y}$ lies in this subspace $\mathcal{U}$, i.e., $\text{rank}(\mathbf{S}) = \text{rank}([\mathbf{S}, \mathbf{y}])$. If $\mathbf{y}$ does not lie in the subspace $\mathcal{U}$, then the error between the quantity $\mathbf{y}_0 = \mathbf{S}\mathbf{p}_0 = \mathbf{S}\left(\mathbf{S}^\mathrm{T}\mathbf{S}\right)^{-1}\mathbf{S}^\mathrm{T}\mathbf{y}$ (calculated using the optimal solution $\mathbf{p}_0$ estimated in the least squares sense) and the actual quantity $\mathbf{y}$ is

$$\mathbf{e}_0 = \mathbf{y} - \mathbf{y}_0 = \mathbf{y} - \mathbf{S}\left(\mathbf{S}^\mathrm{T}\mathbf{S}\right)^{-1}\mathbf{S}^\mathrm{T}\mathbf{y} \,. \tag{1.64}$$

It can now be seen that this error $\mathbf{e}_0$ is orthogonal to the subspace $\mathcal{U}$, since

$$\mathbf{S}^\mathrm{T}\mathbf{e}_0 = \mathbf{S}^\mathrm{T}\mathbf{y} - \mathbf{S}^\mathrm{T}\mathbf{S}\left(\mathbf{S}^\mathrm{T}\mathbf{S}\right)^{-1}\mathbf{S}^\mathrm{T}\mathbf{y} = \mathbf{0} \,. \tag{1.65}$$

Figure 1.7 illustrates this geometrically for $m = 3$ and $n = 2$. The finding that the error

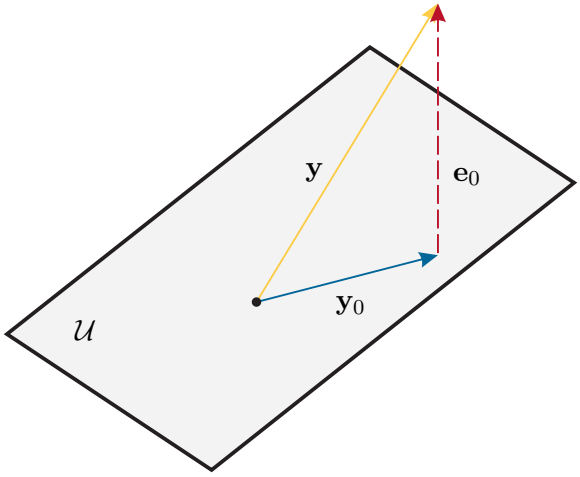


Figure 1.7: On the least squares method.

$\mathbf{e}_0$ is orthogonal to the subspace $\mathcal{U}$ is a special case of the so-called *projection theorem in a Hilbert space*.

Projection Theorem in a Hilbert Space

To illustrate the projection theorem, the concepts of vector space, normed vector space, and Hilbert space will be briefly reviewed below.

Definition 1.1 (Linear Vector Space)**.** A non-empty set $\mathcal{X}$ is called a linear vector space over a (scalar) field K with the binary operations $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ (addition) and $\cdot : K \times \mathcal{X} \to \mathcal{X}$ (scalar multiplication), if the following vector space axioms are satisfied:

(1) The set $\mathcal{X}$ with the operation $+$ is a commutative group, i.e., for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$ the following holds:

(1)	$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$	Commutativity
(2)	$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$	Associativity
(3)	$\mathbf{0} + \mathbf{x} = \mathbf{x}$	Neutral element
(4)	$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$	Inverse element

(2) Scalar multiplication $\cdot$ with scalars a, $b \in K$ satisfies:

(1)	$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$	Distributivity
(2)	$(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$	Distributivity
(3)	$(ab)\mathbf{x} = a(b\mathbf{x})$	Associativity
(4)	$1\mathbf{x} = \mathbf{x} \quad , \quad 0\mathbf{x} = \mathbf{0}$	

Definition 1.2 (Normed Linear Vector Space)**.** A normed linear vector space is a vector space $\mathcal{X}$ over a scalar field K with a real-valued function $\|\mathbf{x}\| : \mathcal{X} \to \mathbb{R}_+$, which assigns to each $\mathbf{x} \in \mathcal{X}$ a real-valued number $\|\mathbf{x}\|$, the so-called *norm* of $\mathbf{x}$, and satisfies the following norm axioms:

(1)	$\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathcal{X}$	Non-negativity
(2)	$\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$	
(3)	$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$	Triangle inequality
(4)	$\|\alpha\mathbf{x}\| = \|\alpha\|\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathcal{X}$ and all $\alpha \in K$	

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Definition 1.3 (Pre-Hilbert Space)**.** Let $\mathcal{X}$ be a linear vector space with the scalar field K. A mapping $\langle \mathbf{x}, \mathbf{y} \rangle : \mathcal{X} \times \mathcal{X} \to K$, which assigns a scalar to any two elements $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, is called an *inner product* if it satisfies the following conditions:

(1) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ Bilinearity
(2) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
(3) $\langle a\mathbf{x}, \mathbf{y} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle$
(4) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$

with $a \in K$.

A complete pre-Hilbert space is called a Hilbert space, cf. e.g. [1.2]. Two vectors $\mathbf{x}$ and $\mathbf{y}$ from a Hilbert space $\mathcal{H}$ are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. A non-empty set $\mathcal{U}$ is called a *subspace* of $\mathcal{H}$ if for all linear combinations of vectors $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{U}$ it holds that $a\mathbf{x} + b\mathbf{y} \in \mathcal{U}$, with scalars a and b. Furthermore, the subspace $\mathcal{U}$ is called *closed* if the limit of every convergent sequence in $\mathcal{U}$ is also in $\mathcal{U}$.

Exercise 1.11. Show that $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ defines a norm according to Definition 1.2.

The following projection theorem now gives a necessary and sufficient condition for solving the following problem: Given is a vector $\mathbf{y}$ in a Hilbert space $\mathcal{H}$. The vector $\mathbf{y}_0$ from a subspace $\mathcal{U}$ of $\mathcal{H}$ is sought that minimizes the norm $\|\mathbf{y} - \mathbf{y}_0\|$.

Theorem 1.1 (Projection Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{U}$ a closed subspace of $\mathcal{H}$. For every vector $\mathbf{y} \in \mathcal{H}$, there exists a unique vector $\mathbf{y}_0 \in \mathcal{U}$ such that*

$$\|\mathbf{y} - \mathbf{y}_0\| \leq \|\mathbf{y} - \mathbf{x}\| \tag{1.66}$$

for all $\mathbf{x} \in \mathcal{U}$. The vector $\mathbf{y}_0$ is the unique minimizing vector if and only if

$$\langle \mathbf{y} - \mathbf{y}_0, \mathbf{x} \rangle = 0 \tag{1.67}$$

for all $\mathbf{x} \in \mathcal{U}$. This means that the error $\mathbf{y} - \mathbf{y}_0$ must be orthogonal to all vectors of the subspace $\mathcal{U}$.

The proof of this theorem can be found in the literature cited at the end of this chapter.

In the following, the projection theorem will be used to solve the optimization problem (1.61). Given is the Hilbert space $\mathcal{H} = \mathbb{R}^m$ with the inner product

$$\langle \mathbf{x}, \mathbf{z} \rangle = \mathbf{x}^{\mathrm{T}} \mathbf{z} = \sum_{j=1}^{m} x_j z_j \tag{1.68}$$

with

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix} \tag{1.69a}$$

$$\mathbf{z}^{\mathrm{T}} = \begin{bmatrix} z_1 & z_2 & \dots & z_m \end{bmatrix} \tag{1.69b}$$

and the closed subspace $\mathcal{U}$ of $\mathcal{H}$, which is spanned by the column vectors $\mathbf{s}_k$, $k = 1, \dots, n$ of the matrix $\mathbf{S}$, i.e., $\mathcal{U} = \operatorname{span}\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$. In the sense of (1.61), we are looking for

that $\mathbf{S}\mathbf{p}_0 = \mathbf{y}_0 \in \mathcal{U}$ which, for a given $\mathbf{y} \in \mathcal{H}$, minimizes the squared norm of the error $\|\mathbf{e}\|_2^2 = \|\mathbf{y} - \mathbf{y}_0\|_2^2$. According to Theorem 1.1, it must therefore hold that

$$\langle \mathbf{y} - \mathbf{S}\mathbf{p}_0, \mathbf{s}_j \rangle = \langle \mathbf{y} - \mathbf{s}_1 p_{0,1} - \mathbf{s}_2 p_{0,2} - \ldots - \mathbf{s}_n p_{0,n}, \mathbf{s}_j \rangle = 0 \quad \text{for all } j = 1, \ldots, n \ . \tag{1.70}$$

In matrix notation, using the properties of the inner product, we obtain

$$\underbrace{\begin{bmatrix} \langle \mathbf{s}_1, \mathbf{s}_1 \rangle & \ldots & \langle \mathbf{s}_n, \mathbf{s}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{s}_1, \mathbf{s}_n \rangle & \ldots & \langle \mathbf{s}_n, \mathbf{s}_n \rangle \end{bmatrix}}_{\mathbf{G}=\mathbf{S}^{\mathrm{T}}\mathbf{S}} \underbrace{\begin{bmatrix} p_{0,1} \\ \vdots \\ p_{0,n} \end{bmatrix}}_{\mathbf{p}_0} = \underbrace{\begin{bmatrix} \langle \mathbf{y}, \mathbf{s}_1 \rangle \\ \vdots \\ \langle \mathbf{y}, \mathbf{s}_n \rangle \end{bmatrix}}_{\mathbf{S}^{\mathrm{T}}\mathbf{y}} \tag{1.71}$$

and thus immediately the solution for $\mathbf{p}_0$ from (1.63). The matrix $\mathbf{G}$ from (1.71) is also called the *Gramian matrix*.

Exercise 1.12. Show that the Gramian matrix from (1.71) is non-singular if and only if the vectors $\mathbf{s}_k$, $k = 1, \ldots, n$, are linearly independent.

Exercise 1.13. Calculate the solution of the quadratic minimization problem from Section 1.2.2 using Theorem 1.1.

Remark: (to Exercise 1.13) The set of all N-tuples $\xi = \{\xi_0, \xi_2, \ldots, \xi_{N-1}\}$ with the inner product

$$\langle x, z \rangle = \frac{1}{N} \sum_{k=0}^{N-1} x_k z_k \quad \text{with} \quad x, z \in \mathcal{H}$$

is used as the Hilbert space $\mathcal{H}$. The set of all N-tuples $\{C\cos(\omega_0 k T_a) + S\sin(\omega_0 k T_a)\}$, $k = 0, \ldots, N-1$ with arbitrary but constant coefficients C and S and fixed angular frequency ω_0 forms a closed subspace $\mathcal{U}$ of $\mathcal{H}$. The quadratic minimization problem from Section 1.2.2 can now be formulated such that we are looking for that $\hat{y} \in \mathcal{U}$ which minimizes the norm (compare with (1.13))

$$\|y - \hat{y}\|^2 = \frac{1}{N} \sum_{k=0}^{N-1} (y_k - \hat{y}_k)^2$$

with

$$\hat{y}_k = \hat{Y}_c \cos(\omega_0 k T_a) + \hat{Y}_s \sin(\omega_0 k T_a)$$

for a given $y \in \mathcal{H}$. According to Theorem 1.1, this is the case if and only if

$$\langle y - \hat{y}, x \rangle = 0 \quad \text{for all } x \in \mathcal{U}$$

or, for the two special cases $x = \cos(\omega_0 k T_a)$ and $x = \sin(\omega_0 k T_a)$, we obtain the

conditions

$$\frac{1}{N}\sum_{k=0}^{N-1}\Big(y_k - \hat{Y}_c\cos(\omega_0 kT_a) - \hat{Y}_s\sin(\omega_0 kT_a)\Big)\cos(\omega_0 kT_a) = 0$$
$$\frac{1}{N}\sum_{k=0}^{N-1}\Big(y_k - \hat{Y}_c\cos(\omega_0 kT_a) - \hat{Y}_s\sin(\omega_0 kT_a)\Big)\sin(\omega_0 kT_a) = 0\ .$$

1.3.3 Least-Squares Identification

The starting point is the ARX model (1.58) without stochastic disturbance, i.e., $(w_k) = (0)$, of the form

$$y_k = \frac{B(\delta)}{A(\delta)}u_k = \delta^{-d}\frac{b_0 + b_1\delta^{-1} + \ldots + b_m\delta^{-m}}{1 + a_1\delta^{-1} + \ldots + a_n\delta^{-n}}u_k \quad \text{with} \quad d \geq 0. \tag{1.72}$$

For a transfer function $G(z)$ with numerator degree m and denominator degree n, d is calculated as $d = n - m$. Note that dead times of the form $z^{-\rho}$, $\rho > 0$, can also be taken into account with the formulation (1.72). The value of the output sequence (y_k) at the k-th sampling time is calculated from (1.72) as

$$y_k = -a_1 y_{k-1} - \ldots - a_n y_{k-n} + b_0 u_{k-d} + b_1 u_{k-d-1} + \ldots + b_m u_{k-d-m} \tag{1.73}$$

or in vector notation

$$y_k = \underbrace{\begin{bmatrix} -y_{k-1} & -y_{k-2} & \ldots & -y_{k-n} & u_{k-d} & u_{k-d-1} & \ldots & u_{k-d-m} \end{bmatrix}}_{\mathbf{s}_k^{\mathrm{T}}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_0 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{p}} \tag{1.74}$$

with the *data vector* $\mathbf{s}_k$ and the *parameter vector* $\mathbf{p}$. In the identification task, the parameter vector $\mathbf{p}$ is now estimated such that a model error, yet to be defined, is minimized. If the model error is chosen as the so-called *generalized equation error* according to Figure 1.8, we obtain the relationship

$$e_k = \hat{A}(\delta)y_k - \hat{B}(\delta)u_k \tag{1.75}$$

with

$$\hat{A}(\delta) = 1 + \hat{a}_1\delta^{-1} + \ldots + \hat{a}_n\delta^{-n} \tag{1.76a}$$
$$\hat{B}(\delta) = \hat{b}_0\delta^{-d} + \hat{b}_1\delta^{-1-d} + \ldots + \hat{b}_m\delta^{-m-d} \tag{1.76b}$$

and the estimated coefficients of the numerator and denominator polynomials $\hat{b}_j$, $j = 0, \ldots, m$ and $\hat{a}_k$, $k = 1, \ldots, n$. Substituting (1.76) into (1.75) and combining the coefficients

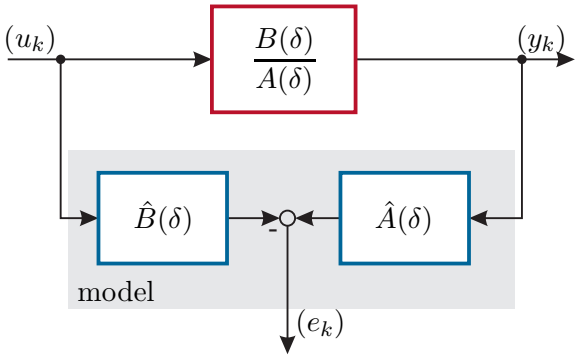


Figure 1.8: On the generalized equation error.

to be estimated in the *parameter estimate vector* $\hat{\mathbf{p}} = \begin{bmatrix}\hat{a}_1 & \dots & \hat{a}_n & \hat{b}_0 & \dots & \hat{b}_m\end{bmatrix}^{\mathrm{T}}$, we obtain

$$e_k = y_k - \mathbf{s}_k^{\mathrm{T}}\hat{\mathbf{p}} \tag{1.77}$$

with the *data vector* $\mathbf{s}_k$ according to (1.74). From (1.77), it can be immediately seen that the estimate $\hat{y}_k$ of y_k is calculated as $\hat{y}_k = \mathbf{s}_k^{\mathrm{T}}\hat{\mathbf{p}}$. By combining $j = 0, \dots, N$ measurements, (1.77) can be extended as follows:

$$\underbrace{\begin{bmatrix} e_0 \\ \vdots \\ e_N \end{bmatrix}}_{\mathbf{e}_N} = \underbrace{\begin{bmatrix} y_0 \\ \vdots \\ y_N \end{bmatrix}}_{\mathbf{y}_N} - \underbrace{\begin{bmatrix} \mathbf{s}_0^{\mathrm{T}} \\ \vdots \\ \mathbf{s}_N^{\mathrm{T}} \end{bmatrix}}_{\mathbf{S}_N} \hat{\mathbf{p}}_N \tag{1.78}$$

where the index N of the parameter estimate vector $\hat{\mathbf{p}}$ indicates that $N+1$ measurements are used to estimate $\mathbf{p}$. For $N > n+m$ with m and n according to (1.72) and $\mathrm{rank}(\mathbf{S}_N) \neq \mathrm{rank}([\mathbf{S}_N, \mathbf{y}_N])$, the system of equations (1.78) has no solution for $\mathbf{e}_N = \mathbf{0}$. Following the considerations of Section 1.3.2, the solution of (1.78) in the least squares sense (see (1.63)) is

$$\hat{\mathbf{p}}_N = \left(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\right)^{-1}\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N \ . \tag{1.79}$$

Remark: Regarding the choice of notation, it should be noted that $j = 0$ does not necessarily refer to the starting time of the measurement $k = 0$, but to the time k at which the entry u_{k-d} in the data vector $\mathbf{s}_k^{\mathrm{T}}$ is first non-zero.

It should be emphasized again that only the specific choice of the generalized equation error according to Figure 1.8 leads to a parametrically linear estimation problem. If, for example, the so-called *output error* shown in Figure 1.9

$$e_k = y_k - \hat{y}_k = y_k - \frac{\hat{B}(\delta)}{\hat{A}(\delta)}u_k = y_k - \delta^{-d}\frac{\hat{b}_0 + \hat{b}_1\delta^{-1} + \ldots + \hat{b}_m\delta^{-m}}{1 + \hat{a}_1\delta^{-1} + \ldots + \hat{a}_n\delta^{-n}}u_k \tag{1.80}$$

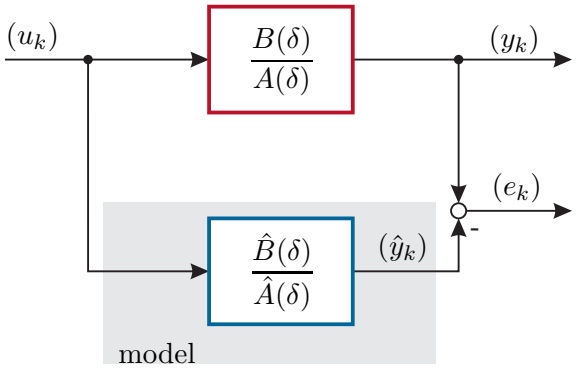


Figure 1.9: On the output error.

were used, then from

$$\hat{y}_k = -\hat{a}_1\hat{y}_{k-1} - \ldots - \hat{a}_n\hat{y}_{k-n} + \hat{b}_0 u_{k-d} + \hat{b}_1 u_{k-d-1} + \ldots + \hat{b}_m u_{k-d-m} \tag{1.81}$$

it can already be seen for $k = 0, \ldots, d+1$ that

$$\begin{aligned} \hat{y}_k &= 0 \quad \text{for } k = 0, 1, \ldots, d-1 \\ \hat{y}_d &= \hat{b}_0 u_0 \\ \hat{y}_{d+1} &= -\hat{a}_1\hat{y}_d + \hat{b}_0 u_1 + \hat{b}_1 u_0 = \left(\hat{b}_1 - \hat{a}_1\hat{b}_0\right)u_0 + \hat{b}_0 u_1, \end{aligned} \tag{1.82}$$

that in this case it is a parametrically nonlinear estimation problem, which is orders of magnitude more difficult to solve.

Before specifying how a stochastic disturbance (v_k) affects the result of the parameter estimation $\hat{\mathbf{p}}$ from (1.79), two fundamental properties of parameter estimation methods are defined.

Definition 1.4 (Unbiased Estimate)**.** If an estimate for an arbitrary number N of measurements yields a systematic error

$$\mathrm{E}(\hat{\mathbf{p}}_N - \mathbf{p}) = \mathrm{E}(\hat{\mathbf{p}}_N) - \mathbf{p} = \mathbf{b} \neq \mathbf{0} \tag{1.83}$$

then this error is called *bias*. For an *unbiased* estimate, therefore,

$$\mathrm{E}(\hat{\mathbf{p}}_N) = \mathbf{p}\ . \tag{1.84}$$

Definition 1.5 (Consistent Estimate)**.** An estimate is called *consistent* if the estimate becomes more accurate the larger the number N of measurements, i.e., if

$$\lim_{N\to\infty} \mathrm{E}(\hat{\mathbf{p}}_N) = \mathbf{p}\ . \tag{1.85}$$

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An estimate is called *consistent in the mean square* if, in addition to (1.85), the condition

$$\lim_{N\to\infty} \operatorname{cov}(\hat{\mathbf{p}}_N) = \lim_{N\to\infty} \mathrm{E}\Big([\hat{\mathbf{p}}_N - \mathbf{p}][\hat{\mathbf{p}}_N - \mathbf{p}]^{\mathrm{T}}\Big) = \mathbf{0} \tag{1.86}$$

is satisfied.

Remark: Note that a consistent estimate says nothing about the quality of the estimate for finite N. It is therefore possible that a consistent estimate is biased for finite N.

To analyze the influence of a stochastic disturbance, it is assumed that the measured output variable y_k is composed of the undisturbed output $\bar{y}_k$ and a stochastic disturbance v_k in the form $y_k = \bar{y}_k + v_k$, see Figure 1.3. The undisturbed output is calculated according to (1.74) as

$$\bar{y}_k = \underbrace{\begin{bmatrix} -\bar{y}_{k-1} & -\bar{y}_{k-2} & \dots & -\bar{y}_{k-n} & u_{k-d} & u_{k-d-1} & \dots & u_{k-d-m} \end{bmatrix}}_{\bar{\mathbf{s}}_k^{\mathrm{T}}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_0 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{p}} . \tag{1.87}$$

Replacing the entries $\bar{y}_j$ in (1.87) by $\bar{y}_j = y_j - v_j$, we obtain

$$\bar{y}_k = \mathbf{s}_k^{\mathrm{T}}\mathbf{p} + \mathbf{n}_k^{\mathrm{T}}\mathbf{p} \tag{1.88}$$

with

$$\mathbf{s}_k^{\mathrm{T}} = \begin{bmatrix} -y_{k-1} & -y_{k-2} & \dots & -y_{k-n} & u_{k-d} & u_{k-d-1} & \dots & u_{k-d-m} \end{bmatrix} \tag{1.89a}$$

$$\mathbf{n}_k^{\mathrm{T}} = \begin{bmatrix} v_{k-1} & v_{k-2} & \dots & v_{k-n} & 0 & 0 & \dots & 0 \end{bmatrix} . \tag{1.89b}$$

By combining $k = 0, \dots, N$ measurements and using $y_k = \bar{y}_k + v_k$, the following formulation can be found:

$$\underbrace{\begin{bmatrix} y_0 \\ \vdots \\ y_N \end{bmatrix}}_{\mathbf{y}_N} = \underbrace{\begin{bmatrix} \mathbf{s}_0^{\mathrm{T}} \\ \vdots \\ \mathbf{s}_N^{\mathrm{T}} \end{bmatrix}}_{\mathbf{S}_N} \mathbf{p} + \underbrace{\begin{bmatrix} \mathbf{n}_0^{\mathrm{T}} \\ \vdots \\ \mathbf{n}_N^{\mathrm{T}} \end{bmatrix}}_{\mathbf{N}_N} \mathbf{p} + \underbrace{\begin{bmatrix} v_0 \\ \vdots \\ v_N \end{bmatrix}}_{\mathbf{v}_N} . \tag{1.90}$$

If we now perform a least-squares identification based on the known part $\mathbf{y}_N = \mathbf{S}_N\mathbf{p}$, we obtain

$$\hat{\mathbf{p}} = \Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1}\mathbf{S}_N^{\mathrm{T}}\mathbf{y} = \Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1}\mathbf{S}_N^{\mathrm{T}}(\mathbf{S}_N\mathbf{p} + \mathbf{N}_N\mathbf{p} + \mathbf{v}_N) \tag{1.91}$$

or, after a short calculation,

$$\hat{\mathbf{p}} = \mathbf{p} + \left(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\right)^{-1}\mathbf{S}_N^{\mathrm{T}}(\mathbf{N}_N\mathbf{p} + \mathbf{v}_N) \ . \tag{1.92}$$

The expected value of the estimation error is thus

$$\mathrm{E}(\hat{\mathbf{p}}) - \mathbf{p} = \mathrm{E}\left(\left(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\right)^{-1}\mathbf{S}_N^{\mathrm{T}}(\mathbf{N}_N\mathbf{p} + \mathbf{v}_N)\right) = \mathbf{b}, \tag{1.93}$$

with the bias $\mathbf{b}$. The estimate $\hat{\mathbf{p}}$ of $\mathbf{p}$ is unbiased if and only if

$$\mathbf{b} = \mathbf{0} \tag{1.94}$$

Furthermore, the estimate is obviously consistent if

$$\lim_{N\to\infty} \mathbf{b} = \mathbf{0} \tag{1.95}$$

is satisfied. The following theorem now provides information on the requirements that must be placed on the stochastic disturbance v_k and the model structure for these two conditions to be satisfied.

Theorem 1.2 (Unbiased and Consistent Least-Squares Identification)**.** *The least-squares identification of the parameter* $\mathbf{p}$ *for the identification problem according to* (1.90) *or according to Figure 1.3 is unbiased and consistent if the stochastic disturbance* v_k *satisfies the* Yule-Walker equation *of an autoregressive signal process of the form*

$$v_k + a_1 v_{k-1} + a_2 v_{k-2} + \ldots + a_n v_{k-n} = w_k, \tag{1.96}$$

with zero-mean white noise w_k *and the coefficients* a_j, $j = 1, \ldots, n$, *of the denominator of the transfer function to be identified.*

That is, the disturbance signal v_k must have been generated by filtering white noise w_k through a filter with the transfer function $1/A(\delta)$. This structure corresponds exactly to the model structure of an ARX model defined in Section 1.3.1, see Figure 1.10. It can also be shown that in this case the least-squares identification is consistent in the mean square. For a proof of Theorem 1.2, the reader is referred to the literature, in particular [1.3].

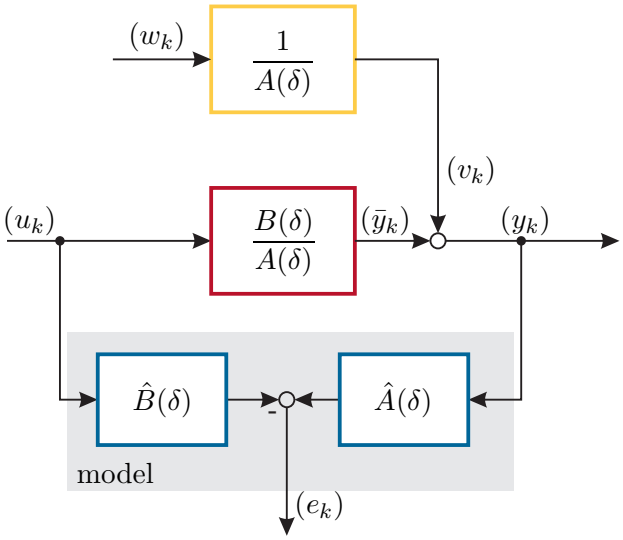


Figure 1.10: On least-squares identification with stochastic disturbance.

1.3.4 Recursive Least-Squares (RLS) Identification

The parameter estimation (1.79) is not suitable for online operation due to the ever-increasing dimensions of the measurement vector $\mathbf{y}_N$ and the matrix $\mathbf{S}_N$. In the following, a *recursive method* is given based on (1.79), which improves the estimate $\hat{\mathbf{p}}$ of the parameter vector $\mathbf{p}$ with each new measurement. According to (1.79), the optimal estimate for $N+1$ measurements is

$$\hat{\mathbf{p}}_N = \left(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\right)^{-1}\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N \tag{1.97}$$

and for $N+2$ measurements

$$\hat{\mathbf{p}}_{N+1} = \left(\mathbf{S}_{N+1}^{\mathrm{T}}\mathbf{S}_{N+1}\right)^{-1}\mathbf{S}_{N+1}^{\mathrm{T}}\mathbf{y}_{N+1}\ . \tag{1.98}$$

Partitioning the data matrix $\mathbf{S}_{N+1}$ and the measurement vector $\mathbf{y}_{N+1}$ in the form

$$\mathbf{S}_{N+1} = \begin{bmatrix}\mathbf{S}_N \\ \mathbf{s}_{N+1}^{\mathrm{T}}\end{bmatrix} \quad \text{and} \quad \mathbf{y}_{N+1} = \begin{bmatrix}\mathbf{y}_N \\ y_{N+1}\end{bmatrix}, \tag{1.99}$$

then $\hat{\mathbf{p}}_{N+1}$ is

$$\begin{aligned}\hat{\mathbf{p}}_{N+1} &= \left(\begin{bmatrix}\mathbf{S}_N^{\mathrm{T}} & \mathbf{s}_{N+1}\end{bmatrix}\begin{bmatrix}\mathbf{S}_N \\ \mathbf{s}_{N+1}^{\mathrm{T}}\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}\mathbf{S}_N^{\mathrm{T}} & \mathbf{s}_{N+1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_N \\ y_{N+1}\end{bmatrix}\right) \\ &= \left(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N + \mathbf{s}_{N+1}\mathbf{s}_{N+1}^{\mathrm{T}}\right)^{-1}\left(\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N + \mathbf{s}_{N+1}y_{N+1}\right)\ .\end{aligned} \tag{1.100}$$

Exercise 1.14. Show the validity of (1.100).

For further calculation, the following auxiliary theorem on *matrix inversion* is required:

Theorem 1.3 (On Matrix Inversion). *If* $\mathbf{A}$, $\mathbf{C}$, *and* $(\mathbf{A}+\mathbf{BCD})$ *are non-singular square matrices, then*

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\Big(\mathbf{C}^{-1}+\mathbf{D}\mathbf{A}^{-1}\mathbf{B}\Big)^{-1}\mathbf{D}\mathbf{A}^{-1} \ . \tag{1.101}$$

Applying Theorem 1.3 to $\mathbf{F} = \Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N + \mathbf{s}_{N+1}\mathbf{s}_{N+1}^{\mathrm{T}}\Big)$ with $\mathbf{A} = \mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N$, $\mathbf{B} = \mathbf{s}_{N+1}$, $\mathbf{C} = 1$, and $\mathbf{D} = \mathbf{s}_{N+1}^{\mathrm{T}}$, we obtain

$$\mathbf{F}^{-1} = \Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1} - \Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1}\mathbf{s}_{N+1}\Big(1+\mathbf{s}_{N+1}^{\mathrm{T}}\Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1}\mathbf{s}_{N+1}\Big)^{-1}\mathbf{s}_{N+1}^{\mathrm{T}}\Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1} \ . \tag{1.102}$$

With the abbreviations

$$\mathbf{P}_N = \Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N\Big)^{-1} \tag{1.103a}$$

$$\mathbf{P}_{N+1} = \Big(\mathbf{S}_{N+1}^{\mathrm{T}}\mathbf{S}_{N+1}\Big)^{-1} \tag{1.103b}$$

$$\mathbf{k}_{N+1} = \frac{\mathbf{P}_N\mathbf{s}_{N+1}}{\Big(1+\mathbf{s}_{N+1}^{\mathrm{T}}\mathbf{P}_N\mathbf{s}_{N+1}\Big)} \tag{1.103c}$$

according to (1.102) we have

$$\mathbf{P}_{N+1} = \mathbf{P}_N - \mathbf{k}_{N+1}\mathbf{s}_{N+1}^{\mathrm{T}}\mathbf{P}_N \tag{1.104}$$

and

$$\mathbf{P}_{N+1}\mathbf{s}_{N+1} = \frac{\mathbf{P}_N\mathbf{s}_{N+1}\Big(1+\mathbf{s}_{N+1}^{\mathrm{T}}\mathbf{P}_N\mathbf{s}_{N+1}\Big) - \mathbf{P}_N\mathbf{s}_{N+1}\mathbf{s}_{N+1}^{\mathrm{T}}\mathbf{P}_N\mathbf{s}_{N+1}}{\Big(1+\mathbf{s}_{N+1}^{\mathrm{T}}\mathbf{P}_N\mathbf{s}_{N+1}\Big)} = \mathbf{k}_{N+1} \ . \tag{1.105}$$

Substituting the relationships (1.103)–(1.105) into (1.100), we obtain

$$\hat{\mathbf{p}}_{N+1} = \mathbf{P}_{N+1}\Big(\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N + \mathbf{s}_{N+1}y_{N+1}\Big) = \mathbf{P}_N\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N - \mathbf{k}_{N+1}\mathbf{s}_{N+1}^{\mathrm{T}}\mathbf{P}_N\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N + \mathbf{k}_{N+1}y_{N+1} \tag{1.106}$$

or with (1.97)

$$\hat{\mathbf{p}}_{N+1} = \hat{\mathbf{p}}_N + \mathbf{k}_{N+1}\Big(y_{N+1} - \mathbf{s}_{N+1}^{\mathrm{T}}\hat{\mathbf{p}}_N\Big) \ . \tag{1.107}$$

The recursive least-squares identification can thus be summarized as follows:

(1) Suitable initial values $\hat{\mathbf{p}}_{-1}$ and $\mathbf{P}_{-1}$ are chosen (see discussion below).

(2) For the sampling times $j = 0, 1, \ldots$, y_j is measured and the data vector is set up according to (1.74):

$$\mathbf{s}_j^{\mathrm{T}} = \begin{bmatrix} -y_{j-1} & -y_{j-2} & \ldots & -y_{j-n} & u_{j-d} & u_{j-d-1} & \ldots & u_{j-d-m} \end{bmatrix} \tag{1.108}$$

The parameter vector $\mathbf{p}$ can then be estimated online using $j+1$ measurements with the iteration rule

$$\mathbf{k}_j = \frac{\mathbf{P}_{j-1}\mathbf{s}_j}{\left(1 + \mathbf{s}_j^{\mathrm{T}}\mathbf{P}_{j-1}\mathbf{s}_j\right)} \tag{1.109a}$$

$$\mathbf{P}_j = \mathbf{P}_{j-1} - \mathbf{k}_j\mathbf{s}_j^{\mathrm{T}}\mathbf{P}_{j-1} \tag{1.109b}$$

$$\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_{j-1} + \mathbf{k}_j\left(y_j - \mathbf{s}_j^{\mathrm{T}}\hat{\mathbf{p}}_{j-1}\right) \tag{1.109c}$$

In the last step, the question of *suitable initial values* $\hat{\mathbf{p}}_{-1}$ and $\mathbf{P}_{-1}$ must still be clarified. To this end, the iteration rule for $\mathbf{P}_k^{-1}$ (see (1.100) and (1.103)) is first considered:

$$\mathbf{P}_{k+1}^{-1} = \mathbf{P}_k^{-1} + \mathbf{s}_{k+1}\mathbf{s}_{k+1}^{\mathrm{T}} \tag{1.110}$$

and this is iterated for $k = -1, 0, 1, \ldots, N, \ldots$

$$\mathbf{P}_0^{-1} = \mathbf{P}_{-1}^{-1} + \mathbf{s}_0\mathbf{s}_0^{\mathrm{T}} \tag{1.111a}$$

$$\mathbf{P}_1^{-1} = \mathbf{P}_{-1}^{-1} + \mathbf{s}_0\mathbf{s}_0^{\mathrm{T}} + \mathbf{s}_1\mathbf{s}_1^{\mathrm{T}} \tag{1.111b}$$

$$\vdots \quad \vdots$$

$$\mathbf{P}_N^{-1} = \mathbf{P}_{-1}^{-1} + \sum_{j=0}^{N}\mathbf{s}_j\mathbf{s}_j^{\mathrm{T}} = \mathbf{P}_{-1}^{-1} + \mathbf{S}_N^{\mathrm{T}}\mathbf{S}_N \ . \tag{1.111c}$$

It can be seen that with the choice

$$\mathbf{P}_{-1} = \alpha\mathbf{E} \tag{1.112}$$

for large values of α, $\lim_{\alpha\to\infty}\mathbf{P}_{-1}^{-1} = \mathbf{0}$ holds, and thus (1.111) agrees with the corresponding expression for the non-recursive estimate from (1.79). For large values of α in (1.112), $\mathbf{P}_{-1}^{-1}$ therefore has a negligible influence on the recursively calculated $\mathbf{P}_N$. The iteration rule for $\hat{\mathbf{p}}_k$ is, using (1.105) and (1.107),

$$\hat{\mathbf{p}}_{k+1} = \hat{\mathbf{p}}_k + \mathbf{P}_{k+1}\mathbf{s}_{k+1}\left(y_{k+1} - \mathbf{s}_{k+1}^{\mathrm{T}}\hat{\mathbf{p}}_k\right) \ . \tag{1.113}$$

If this iteration is now carried out for $k = -1, 0, 1, \ldots, N, \ldots$, then using (1.111) we obtain

$$\hat{\mathbf{p}}_0 = \hat{\mathbf{p}}_{-1} + \mathbf{P}_0\mathbf{s}_0\left(y_0 - \mathbf{s}_0^{\mathrm{T}}\hat{\mathbf{p}}_{-1}\right) = \mathbf{P}_0\left(\mathbf{s}_0 y_0 + \underbrace{\left(\mathbf{P}_0^{-1} - \mathbf{s}_0\mathbf{s}_0^{\mathrm{T}}\right)}_{\mathbf{P}_{-1}^{-1}}\hat{\mathbf{p}}_{-1}\right) \tag{1.114a}$$

$$\hat{\mathbf{p}}_1 = \mathbf{P}_1\left(\mathbf{s}_1 y_1 + \mathbf{P}_0^{-1}\hat{\mathbf{p}}_0\right) = \mathbf{P}_1\left(\mathbf{s}_1 y_1 + \mathbf{s}_0 y_0 + \mathbf{P}_{-1}^{-1}\hat{\mathbf{p}}_{-1}\right) \tag{1.114b}$$

$$\vdots \quad \vdots$$

$$\hat{\mathbf{p}}_N = \mathbf{P}_N\left(\sum_{j=0}^{N}\mathbf{s}_j y_j + \mathbf{P}_{-1}^{-1}\hat{\mathbf{p}}_{-1}\right) = \mathbf{P}_N\left(\mathbf{S}_N^{\mathrm{T}}\mathbf{y}_N + \mathbf{P}_{-1}^{-1}\hat{\mathbf{p}}_{-1}\right) \ . \tag{1.114c}$$

As can be seen from (1.114), the choice of $\mathbf{P}_{-1}^{-1}$ from (1.112) for large α implies that the initial value $\hat{\mathbf{p}}_{-1}$ can be chosen freely and yet $\hat{\mathbf{p}}_N$ from (1.114) coincides with the result of the non-recursive estimate from (1.79). For simplicity, $\hat{\mathbf{p}}_{-1} = \mathbf{0}$ is therefore often chosen.

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1.3.5 Weighted Least Squares Method

In the weighted least squares method, we seek a solution to the overdetermined linear system of equations (see (1.60))

$$\mathbf{y}_N = \mathbf{S}_N \mathbf{p} \tag{1.115}$$

such that the *weighted quadratic error*

$$\sum_{j=0}^{N} \alpha_j e_j^2 \quad , \quad e_j = y_j - \mathbf{s}_j^{\mathrm{T}} \mathbf{p} \tag{1.116}$$

with the sequence of positive weighting coefficients α_j ($\alpha_j > 0$ for all j) is minimized with respect to $\mathbf{p}$. It can be seen that by choosing

$$\tilde{y}_j = \sqrt{\alpha_j} y_j \quad \text{and} \quad \tilde{\mathbf{s}}_j^{\mathrm{T}} = \sqrt{\alpha_j} \mathbf{s}_j^{\mathrm{T}} \tag{1.117}$$

the optimization problem (1.116) can be transformed into the classical least-squares problem according to (1.61)

$$\min_{\mathbf{p}} \sum_{j=0}^{N} \tilde{e}_j^2 = \min_{\mathbf{p}} \|\tilde{\mathbf{e}}_N\|_2^2 \quad \text{with} \quad \tilde{\mathbf{e}}_N = \tilde{\mathbf{y}}_N - \tilde{\mathbf{S}}_N \mathbf{p} \tag{1.118}$$

The corresponding recursive estimator according to (1.109) is therefore

$$\tilde{\mathbf{k}}_j = \frac{\tilde{\mathbf{P}}_{j-1} \tilde{\mathbf{s}}_j}{\left(1 + \tilde{\mathbf{s}}_j^{\mathrm{T}} \tilde{\mathbf{P}}_{j-1} \tilde{\mathbf{s}}_j\right)} \tag{1.119a}$$

$$\tilde{\mathbf{P}}_j = \tilde{\mathbf{P}}_{j-1} - \tilde{\mathbf{k}}_j \tilde{\mathbf{s}}_j^{\mathrm{T}} \tilde{\mathbf{P}}_{j-1} \tag{1.119b}$$

$$\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_{j-1} + \tilde{\mathbf{k}}_j \left(\tilde{y}_j - \tilde{\mathbf{s}}_j^{\mathrm{T}} \hat{\mathbf{p}}_{j-1}\right) . \tag{1.119c}$$

Substituting the relationship (1.117) and the transformation

$$\tilde{\mathbf{k}}_j = \frac{\mathbf{k}_j}{\sqrt{\alpha_j}} \quad \text{and} \quad \tilde{\mathbf{P}}_j = \mathbf{P}_j \tag{1.120}$$

into (1.119), we obtain

$$\mathbf{k}_j = \frac{\mathbf{P}_{j-1} \mathbf{s}_j}{\left(\frac{1}{\alpha_j} + \mathbf{s}_j^{\mathrm{T}} \mathbf{P}_{j-1} \mathbf{s}_j\right)} \tag{1.121a}$$

$$\mathbf{P}_j = \mathbf{P}_{j-1} - \mathbf{k}_j \mathbf{s}_j^{\mathrm{T}} \mathbf{P}_{j-1} \tag{1.121b}$$

$$\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_{j-1} + \mathbf{k}_j \left(y_j - \mathbf{s}_j^{\mathrm{T}} \hat{\mathbf{p}}_{j-1}\right) . \tag{1.121c}$$

A special choice for α_j is given by

$$\alpha_j = q^{N-j} \quad \text{with} \quad 0 < q \leq 1 \tag{1.122}$$

Substituting (1.122) into the cost function of (1.116)

$$\sum_{j=0}^{N} q^{N-j} e_j^2 = \|\mathbf{e}_N\|_Q^2 = \mathbf{e}_N^{\mathrm{T}} \mathbf{Q}_N \mathbf{e}_N \quad \text{with} \quad \mathbf{Q}_N = \begin{bmatrix} q^N & 0 & \cdots & 0 & 0 \\ 0 & q^{N-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{1.123}$$

it is evident that with this special choice of α_j, the equation errors are weighted less the further back they lie. This is also referred to as an *exponentially decaying memory* with the *memory factor* q. For this memory factor q, values in the range of $0.9 < q < 0.995$ have proven useful in practical application. Note that this method is also suitable for the identification of *slowly time-varying systems*, where the speed of the parameter changes of the system naturally determines the choice of q, i.e., the speed of forgetting old measurement values.

Exercise 1.15. In your opinion, where does the name exponentially decaying memory come from?

The recursive least-squares estimator with exponentially decaying memory follows directly from (1.121) with $\alpha_j = q^{N-j}$ and the transformation $\mathbf{P}_j \to \mathbf{P}_j / q^{N-j}$ to

$$\mathbf{k}_j = \frac{\mathbf{P}_{j-1}\mathbf{s}_j}{\left(q + \mathbf{s}_j^{\mathrm{T}} \mathbf{P}_{j-1} \mathbf{s}_j\right)} \tag{1.124a}$$

$$\mathbf{P}_j = \left(\mathbf{P}_{j-1} - \mathbf{k}_j \mathbf{s}_j^{\mathrm{T}} \mathbf{P}_{j-1}\right) \frac{1}{q} \tag{1.124b}$$

$$\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_{j-1} + \mathbf{k}_j \left(y_j - \mathbf{s}_j^{\mathrm{T}} \hat{\mathbf{p}}_{j-1}\right) . \tag{1.124c}$$

Exercise 1.16. Verify the correctness of (1.124).

Instead of the diagonal matrix $\mathbf{Q}_N$ in (1.123), any arbitrary positive definite *weighting matrix* $\mathbf{Q}_N > 0$ could be chosen. In this case, not the Euclidean norm of the error $\|\mathbf{e}_N\|_2^2$, but a weighted Euclidean norm of the form

$$\|\mathbf{e}_N\|_Q^2 = \mathbf{e}_N^{\mathrm{T}} \mathbf{Q}_N \mathbf{e}_N \tag{1.125}$$

is minimized. It is now easy to see that this problem

$$\min_{\mathbf{p}} \|\mathbf{e}_N\|_Q^2 \quad \text{with} \quad \mathbf{e}_N = \mathbf{y}_N - \mathbf{S}_N \mathbf{p} \tag{1.126}$$

by applying the projection theorem of Theorem 1.1 with the inner product

$$\langle \mathbf{x}, \mathbf{z} \rangle = \mathbf{x}^{\mathrm{T}} \mathbf{Q} \mathbf{z} \tag{1.127}$$

leads to the solution

$$\hat{\mathbf{p}}_N = \left(\mathbf{S}_N^{\mathrm{T}} \mathbf{Q}_N \mathbf{S}_N\right)^{-1} \mathbf{S}_N^{\mathrm{T}} \mathbf{Q}_N \mathbf{y}_N \tag{1.128}$$

Exercise 1.17. Show the validity of (1.128). Furthermore, check whether the error $\mathbf{e}_N = \mathbf{y}_N - \mathbf{S}_N\hat{\mathbf{p}}_N$ is orthogonal to the optimal solution (1.128) in the sense of the inner product (1.127).

Exercise 1.18. Show that every positive definite matrix $\mathbf{Q}$ has exactly one positive definite square root $\mathbf{W}$, i.e., $\mathbf{Q} = \mathbf{W}^{\mathrm{T}}\mathbf{W}$.

Remark: (to Exercise 1.18) Use the fact that every symmetric matrix $\mathbf{Q}$ can be brought to diagonal form $\mathbf{D}$ by a similarity transformation, so

$$\mathbf{D} = \mathbf{T}\mathbf{Q}\mathbf{T}^{\mathrm{T}} \quad \text{with} \quad \mathbf{T}\mathbf{T}^{\mathrm{T}} = \mathbf{E} \; .$$

Exercise 1.19. Prove that (1.125) is a norm in the sense of Definition 1.2.

Exercise 1.20. Given is the continuous transfer function

$$G(s) = \frac{K}{(sT_1 + 1)(sT_2 + 1)}$$

with the parameters $K, T_1, T_2 > 0$. Determine a state-space representation for $G(s)$ and implement it in Matlab/Simulink for a time-varying parameter T_2 and constant parameters K and T_1. Subsequently, determine the corresponding z-transfer function for $G(s)$ for a sampling time of $T_a = 0.25\,\mathrm{s}$. Identify the coefficients a_j and b_j of the denominator and numerator polynomials of the discrete-time transfer function using the recursive least-squares algorithm. Choose a multiple 3-2-1 step as the input signal and for the parameters $K = 1$, $T_1 = 7.5\,\mathrm{s}$, and a step change of T_2 from $5\,\mathrm{s}$ to $2.5\,\mathrm{s}$. Investigate the influence of the forgetting factor q and compare the result for $q = 1$ with the result of the off-line least-squares procedure.

1.3.6 Least-Mean Squares (LMS) Identification

Another way to solve the system of equations (1.71)

$$\left(\sum_{j=0}^{N} \mathbf{s}_j \mathbf{s}_j^{\mathrm{T}}\right) \hat{\mathbf{p}}_N = \mathbf{S}_N^{\mathrm{T}} \mathbf{S}_N \hat{\mathbf{p}}_N = \mathbf{S}_N^{\mathrm{T}} \mathbf{y}_N = \sum_{j=0}^{N} \mathbf{s}_j y_j \tag{1.129}$$

online, is given by the so-called *Least-Mean Squares (LMS) algorithm*, also known as the *stochastic gradient method*. The parameter vector $\mathbf{p}$ is recursively estimated in the form

$$\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_{j-1} + \mu_j \mathbf{s}_j \underbrace{\left(y_j - \mathbf{s}_j^{\mathrm{T}} \hat{\mathbf{p}}_{j-1}\right)}_{e_j} \tag{1.130}$$

with a suitably chosen initial value $\hat{\mathbf{p}}_{-1}$ and the estimation error at the j-th time instant e_j. Using the *(time-varying) parameter* μ_j ($\mu_j \geq 0$ for all $j \geq 0$), it is possible to adjust the convergence speed and the sensitivity to noise to a certain extent. Typically, a constant value $\mu_j = \bar{\mu}$ is chosen for μ_j, whereby the parameter estimator (1.130) is able to react faster to changes in the parameter vector $\mathbf{p}$ (time-varying system) for larger values of $\bar{\mu}$ and is less sensitive to measurement noise for smaller values of $\bar{\mu}$. To make the convergence speed independent of the signal level of the entries in $\mathbf{s}_j$, it is common to choose the (time-varying) parameter μ_j in the LMS algorithm (1.130) in the form

$$\mu_j = \frac{\bar{\mu}}{\mathbf{s}_j^{\mathrm{T}} \mathbf{s}_j} \tag{1.131}$$

or

$$\mu_j = \frac{\bar{\mu}}{l_j} \quad \text{with} \quad l_{j+1} = l_j + \bar{\mu}\left(\mathbf{s}_j^{\mathrm{T}} \mathbf{s}_j - l_j\right) \tag{1.132}$$

with the initial value $l_{-1} = \mathbf{s}_{-1}^{\mathrm{T}} \mathbf{s}_{-1} > 0$. Without proof, it should be noted that the convergence of (1.130) can be guaranteed for sufficiently small $\bar{\mu}$.

Note 1.2. The LMS algorithm is very frequently used in applications for the adaptive filtering of signals (*adaptive noise cancellation*), such as echo compensation of transmitted signals. Consider the arrangement in Figure 1.11, where the measured signal sequence (d_k) consists of a useful signal component (r_k) to be identified and a non-measurable noise component $G_1(\delta)m_k$ in the form

$$d_k = G_1(\delta) m_k + r_k \tag{1.133}$$

with the noise signal (m_k) and the unknown transfer operator $G_1(\delta)$. Furthermore, it is assumed that the noise signal (m_k) and the useful signal (r_k) are uncorrelated. The noise signal (m_k) is not directly measured but can be indirectly captured via the signal (n_k) with

$$n_k = G_2(\delta) m_k \tag{1.134}$$

and the unknown transfer operator $G_2(\delta)$. If the two transfer operators $G_1(\delta)$ and $G_2(\delta)$ were known, then the useful signal could be reconstructed from knowledge of

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(d_k) and (n_k) in the form

$$r_k = d_k - \underbrace{G_1(\delta)G_2^{-1}(\delta)}_{\tilde{G}(\delta)} n_k \tag{1.135}$$

Note that $\tilde{G}(\delta) = G_1(\delta)G_2^{-1}(\delta)$ is generally a non-causal transfer operator.

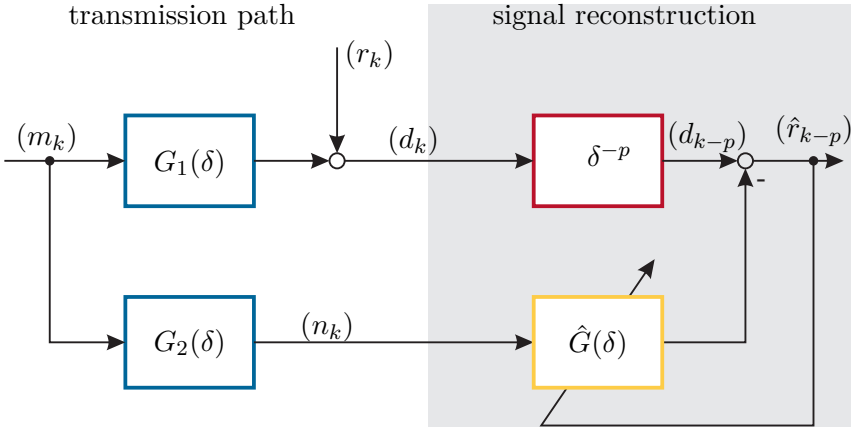


Figure 1.11: On adaptive signal filtering with the LMS algorithm.

Exercise 1.21. Under what conditions on $G_1(\delta)$ and $G_2(\delta)$ is $\tilde{G}(\delta)$ a causal transfer operator?

For this reason, a positive integer number $p > 0$ is chosen such that the transfer operator

$$G(\delta) = \tilde{G}(\delta)\delta^{-p} \tag{1.136}$$

is always causal. Replacing $\tilde{G}(\delta)$ in (1.135) by $G(\delta)$ according to (1.136), we obtain

$$r_k = d_k - G(\delta)\delta^p n_k \tag{1.137a}$$
$$\delta^{-p} r_k = \delta^{-p} d_k - G(\delta) n_k \tag{1.137b}$$
$$r_{k-p} = d_{k-p} - G(\delta) n_k \ . \tag{1.137c}$$

Since the transfer operator $G(\delta)$ in (1.137) is unknown, a transfer operator $\hat{G}(\delta)$ to be identified is substituted for $G(\delta)$. In most applications, an MA model (FIR filter) according to (1.47), (1.48) of the form

$$\hat{G}(\delta) = \hat{C}(\delta) = \hat{c}_0 + \hat{c}_1\delta^{-1} + \ldots + \hat{c}_q\delta^{-q} \tag{1.138}$$

is used in combination with the LMS algorithm. Of course, the transfer operator $G(\delta)$ is only describable by an MA model in exceptional cases, which is why a very high model order q of the MA model is required for good reconstruction of the useful signal (r_k). Substituting the expression $\hat{G}(\delta)$ from (1.138) for $G(\delta)$ in (1.137), the

estimate of the useful signal $\hat{r}_{k-p}$ is calculated as

$$\hat{r}_{k-p} = d_{k-p} - \underbrace{\begin{bmatrix} n_k & n_{k-1} & n_{k-2} & \dots & n_{k-q} \end{bmatrix}}_{\mathbf{s}_k^{\mathrm{T}}} \underbrace{\begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \vdots \\ \hat{c}_q \end{bmatrix}}_{\hat{\mathbf{p}}} \tag{1.139}$$

with the estimate $\hat{\mathbf{p}}$ of the parameter vector and the data vector $\mathbf{s}_k^{\mathrm{T}}$. Applying the LMS algorithm (1.130), the parameter estimator becomes

$$\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_{j-1} + \mu_j \mathbf{s}_j \underbrace{\left(d_{j-p} - \mathbf{s}_j^{\mathrm{T}} \hat{\mathbf{p}}_{j-1} \right)}_{\hat{r}_{j-p}} \tag{1.140}$$

with the parameter μ_j according to (1.131) or (1.132) and a suitably chosen initial value $\hat{\mathbf{p}}_{-1}$. Figure 1.11 provides a graphical illustration of the algorithm.

Exercise 1.22. Given is a system according to Figure 1.11 with the measured signal sequence (d_k), in which a periodic useful signal $(s_k) = (20\sin(3t - \pi/4))$ is hidden. The noise signal (m_k) is white noise, the sampling time is 0.1 s, and the two transfer operators are

$$G_1(\delta) = 10\frac{\delta + 2}{0.8 + \delta + \delta^2} \quad \text{and} \quad G_2(\delta) = \frac{1 - 2\delta}{(\delta + 0.2)^2(\delta - 0.8)} \ .$$

Design an online algorithm according to the LMS method for extracting the useful signal using an MA model. Perform these calculations in Matlab and also check the results in the frequency domain using the FFT. Specify different orders of the MA models and change the signal delay p according to (1.136). How do these specifications and changes affect the result?

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