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PERFORMANCE GRADIENTS IN AUTOMATIC CONTROL ANALYSIS
A Catalogue of Correspondences

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Abstract: *A comprehensive collection of relevant formulas and correspondencies is given for differential quotients of expressions in norm, trace, determinant and eigenvalue with respect to vectors and matrices as often used in automatic control.*

The results are provided for several indices of performance: Stability radius, entirety of eigenvalues, integral criteria of squared structures as well as classical control setups and dynamic interaction of multiple-input multiple-output systems.

The controllers are investigated in many representations: Transfer functions, state space, polynomial matrices, three-dimensional arrays and s-power-oriented coefficient matrices.

Both continuous-time and discrete-time representations are in use.

Keywords: Stability margin, transfer matrices, polynomial matrices, power-oriented matrices, three-dimensional arrays, square energy performance, per-unit square index of performance, interaction energy gradient, perturbed systems, robustness, robustification.

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1 Introduction

Several properties of control systems are given by scalars, e.g., eigenvalues, singular values, norms, Frobenius norms, traces, determinants, condition numbers, indices of performance etc.

The paper addresses several performance criteria as used in the field of automatic control. In most cases they are utilized to determine selected gradients of the controller implementations. Optimal design of the controller is intended.

The results are provided for several indices of performance: Stability radius, entirety of eigenvalues, integral criteria of squared structures as well as classical control setups and dynamic interaction of multiple-input multiple-output systems.

The controllers are investigated in many representations: Transfer function, state space, polynomial matrices, three-dimensional arrays and s-power-oriented coefficient matrices.

Both continuous-time and discrete-time representations are in use.

There are several disadvantages for gradients if they are considered from purely mathematically abstract viewpoint, such as selection of a specific performance index, unknown stepsize, vague stopping condition, the local minimum problem.

There is also a plenty of advantages if the problems are considered based on a good physical, chemical or economic insight, selection of various performance indices and a combination of their gradients, the free and process-oriented selection of step size and accuracy-oriented stopping condition. Process uncertainties and robustness considerations can be included without difficulty.

In a multitude of control system references, the stability border of a control system is investigated. By way of contrast, this paper addresses the gradients starting from an arbitrary nonoptimum controller assumption which might arise on the occasion of some refurbishment of an automatic control system.

Intentionally, throughout the paper several details and some redundant remarks are included to avoid any unclearness, e.g., as far as matrix dimensions, transpositions etc. are concerned.

The formulas are collected from numerous publications. Several are selected from recent publications of the author. A former article (*Weinmann, A., 2001b*), which comprises only a quarter of the present article, was an initial basis.

1.1 Notations

Many correspondences in this paper are influenced by the intention of straightforward optimal control system design. optimization. Among these expressions are the parameter vector $\mathbf{p}$, the matrix $\mathbf{K}$ of the controller in state space or a general matrix $\mathbf{M}$.

The symbols p and $\mathbf{p}$ are used for general scalar and vector-valued parameters, respectively. For a general matrix the term $\mathbf{M}$ or $\mathbf{X}$ is chosen; the symbol $\mathbf{P}$ is avoided since it is commonly used as the Lyapunov matrix. If in practice an expression is only used associated with controller matrix then $\mathbf{M}$ is replaced by $\mathbf{K}$.

The following notations are adopted. The set of real and complex matrices is $\mathcal{R}^{m\times n}$ and $\mathcal{C}^{f\times z}$, respectively, where the superscript denotes the dimension. Alternatively, the dimension is often denoted by $\mathbf{e}^{[r]}$, $\mathbf{A}^{[n\times m]}$ or $\mathbf{A}^{(n\times m)}$ or $\in \mathcal{R}^{n\times m}$. $\Re e$ and $\Im m$ assign the real and imaginary part of a complex quantity. The identity matrix of dimension $n\times n$ is $\mathbf{I}_n$. The meaning of $i=\{1,4\}$ is all the integers between 1 and 4. The entire continuum between 1 and r is termed $k=[1,r]$. Hence, an interval polynomial is, e.g., $s^3+[1,2]s^2+5s+3$ (*Barmish, B.R., 1994*). A set of a polynomial family is termed $\mathcal{P}=\{p(s,q) : q\in Q\}$.

Another notation example: For the stable open-loop system F_o and unity feedback, the Nyquist stability criterion can be written as $F_o(j\omega)\notin(-\infty,-1]\forall\omega\in\mathcal{R}$.

The jth column (row) of a matrix $\mathbf{M}$ is $M_{\cdot j}$ $(M_{i\cdot})$, corresponding to MATLAB `M(:, j)` and `M(i, :)`, respectively.

The meaning of $\overset{(61)}{=}$ is that for evolving the current execution, the result of Eq.(61) was utilized. For the sake of abbreviation $[\mathbf{M};\ \mathbf{N}] \triangleq \begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix}$ is used for columnizing. "col" is an operator concatenating all the column partitions of a matrix.

2 Algebraic and Combinatoric Matrix Correspondences

In general, the elements of all vectors $\mathbf{p}$ or matrices $\mathbf{M}$, $\mathbf{K}$ are considered as independent variables with no interelement dependencies. Considering a derivative of a function with respect to the vector $\mathbf{p}$ or the matrix $\mathbf{M}$, vector or matrix interrelations in $\mathbf{p}$ or $\mathbf{M}$, respectively, must not exist, in general.

Interrelation dependencies are only addressed in a specific Section 25.

For the sake of abbreviation,

$$(\mathbf{A}^T)^{-1} \equiv \mathbf{A}^{-T} \tag{1}$$

is used. The superscript ^{H} („Hermite“) means conjugate transpose

$$\mathbf{A}^H = (\mathbf{A}^*)^T \; . \tag{2}$$

$$\text{Kronecker delta} \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \; . \tag{3}$$

$$\text{Unit vector } \mathbf{e}_k : \mathbf{e}_k = (0, 0, \ldots 1, \ldots, 0, 0)^T = \mathbf{e}_k^{(m\times 1)} \quad (1 \text{ at position } \; k \; \text{ only}) \tag{4}$$

$$\text{Kronecker matrix } \mathbf{E}_{ik} : \quad \mathbf{E}_{ik}^{(n\times m)} = \mathbf{e}_i \mathbf{e}_k^T \in \mathcal{R}^{n\times m} \; . \tag{5}$$

Selection via unit vector and Kronecker matrix: The unit vector supports a selection of a row/column or element of $\mathbf{A}$

$$\mathbf{e}_i^T \mathbf{A} = A_{i\cdot}, \quad \mathbf{A}\mathbf{e}_k = A_{\cdot k} \; , \; \mathbf{e}_j^T \mathbf{A} \mathbf{e}_i = A_{ji} \; . \tag{6}$$

Kronecker matrix provides a useful selection. Premultiplication of a matrix $\mathbf{A}$ by $\mathbf{E}_{ij}$ yields the matrix $\mathbf{V}$

$$\mathbf{V} = \mathbf{E}_{ij}\mathbf{A} = \begin{cases} \text{null matrix } \mathbf{0} \\ \text{except the } i\text{th row } V_{i\cdot} = A_{j\cdot} \end{cases} , \tag{7}$$

i.e., a matrix of zeros where the jth row of $\mathbf{A}$ is assigned to the ith row.

Postmultiplication of a matrix $\mathbf{A}$ by $\mathbf{E}_{ij}$ yields the matrix $\mathbf{W}$

$$\mathbf{W} = \mathbf{A}\mathbf{E}_{ij} = \begin{cases} \text{null matrix } \mathbf{0} \\ \text{except the } j\text{th column } W_{\cdot j} = A_{\cdot i} \end{cases} \tag{8}$$

i.e., a matrix of zeros where the ith column of $\mathbf{A}$ is displaced to the jth column.

Changing the order in a product of symmetric matrices: For $\mathbf{V} = \mathbf{V}^T$ and $\mathbf{W} = \mathbf{W}^T$ the product $\mathbf{VW}$ is not symmetric $\mathbf{VW} \neq (\mathbf{VW})^T = \mathbf{WV}$.

For $\mathbf{A}$, $\mathbf{B}$ commutative, one has for the trace tr (sum of main-diagonal elements of a square matrix)

$$\mathrm{tr}[\mathbf{AB}] = \mathrm{tr}[\mathbf{BA}] \tag{9}$$

$$\sum_{k=1}^{n} \mathbf{E}_{kk}\mathbf{A}^{(n\times n)} = \mathbf{A} \tag{10}$$

$$\mathrm{tr}[\mathbf{E}_{ij}] = I_{ji} = \delta_{ij} \quad (= 1 \;\text{ for } i = j\text{; and } = 0 \text{ for } i \neq j) \tag{11}$$

$$\mathrm{tr}[\mathbf{M}\mathbf{E}_{ij}] = M_{ji} = (\mathbf{M}^T)_{ij} \; . \tag{12}$$

Note that $e^{\mathbf{A}(t+\tau)} = e^{\mathbf{A}t}e^{\mathbf{A}\tau}$ but $e^{\mathbf{A}+\mathbf{BK}} \neq e^{\mathbf{A}}e^{\mathbf{BK}}$ even for matrices small in norm sense.

The Kronecker product $\otimes$ of two matrices (or direct product, tensor product) is defined by a partitioned matrix whose (i,j)-partition is $A_{ij}\mathbf{B}$

$$\mathbf{A}\otimes\mathbf{B} \triangleq \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} & \dots & A_{1n}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & A_{m2}\mathbf{B} & \dots & A_{mn}\mathbf{B} \end{pmatrix} = \mathbf{matrix}[A_{ij}\mathbf{B}] \,, \qquad \begin{array}{l} \mathbf{A} \in \mathcal{C}^{n\times m} \\ \mathbf{B} \in \mathcal{C}^{r\times s} \\ \mathbf{A}\otimes\mathbf{B} \in \mathcal{C}^{nr\times ms} \;. \end{array} \tag{13}$$

The Kronecker sum $\oplus$ is defined by (*Brewer, J.W., 1978*)

$$\mathbf{N}\oplus\mathbf{M} = \mathbf{N}\otimes\mathbf{I}_m + \mathbf{I}_n\otimes\mathbf{M} \,; \qquad \mathbf{N}\in\mathcal{C}^{n\times n},\;\; \mathbf{M}\in\mathcal{C}^{m\times m},\;\; \mathbf{N}\oplus\mathbf{M}\in\mathcal{C}^{mn\times mn} \;. \tag{14}$$

The matrix $\mathbf{U}_{r,m}$ is the permutation matrix in Kronecker matrix sense; it is always square and consists of entries zero except one solitary digit one in *each* row and column. $\mathbf{E}_{ij}$ is the Kronecker matrix of Eq.(5) with entries zero except a *solitary* digit one at position i,j

$$\mathbf{U}_{k,l} \triangleq \mathbf{U}_{k,l}^{(kl\times kl)} \;\triangleq\; \sum_{i=1}^{k}\sum_{j=1}^{l} \mathbf{E}_{ij}^{(k\times l)} \otimes \mathbf{E}_{ji}^{(l\times k)} = \sum_{i}^{k}\sum_{j}^{l} \mathbf{E}_{ij}^{(k\times l)} \otimes \left(\mathbf{E}_{ij}^{(k\times l)}\right)^{T} . \tag{15}$$

E.g.,

$$\mathbf{E}_{1,3}^{(2\times 4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

$$\mathbf{U}_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{17}$$

Note the orthogonality property $\mathbf{U}_{k,l}^{-1} = \mathbf{U}_{k,l}^{T} = \mathbf{U}_{l,k}$,

$$\mathbf{U}_{k,l}^{T}\mathbf{U}_{k,l} = \mathbf{I}_{(kl)}^{(kl\times kl)} \tag{18}$$

and

$$(\mathbf{I}_r \otimes \mathbf{U}_{k,l})^{-1} = \mathbf{I}_r \otimes \mathbf{U}_{k,l}^{T} = \mathbf{I}_r \otimes \mathbf{U}_{l,k} \;. \tag{19}$$

In addition, one has

$$\frac{\partial \mathbf{M}^T}{\partial \mathbf{M}} = \mathbf{U}_{k,l} \qquad \mathbf{M} \in \mathcal{R}^{k\times l} \tag{20}$$

and

$$\text{col }\mathbf{M}^T = \mathbf{U}_{k,l}\ \text{col}\mathbf{M} \qquad \mathbf{M} \in \mathcal{R}^{k\times l}\ . \tag{21}$$

Self-derivative matrix (rectangular matrix $\partial\mathbf{M}/\partial\mathbf{M}$, only square if $k = l$,) is given by

$$\frac{\partial \mathbf{M}^{(k\times l)}}{\partial \mathbf{M}} = \bar{\mathbf{U}}_{k,l} = \bar{\mathbf{U}}_{k,l}^{(k^2\times l^2)} \triangleq \sum_{i=1}^{k}\sum_{j=1}^{l} \mathbf{E}_{ij}^{(k\times l)} \otimes \mathbf{E}_{ij}^{(k\times l)}\ , \tag{22}$$

e.g.,

$$\bar{\mathbf{U}}_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}\ . \tag{23}$$

Mixed product rule (only applicable if $\mathbf{A}, \mathbf{D}$ and $\mathbf{B}, \mathbf{G}$ are conformable)

$$(\mathbf{A}\otimes\mathbf{B})\ (\mathbf{D}\otimes\mathbf{G}) = (\mathbf{AD})\otimes(\mathbf{BG})\ . \tag{24}$$

The Frobenius norm of the Kronecker product is

$$\|\mathbf{I}_m \otimes \mathbf{M}\|_F = \sqrt{m}\|\mathbf{M}\|_F\ , \tag{25}$$

moreover

$$\|(\text{col }\mathbf{I}_m)\otimes\mathbf{I}_n\| = \sqrt{mn} \tag{26}$$

$$\|\mathbf{x}_0\mathbf{x}_0^T\|_F = \|\text{col}(\mathbf{x}_0\mathbf{x}_0^T)\|_F = \|\mathbf{x}_0\|_F^2 \tag{27}$$

$$\text{tr}[\mathbf{M}^{(n\times n)}] \equiv (\text{col }\mathbf{I}_n)^T\text{col }\mathbf{M} \tag{28}$$

$$\text{col}(\mathbf{HFG}) \equiv (\mathbf{G}^T\otimes\mathbf{H})\text{col}\mathbf{F} \tag{29}$$

$$\text{col}[\mathbf{AM}+\mathbf{MB}] = (\mathbf{B}^T\oplus\mathbf{A})\ \text{col}\,\mathbf{M}. \tag{30}$$

Here the abbreviation “col” (or “vec”) is the column operator, concatenating all the columns of a matrix to an entire column vector.

$$\text{col}(\mathbf{x}\mathbf{y}^T) = \mathbf{y}\otimes\mathbf{x} \tag{31}$$

$$\mathbf{G}^{(k\times l)}\otimes\mathbf{H}^{(n\times m)} \equiv \mathbf{U}_{k,n}\ (\mathbf{H}\otimes\mathbf{G})\ \mathbf{U}_{m,l} \tag{32}$$

$$\text{col }\mathbf{K}^T = \mathbf{U}_{m,r}\text{col }\mathbf{K}\ , \qquad \mathbf{K}\in\mathcal{R}^{m\times r} \tag{33}$$

$$\text{col }\mathbf{I}_n \equiv \sum_{i=1}^{i=n}\mathbf{e}_i\otimes\mathbf{e}_i \tag{34}$$

$$\mathrm{col}(\mathbf{c}\ \mathbf{c}^T) = \mathrm{col}(\mathbf{c} \cdot 1 \cdot \mathbf{c}^T) = \mathbf{c} \otimes \mathbf{c}, \quad \mathrm{loc}_n(\mathbf{x}_0 \otimes \mathbf{x}_0) = \mathbf{x}_0 \mathbf{x}_0^T\ . \tag{35}$$

An algorithm for analytical columnizing and decolumnizing runs as follows: Columnizing $\mathbf{K}_y \in \mathcal{R}^{m\times r}$

$$\mathrm{col}[\mathbf{K}_y] = \sum_{i=1}^{r} \mathbf{e}_i^{(r\times 1)} \otimes [\mathbf{K}_y \cdot (\mathbf{1}^{(1\times r)} \otimes \mathbf{e}_i^{(r\times 1)}) \cdot \mathbf{e}_i^{(r\times 1)}] \ \in \mathcal{R}^{mr} \tag{36}$$

where $\mathbf{1} \stackrel{\triangle}{=} (1\ 1\ 1\ \ldots\ 1) \in \mathcal{R}^{1\times r}$.

Alternatively,

$$\mathrm{col}\ [\mathbf{K}_y] = \left[\sum_{i=1}^{r} (\mathbf{e}_i^{[r]} \otimes \mathbf{I}_m) \mathbf{K}_y (\mathbf{e}_i^{[r]} \otimes \mathbf{1}\right] \mathbf{e}_1^{[r]}\ . \tag{37}$$

Due to the structure

$$\mathrm{col}\ [\mathbf{K}_y] = \mathbf{h}_1^T \mathbf{K}_y \mathbf{l}_1 + \mathbf{h}_2^T \mathbf{K}_y \mathbf{l}_2 + \ldots, \tag{38}$$

arithmetic calculations are feasible, "col" itself is not suitable for differentiation.

$$\frac{\partial \mathrm{col}\, \mathbf{K}_y}{\partial p} = \sum_{i=1}^{r} \mathbf{h}_i^T \frac{\partial \mathbf{K}_y}{\partial p} \mathbf{l}_i = \left[\sum_{i=1}^{r} (\mathbf{e}_i^{[r]} \otimes \mathbf{I}_m) \frac{\partial \mathbf{K}_y}{\partial p} (\mathbf{e}_i^{[r]} \otimes \mathbf{1}\right] \mathbf{e}_1^{[r]}\ . \tag{39}$$

Decolumnizing $\mathbf{h} \in \mathcal{R}^{mr\times 1}$

$$\mathrm{loc}_m[\mathbf{h}] = \sum_{i=1}^{r} (\mathbf{e}_i^{(r\times 1)})^T \otimes \{[(\mathbf{e}_i^{(r\times 1)})^T \otimes \mathbf{I}_m] \cdot \mathbf{h}\}\ . \tag{40}$$

Alternative decolumnizing

$$[\mathrm{loc}_r \mathbf{h}]^T = [\mathbf{I}_m \otimes (\mathbf{h}^{(mr\times 1)})^T][(\mathrm{col}\, \mathbf{I}_m) \otimes \mathbf{I}_r]\ . \tag{41}$$

$$\mathrm{loc}_m[\mathbf{h}^{(mr\times 1)}] = \sum_{i=1}^{r} [(\mathbf{e}_i^r)^T \otimes \mathbf{I}_m][\mathbf{E}_{ii}^{rr} \otimes \mathbf{I}_m] \cdot \mathbf{h} \cdot (\mathbf{e}_i^r)^T = \sum_{i=1}^{r} [(\mathbf{e}_i^r)^T \otimes \mathbf{I}_m] \cdot \mathbf{h} \cdot (\mathbf{e}_i^r)^T \in \mathcal{R}^{m\times r} \tag{42}$$

The dyadic product $\mathbf{a} * \mathbf{b}^T$ is characterized both by

$$\det(\mathbf{a} * \mathbf{b}^T) = 0 \quad \text{and} \quad \mathrm{adj}[\mathbf{a} * \mathbf{b}^T] = \mathbf{0}\ . \tag{43}$$

For the dyadic-like product

$$\det(\mathbf{A}\mathbf{B}^T) = 0,\ \det[\mathbf{C} \otimes (\mathbf{A}\mathbf{B}^T)] = 0, \quad \mathbf{A},\ \mathbf{B}\ \in \mathcal{R}^{n\times m}, \quad m < n,\ \mathbf{C} \text{ arbitrary square.} \tag{44}$$

Linear combination of p separate matrices $\mathbf{M}_i^{[m\times n]}$, $n \geq m$

$$\mathbf{L} \;=\; \sum_{i=1}^{p} \beta_i \mathbf{M}_i = (\boldsymbol{\beta} \otimes \mathbf{I}_m)^T \;\; \text{blcol}[\mathbf{M}_i] \quad \in \mathcal{R}^{m\times n} \;, \quad (45)$$

where $\quad \boldsymbol{\beta} \triangleq \text{col}\ \beta_i \in \mathcal{R}^p, \qquad \text{blcol}[\mathbf{M}_i] \triangleq [\mathbf{M}_1\ ;\ \mathbf{M}_2;\ \vdots\ ;\ \mathbf{M}_p] \quad \in \mathcal{R}^{mp\times n}$ (46)

Sum of rows/columns: Given a matrix $\mathbf{M} \in \mathcal{R}^{n\times m}$, defining $\mathbf{h} = \mathbf{e}_i + \mathbf{e}_k$ (a null row except unitary ones at position i and k), premultiplying yields a row

$$\mathbf{h}^T\mathbf{M} \;=\; M_{i.} + M_{k.} \qquad (47)$$

which is the sum of the ith and the kth row of $\mathbf{M}$. Postmultiplying $\mathbf{Mh}$ yields a sum of columns, correspondingly.

In order to delete a superfluous row μ and column μ in a matrix $\mathbf{M}$, utilize

$$\mathbf{I}_{n,wr\mu}\mathbf{M}\ \mathbf{I}_{n,wr\mu}^T \;, \qquad (48)$$

where $\mathbf{I}_{n,wr\mu}$ is the identity matrix $\mathbf{I}_n$ without row μ.

Hadamard product rules:

$$\mathbf{A} \cdot\!*\mathbf{B} = \mathbf{B} \cdot\!*\ \mathbf{A} = \text{matrix}_{ij}[A_{ij}B_{ij}], \qquad (\mathbf{A} \cdot\!*\mathbf{B})^T = \mathbf{A}^T \cdot\!*\ \mathbf{B}^T = \mathbf{B}^T \cdot\!*\ \mathbf{A}^T \quad (49)$$

differing from the regular matrix product

$$\mathbf{A}^{[n\times m]} = \mathbf{B}^{[n\times p]}\mathbf{C}^{[p\times m]} \qquad A_{ij} = \sum_{\nu=1}^{p} B_{i\nu}C_{\nu j}\ . \qquad (50)$$

$$\frac{\partial}{\partial B_{ij}}[\mathbf{A} \cdot\!*\ \mathbf{B}] = \mathbf{A} \cdot\!*\ \frac{\partial \mathbf{B}}{\partial B_{ij}} = \mathbf{A} \cdot\!*\ \mathbf{E}_{ij} = A_{ij}\mathbf{E}_{ij} \qquad (51)$$

$$\mathbf{A} \cdot\!*\ \mathbf{B} = \mathbf{C}, \qquad \sum_{ij} C_{ij} = (\text{col}\mathbf{A})^T\text{col}\mathbf{B}\ . \qquad (52)$$

For weighted matrices

$$\frac{\partial}{\partial \mathbf{M}}\|\mathbf{W}\ \cdot\!*\ \mathbf{M}\|_F^2 = 2\ \mathbf{W}\ \cdot\!*\ \mathbf{W}\ \cdot\!*\ \mathbf{M} \qquad (53)$$

$$\frac{\partial \det(\mathbf{W}\ \cdot\!*\ \mathbf{M}}{\partial \mathbf{M}} \overset{(117)}{=} \mathbf{W}\ \cdot\!*\ \text{adj}^T(\mathbf{W}\ \cdot\!*\ \mathbf{M})\ . \qquad (54)$$

2.1 Formulas Concerning the Inverse and Partitioned Matrices

Sherman-Morrison-Woodbury formula, or matrix inversion lemma (see Fig. 2.1):

$$(\mathbf{A}+\mathbf{BDC})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}^{-1}+\mathbf{CA}^{-1}\mathbf{B})^{-1}\mathbf{CA}^{-1} \tag{55}$$

$$(\mathbf{A}+\mathbf{UV}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I}+\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^T\mathbf{A}^{-1}. \tag{56}$$

For $\mathbf{A}\in\mathcal{R}^{n\times n},\ \mathbf{B}\in\mathcal{R}^{n\times r},\ \mathbf{C}\in\mathcal{R}^{r\times n},\ \mathbf{D}\in\mathcal{R}^{r\times r}$, note the economizing computational effect, especially for $r \ll n$: If $\mathbf{A}^{-1}$ is given, then due to Eq.(55) the inverse of $(\mathbf{A}+\mathbf{BDC})$ requires the inverse of an (r,r)-matrix, only.

$$(\mathbf{A}+\mathbf{D})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{D}^{-1}+\mathbf{A}^{-1})^{-1}\mathbf{A}^{-1} \tag{57}$$

or, for $\Delta\mathbf{D}$ small in norm sense,

$$(\mathbf{A}+\Delta\mathbf{D})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\cdot\Delta\mathbf{D}\cdot\mathbf{A}^{-1} \tag{58}$$

$$(\mathbf{U}+\mathbf{B}\cdot\Delta\mathbf{D}\cdot\mathbf{C})^{-1} \doteq \mathbf{U}^{-1} - \mathbf{U}^{-1}\mathbf{B}\cdot\Delta\mathbf{D}\cdot\mathbf{CU}^{-1}. \tag{59}$$

Avoiding the inverse of a complex matrix, where only the inverse of a real matrix needed

$$(\mathbf{A}\pm j\mathbf{B})^{-1} = (\mathbf{B}^{-1}\mathbf{A}\mp j\mathbf{I})(\mathbf{AB}^{-1}\mathbf{A}+\mathbf{B})^{-1}. \tag{60}$$

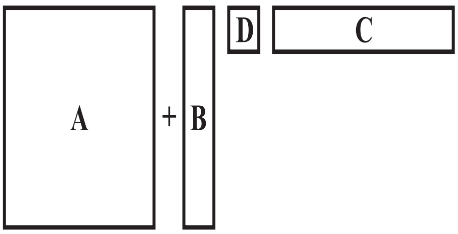

Figure 1: Matrix dimensions where Sherman-Morrison-Woodbury formula is advantageous from the numerical viewpoint

$$\begin{pmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \tag{61}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{BECA}^{-1} & -\mathbf{A}^{-1}\mathbf{BE} \\ -\mathbf{ECA}^{-1} & \mathbf{E} \end{pmatrix}, \tag{62}$$

where $\mathbf{E} = (\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B})^{-1}$;

$$(\mathbf{A}+\mathbf{ab}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\frac{\mathbf{ab}^T}{1+\mathbf{b}^T\mathbf{A}^{-1}\mathbf{a}}\mathbf{A}^{-1} \tag{63}$$

$$(\mathbf{I}-\mathbf{a}\mathbf{b}^T)^{-1} = \mathbf{I} - \frac{1}{\mathbf{a}^T\mathbf{b}-1}\mathbf{a}\mathbf{b}^T \ , \tag{64}$$

$$\det\begin{pmatrix}\mathbf{N} & \mathbf{P}\\ \mathbf{Q} & \mathbf{R}\end{pmatrix} = \det\mathbf{N}\times\det(-\mathbf{Q}\mathbf{N}^{-1}\mathbf{P}+\mathbf{R}) = \det\mathbf{R}\times\det(-\mathbf{P}\mathbf{R}^{-1}\mathbf{Q}+\mathbf{N}) \ . \tag{65}$$

3 Important Norms

The Frobenius or Euler norm is defined as

$$\|\mathbf{x}\|_F \ \triangleq \ \sqrt{|x_1|^2+|x_2|^2+\ldots|x_n|^2} = \sqrt{\mathbf{x}^H\mathbf{x}} \ . \tag{66}$$

Spectral (induced) norm or Hilbert Norm

$$\|\mathbf{G}\|_s \ \triangleq \ \sup_{\mathbf{x}} \frac{\|\mathbf{G}\mathbf{x}\|_F}{\|\mathbf{x}\|_F} \quad \text{where } \mathbf{x}\neq\mathbf{0} \ . \tag{67}$$

Singular value

$$\sigma[\mathbf{G}] \ \triangleq \ +\sqrt{\lambda[\mathbf{G}^H\mathbf{G}]} \tag{68}$$

$$\sigma_{\max} = +\sqrt{\lambda_{\max}[\mathbf{G}^H\mathbf{G}]} \equiv \|\mathbf{G}\|_s \ . \tag{69}$$

The maximum singular value is a generalization of the absolute value.

Referring to the spectral norm and the maximum singular value, one has $\|\mathbf{G}(s)\|_s = \sigma_{\max}[\mathbf{G}(s)] \leq \|\mathbf{G}(s)\|_F$, and the H_∞ norm

$$\|\mathbf{G}(s)\|_\infty \ \triangleq \ \sup_{\omega} \sigma_{\max}[\mathbf{G}(j\omega)] \ , \tag{70}$$

the H_∞ norm can be replaced or limited by the maximum of the Frobenius norm versus frequency $\|\mathbf{G}(s)\|_\infty < \sup_\omega \|\mathbf{G}(s=j\omega)\|_F$. The replacement often simplifies the derivation but only yields a sufficient result.

According to $\frac{1}{\sqrt{n}}\|\mathbf{G}\|_F \leq \sigma_{\max}[\mathbf{G}] \leq \|\mathbf{G}\|_F$ where $\mathbf{G}\in\mathcal{C}^{n\times n}$, the extent of overbounding is assessed as $\|\mathbf{G}\|_F \leq \sqrt{n}\ \sigma_{\max}[\mathbf{G}]$. For $n=3$ e.g., overbounding is less than 73%.

4 Derivative of a Scalar-Valued Function with Respect to a Scalar

Consider the scalar parameter p (*Vetter, W.J., et al., 1972*)

$$\frac{\partial \det\mathbf{A}}{\partial p} = \mathrm{tr}\ [\frac{\partial\mathbf{A}}{\partial p}\ \mathbf{adj}\ \mathbf{A}] = \mathrm{tr}[\frac{\partial\mathbf{A}}{\partial p}\mathbf{A}^{-1}]\cdot\det\mathbf{A} \ . \tag{71}$$

Derivation of $\det(p\mathbf{I}_n - \mathbf{A}),\ \mathbf{A} \in \mathcal{C}^{n\times n},\ p$ scalar, yields

$$\frac{\partial}{\partial p} \det\ (p\mathbf{I}_n - \mathbf{A}) = \sum_{i=1}^{n} \det\ (p\mathbf{I}_{n-1} - \mathbf{A}_{red\ ii}), \tag{72}$$

where $\mathbf{A}_{red\ ii}$ is a square matrix evolved from $\mathbf{A}$ when deleting the row i and the column i .

For the matrix product one has

$$\frac{\partial}{\partial p}\ \mathrm{tr}[\mathbf{A}^T(p)\mathbf{A}(p)] = \frac{\partial}{\partial p}\|\mathbf{A}(p)\|_F^2 = 2\ \mathrm{tr}[\mathbf{A}^T(p)\frac{\partial \mathbf{A}(p)}{\partial p}]\ . \tag{73}$$

Eventually, for an inverse expression

$$\frac{\partial}{\partial p}\ \mathrm{tr}\ [(p\mathbf{I} + \mathbf{C})^{-2}\mathbf{D}] = -2\ \mathrm{tr}\ [(p\mathbf{I} + \mathbf{C})^{-3}\mathbf{D}] \tag{74}$$

and for the second-order adjoint

$$\mathrm{tr}[\mathbf{A}] = \mathrm{tr}[\mathrm{adj}(\mathbf{A})] \quad \text{only for } \mathbf{A} \in \mathcal{R}^{2\times 2}\ . \tag{75}$$

Derivative of the trace of the inverse:

$$\mathrm{tr}[\mathbf{A}^{-1}] = \sum_{k=1}^{n} \mathbf{e}_k^T\mathbf{A}^{-1}\mathbf{e}_k \tag{76}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial f} \overset{(181)}{=} -\sum_{k=1}^{n} \mathbf{e}_k^T\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial f}\mathbf{A}^{-1}\mathbf{e}_k \tag{77}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial f} = -\sum_{k=1}^{n} \mathrm{tr}[\mathbf{e}_k^T\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial f}\mathbf{A}^{-1}\mathbf{e}_k] \tag{78}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial f} \overset{(9)}{=} -\sum_{k=1}^{n} \mathrm{tr}[\mathbf{A}^{-1}\mathbf{e}_k\mathbf{e}_k^T\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial f}] \tag{79}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial f} \overset{(5)}{=} -\sum_{k=1}^{n} \mathrm{tr}[\mathbf{A}^{-1}\mathbf{E}_{kk}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial f}] \tag{80}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial f} = -\mathrm{tr}[(\sum_{k=1}^{n} \mathbf{A}^{-1}\mathbf{E}_{kk}\mathbf{A}^{-1})\frac{\partial \mathbf{A}}{\partial f}] \tag{81}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial f} \overset{(10)}{=} -\mathrm{tr}[\mathbf{A}^{-2}\frac{\partial \mathbf{A}}{\partial f}]\ . \tag{82}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial A_{ij}} = -\mathrm{tr}[\mathbf{A}^{-2}\mathbf{E}_{ij}] \overset{(12)}{=} -(\mathbf{A}^{-2})_{ji} \tag{83}$$

The Taylor expansion (power-series expansion) of a scalar f for $\mathbf{x} \in \mathcal{R}^2$ and $\mathbf{x}$ close to $\mathbf{x}_o$ is

$$f(\mathbf{x}) = f(\mathbf{x}_o) \quad + \quad \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}\Big|_{x_1=x_{1o},\ x_2=x_{2o}}(x_i - x_{io})$$

$$+ \quad 0.5\sum_{i=1}^{2}\frac{\partial^2 f}{\partial x_i^2}\Big|_{x_1=x_{1o},\ x_2=x_{2o}}(x_i - x_{io})^2$$
$$+ \quad \frac{\partial^2 f}{\partial x_1 \partial x_2}\Big|_{x_1=x_{1o},\ x_2=x_{2o}}(x_1 - x_{1o})(x_2 - x_{2o}) \ + \dots \tag{84}$$

The Taylor expansion (close to $\mathbf{p}_o$) of a matrix $\mathbf{A} \in \mathcal{R}^{n\times m}$, which depends on a vector $\mathbf{p} \in \mathcal{R}^r$, is

$$\mathbf{A}(\mathbf{p}) \doteq \mathbf{A}(\mathbf{p}_o) + \sum_{i=1} \frac{1}{i!}[(\mathbf{p}-\mathbf{p}_o)^{T,[[i]]} \otimes \mathbf{I}_n]\Big[\frac{\partial^i}{\partial \mathbf{p}^i}\mathbf{A}(\mathbf{p})\Big]_{\mathbf{p}=\mathbf{p}_o} , \tag{85}$$

where the superscript $^{[[i]]}$ terms the ith Kronecker power

$$(\mathbf{p}-\mathbf{p}_o)^{[[i]]} = (\mathbf{p}-\mathbf{p}_o) \otimes (\mathbf{p}-\mathbf{p}_o) \otimes \dots \otimes (\mathbf{p}-\mathbf{p}_o) \in \mathcal{R}^{ri} \tag{86}$$

(*Vetter, W.J., 1970*).

5 Derivative of a Scalar-Valued Function with Respect to a Vector

The derivative of a function with respect to a vector or a matrix is a definition of a scheme, only. This abbreviation is suitable when cascades of calculations are taken into consideration.

The vector derivative operator (or gradient or nabla operator) with respect to the vector $\mathbf{p} = \mathbf{vector}[p_i]$ is termed

$$\frac{\partial}{\partial \mathbf{p}} \triangleq \begin{pmatrix} \frac{\partial}{\partial p_1} \\ \frac{\partial}{\partial p_2} \\ \vdots \\ \frac{\partial}{\partial p_n} \end{pmatrix} \triangleq \mathbf{grad_p} = \nabla_p \ . \tag{87}$$

Derivative of an inner vector product

$$\frac{\partial}{\partial \mathbf{p}} \ \mathbf{a}^T(\mathbf{p}) \ \mathbf{b}(\mathbf{p}) = \mathbf{vector}_i \ [\Big(\frac{\partial \mathbf{a}}{\partial p_i}\Big)^T \mathbf{b} + \Big(\frac{\partial \mathbf{b}}{\partial p_i}\Big)^T \mathbf{a}] = \Big(\frac{\partial \mathbf{a}^T}{\partial \mathbf{p}}\Big)\mathbf{b} + \Big(\frac{\partial \mathbf{b}^T}{\partial \mathbf{p}}\Big)\mathbf{a} \ . \tag{88}$$

For a square term in $\mathbf{p}$, we have

$$\frac{\partial}{\partial \mathbf{p}}\mathbf{p}^T\mathbf{A}\mathbf{p} = (\mathbf{A}+\mathbf{A}^T)\mathbf{p} \qquad \text{if } \ \mathbf{A} \ \text{ independent of } \mathbf{p} \ . \tag{89}$$

$$\text{If } \ \mathbf{Q} = \mathbf{Q}^T \text{independent of } \mathbf{p} \ , \quad \frac{\partial}{\partial \mathbf{p}} \ \mathbf{a}^T(\mathbf{p}) \ \mathbf{Q} \ \mathbf{a}(\mathbf{p}) = \frac{\partial}{\partial \mathbf{p}} \mathrm{tr} \ \mathbf{Q}\mathbf{a}\mathbf{a}^T = 2\frac{\partial \mathbf{a}^T}{\partial \mathbf{p}}\mathbf{Q}\mathbf{a} \ . \tag{90}$$

From above, if $\mathbf{a} = \mathbf{R}\mathbf{p}$, $\mathbf{R}$ constant

$$\frac{\partial}{\partial \mathbf{p}}\mathbf{p}^T\mathbf{R}^T\mathbf{Q}\mathbf{R}\mathbf{p} = 2\mathbf{R}^T\mathbf{Q}\mathbf{R}\mathbf{p} \ . \tag{91}$$

5.1 Minimizing a Square Expression

$$\begin{aligned}\text{Resolve} \quad I_r(\mathbf{x}) &= 0.5\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + a \quad \to \quad \min_{\mathbf{x}} && (92)\\ \text{gradient} \quad \frac{\partial I_r}{\partial \mathbf{x}} &= \mathbf{Q}\mathbf{x} + \mathbf{b} && (93)\\ \text{Hesse matrix} \quad \frac{\partial^2 I_r}{\partial \mathbf{x}^2} \triangleq \frac{\partial^2 I_r}{\partial \mathbf{x}\partial \mathbf{x}^T} &= \mathbf{Q} > 0 \;, && (94)\end{aligned}$$

i.e., a positive definite Hesse matrix is required for the minimum, see also Eq.(238).

Define $\mathbf{Q}_i \in \mathcal{R}^{i\times i}, \quad i = \{1,\ldots,n\}$ all the successive northwest subdeterminants (main principle minors) of the matrix $\mathbf{Q}$. Then, irrespective if n is odd or even, the Sylvester conditions for positive (negative) definiteness of $\mathbf{Q}$ are

$$\mathbf{Q} > 0 \;:\; \det \mathbf{Q}_i > 0; \qquad \mathbf{Q} < 0 \;:\; (-1)^i \det \mathbf{Q}_i > 0 \;\; \forall i \;. \tag{95}$$

5.2 Minimizing a Square Expression Subject to a Condition

Find the $\min_{\mathbf{a}} \mathbf{a}^T\mathbf{Q}\mathbf{a}$ s.t. the equality condition $\mathbf{V}\mathbf{a} - \mathbf{b} = \mathbf{0}$, where $\mathbf{V} \in \mathcal{R}^{m\times n}, \;\; n > m$ and $\mathbf{Q}^T = \mathbf{Q} > 0$. Using the vector Lagrange multipier $\boldsymbol{\lambda}$, the optimum results are $\boldsymbol{\lambda}^\star = -2(\mathbf{V}\mathbf{Q}^{-1}\mathbf{V}^T)^{-1}\mathbf{b}, \quad \mathbf{a}^\star = \mathbf{Q}^{-1}\mathbf{V}^T(\mathbf{V}\mathbf{Q}^{-1}\mathbf{V}^T)^{-1}\mathbf{b}$ and the minimum of the square expression $\mathbf{b}^T(\mathbf{V}\mathbf{Q}^{-1}\mathbf{V}^T)^{-1}\mathbf{b}$.

5.3 Optimal Reference Model

Find the controller $\mathbf{K}$ to approximate the model $\mathbf{A}_{cl,ref} \in \mathcal{R}^{n\times n}$ s.t. $k_{WK} = \|\mathbf{K}\mathbf{W}_K\|_F$ (*Weinmann, A., 2006d*). Including the weighting matrices $\mathbf{W}_A$ and $\mathbf{W}_K$,

$$\|\mathbf{W}_A(\mathbf{A}_{cl,ref} - \mathbf{A} - \mathbf{B}\mathbf{K})\|_F^2 + \lambda\|\mathbf{K}\mathbf{W}_K\|_F^2 \to \min_{\mathbf{K}} \qquad \mathbf{B},\; \mathbf{W}_K \in \mathcal{R}^{n\times m} \;. \tag{96}$$

$$\begin{aligned}\text{The result is} \quad \mathbf{K}^\star = \mathbf{L}^{-1}\mathbf{F} \;, \quad \text{where } \mathbf{L} &\triangleq \lambda\mathbf{W}_K\mathbf{W}_K^T + \mathbf{B}^T\mathbf{W}_A^T\mathbf{W}_A\mathbf{B} && (97)\\ \mathbf{F} &\triangleq \mathbf{B}^T\mathbf{W}_A^T\mathbf{W}_A(\mathbf{A}_{cl,ref} - \mathbf{A}) && (98)\\ k_{WK}^2 &= \mathrm{tr}\{\mathbf{W}_K^T\mathbf{F}\mathbf{L}^{-T}\mathbf{L}^{-1}\mathbf{F}\mathbf{W}_K\} \rightsquigarrow \lambda \;. && (99)\end{aligned}$$

6 Derivative of a Scalar-Valued Function with Respect to a Matrix

$$\frac{\partial c(\mathbf{M})}{\partial \mathbf{M}^T} = \Big(\frac{\partial c(\mathbf{M})}{\partial \mathbf{M}}\Big)^T \tag{100}$$

6.1 Eigenvalue Differential Quotients

The eigenvalue derivative with respect to its generating matrix $\mathbf{A}$ (sensitivity of the eigenvalues) is

$$\frac{\partial \lambda_i[\mathbf{A}]}{\partial \mathbf{A}} = \frac{\partial(\mathbf{a}_i^{\triangleleft H}\mathbf{A}\mathbf{a}_i)}{\partial \mathbf{A}} = \frac{\partial \text{ tr } [\mathbf{A}\mathbf{a}_i\mathbf{a}_i^{\triangleleft H}]}{\partial \mathbf{A}} = (\mathbf{a}_i\mathbf{a}_i^{\triangleleft H})^T = \mathbf{a}_i^{\triangleleft *}\mathbf{a}_i^T \ , \tag{101}$$

where $\mathbf{a}_i$ is the right eigenvector of $\mathbf{A}$ associated with $\lambda_i[\mathbf{A}]$ and $\mathbf{a}_i^{\triangleleft}$ is the left eigenvector. Alternatively, one can derive

$$\frac{\partial \lambda_i[\mathbf{A}]}{\partial \mathbf{A}} = \frac{\partial(\mathbf{a}_i^{\triangleleft H}\mathbf{A}\mathbf{a}_i)}{\partial \mathbf{A}} = \frac{\partial(\mathbf{a}_i^T\mathbf{A}^T\mathbf{a}_i^{\triangleleft *})}{\partial \mathbf{A}} \tag{102}$$

$$= \frac{\partial \text{ tr } [\mathbf{a}_i^T\mathbf{A}^T\mathbf{a}_i^{\triangleleft *}]}{\partial \mathbf{A}} = \frac{\partial \text{ tr } [\mathbf{A}^T\mathbf{a}_i^{\triangleleft *}\mathbf{a}_i^T]}{\partial \mathbf{A}} = \mathbf{a}_i^{\triangleleft *}\mathbf{a}_i^T \tag{103}$$

$$\text{or} \quad \frac{\partial \lambda_i[\mathbf{A}]}{\partial \mathbf{A}} = \frac{\text{adj}^T(\lambda_i\mathbf{I}_n - \mathbf{A})}{\text{tr adj } (\lambda_i\mathbf{I}_n - \mathbf{A})}, \quad \text{where} \quad \det\,(\lambda_i[\mathbf{A}]\mathbf{I}_n - \mathbf{A}) = 0 \ . \tag{104}$$

The sensitivity of the closed-loop eigenvalues $\lambda_i[\mathbf{A}]$ with respect to any matrix $\mathbf{K}$ is

$$\frac{\partial \lambda_i[\mathbf{A}]}{\partial \mathbf{K}} = (\mathbf{I}_m \otimes \mathbf{a}_i^{\triangleleft * T})\frac{\partial \mathbf{A}}{\partial \mathbf{K}}(\mathbf{I}_n \otimes \mathbf{a}_i) \ . \tag{105}$$

For an eigenvalue at the origin $\lambda_i = 0$

$$\frac{\text{adj}^T\mathbf{M}}{\text{tr adj}\mathbf{M}} = \frac{\mathbf{M}^{-T}\det\mathbf{M}}{\text{tr}[\mathbf{M}^{-1}]\det\mathbf{M}} = \frac{\mathbf{M}^{-T}}{\text{tr}[\mathbf{M}^{-1}]} \ . \tag{106}$$

6.2 Derivative of the Singular Values

Referring to Eq.(126),

$$\frac{\partial \sigma_i}{\partial \mathbf{M}} = \frac{\partial \mathbf{u}_i^H\mathbf{M}\mathbf{v}_i}{\partial \mathbf{M}} = \mathbf{u}_i^*\mathbf{v}_i^T \tag{107}$$

$$\frac{\partial \sigma_i}{\partial(\Re e\ \mathbf{M})} = \Re e\ \mathbf{u}_i^*\mathbf{v}_i^T = \Re e\ \mathbf{u}_i\mathbf{v}_i^H \qquad \frac{\partial \sigma_i}{\partial(\Im m\ \mathbf{M})} = -\Im m\ \mathbf{u}_i^*\mathbf{v}_i^T = \Im m\ \mathbf{u}_i\mathbf{v}_i^H \tag{108}$$

$$\frac{\partial(\sigma_i^2)}{\partial(\Re e\ \mathbf{M})} = -(\Re e\ \mathbf{M})(\mathbf{\Lambda}_1^T + \mathbf{\Lambda}_1) - (\Im m\ \mathbf{M})(\mathbf{\Lambda}_2^T - \mathbf{\Lambda}_2) \tag{109}$$

$$\frac{\partial(\sigma_i^2)}{\partial(\Im m\ \mathbf{M})} = -(\Im m\ \mathbf{M})(\mathbf{\Lambda}_1^T + \mathbf{\Lambda}_1) + (\Re e\ \mathbf{M})(\mathbf{\Lambda}_2^T - \mathbf{\Lambda}_2) \ , \tag{110}$$

$$\text{where} \quad \mathbf{\Lambda}_1 = -\frac{\partial \lambda[\mathbf{M}^H\mathbf{M}]}{\partial(\Re e\ \mathbf{M}^H\mathbf{M})} = -\Re e\ \mathbf{v}_i^*\mathbf{v}_i^T, \qquad \mathbf{\Lambda}_2 = -\frac{\partial \lambda[\mathbf{M}^H\mathbf{M}]}{\partial(\Im m\ \mathbf{M}^H\mathbf{M})} = \Im m\ \mathbf{v}_i^*\mathbf{v}_i^T \tag{111}$$

(*Sevaston, G.E., and Longman, R.W., 1988; Freudenberg, J.S., et al. 1982*).

6.3 Derivative of the Spectral Condition Number

From

$$\kappa_s[\mathbf{M}] = \frac{\sigma_{\max}[\mathbf{M}]}{\sigma_{\min}[\mathbf{M}]} \qquad \text{derive} \qquad \Delta\kappa_s[\mathbf{M}] = \frac{\sigma_{\min}\Delta\sigma_{\max} - \sigma_{\max}\Delta\sigma_{\min}}{\sigma_{\min}^2} \; . \tag{112}$$

For $\partial\sigma_{\max}/\partial\mathbf{M} \triangleq \mathbf{u}_a\mathbf{v}_a^T$ and $\partial\sigma_{\min}/\partial\mathbf{M} \triangleq \mathbf{u}_i\mathbf{v}_i^T$, where $\mathbf{v}_i, \mathbf{v}_a, \mathbf{u}_i, \mathbf{u}_a$ are the real eigenvectors of $\mathbf{M}^H\mathbf{M}$ and $\mathbf{M}\mathbf{M}^H$ associated with $\sigma_{\min}$ (subscript i) and $\sigma_{\max}$ (subscript a), respectively, one has

$$\Delta\sigma_{\max} = \text{tr}\;[\Delta\sigma_{\max}] = \text{tr}\;[\mathbf{v}_a\mathbf{u}_a^T\Delta\mathbf{M}] \tag{113}$$

and

$$\Delta\sigma_{\min} = \text{tr}\;[\mathbf{v}_i\mathbf{u}_i^T\Delta\mathbf{M}] \tag{114}$$

$$\Delta\kappa_s[\mathbf{M}] = \frac{\text{tr}\;[(\sigma_{\min}[\mathbf{M}]\mathbf{v}_a\mathbf{u}_a^T - \sigma_{\max}[\mathbf{M}]\mathbf{v}_i\mathbf{u}_i^T)\Delta\mathbf{M}]}{\sigma_{\min}^2[\mathbf{M}]} \tag{115}$$

or

$$\frac{\partial\kappa_s[\mathbf{M}]}{\partial\mathbf{M}} = \frac{\sigma_{\min}[\mathbf{M}]\mathbf{u}_a\mathbf{v}_a^T - \sigma_{\max}[\mathbf{M}]\mathbf{u}_i\mathbf{v}_i^T}{\sigma_{\min}^2[\mathbf{M}]} \; . \tag{116}$$

6.4 Differential Quotients of Determinant Expressions

Some determinant derivative properties are (*Athans, M., 1968; Vetter, W.J., 1971*)

$$\frac{\partial}{\partial\mathbf{M}}\;\det\mathbf{M} = (\mathbf{M}^{-1})^T\;\det\;\mathbf{M} = \mathbf{adj}^T(\mathbf{M}), \tag{117}$$

$$\frac{\partial}{\partial\mathbf{M}}\;\det\;(\mathbf{M}^n) = n(\mathbf{M}^{-1})^T(\det\;\mathbf{M})^n, \tag{118}$$

$$\frac{\partial}{\partial\mathbf{M}}\det\;e^{\mathbf{M}} = \det\;e^{\mathbf{M}}, \tag{119}$$

$$\frac{\partial}{\partial\mathbf{M}}\;\log\;\det\;\mathbf{M} = (\mathbf{M}^{-1})^T \; . \tag{120}$$

If $\mathbf{V}$ is independent of $\mathbf{M}$, $\qquad \dfrac{\partial}{\partial\mathbf{M}}\det\;(\mathbf{M}\mathbf{V}\mathbf{M}^T) = 2(\mathbf{M}\mathbf{V}\mathbf{M}^T)^{-1}\mathbf{M}\mathbf{V}$. (121)

If $\mathbf{A}$ and $\mathbf{B}$ are square and independent of $\mathbf{M}$, and $\mathbf{M}$ is square, then

$$\frac{\partial}{\partial\mathbf{M}}\det\;(\mathbf{A}\mathbf{M}\mathbf{B}) = (\mathbf{M}^{-1})^T\det\;(\mathbf{A}\mathbf{M}\mathbf{B})\;, \tag{122}$$

$$\frac{\partial\det[\mathbf{A}(\mathbf{H}+\mathbf{M})\mathbf{B}]}{\partial\mathbf{M}} = (\mathbf{H}+\mathbf{M})^{-T}\det[\mathbf{A}(\mathbf{H}+\mathbf{M})\mathbf{B}]\;. \tag{123}$$

Mostly used for controller coefficient matrix $\mathbf{M} = \mathbf{K}$, if $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{C}$ are independent of $\mathbf{M} = \mathbf{K}$, one finds

$$\frac{\partial}{\partial \mathbf{K}} \det(s\mathbf{I}_n - \mathbf{A} - \mathbf{BKC}) = -\mathbf{B}^T[\mathbf{adj}(s\mathbf{I}_n - \mathbf{A} - \mathbf{BKC})]^T\mathbf{C}^T \tag{124}$$

$$\frac{\partial \det(\mathbf{BK}_y\mathbf{C})}{\partial \mathbf{K}_y} = [\mathbf{C} \cdot \mathrm{adj}(\mathbf{BK}_y\mathbf{C}) \cdot \mathbf{B}]^T \ . \tag{125}$$

6.5 Trace and Frobenius Norm Gradients

Some trace derivative properties are (*Athans, M., 1968*; *Vetter, W.J., 1971*; *Brewer, J.W., 1978; Weinmann, A., 1991*)

$$\frac{\partial}{\partial \mathbf{M}}\mathbf{a}^T\mathbf{M}\mathbf{b} = \mathbf{b}\mathbf{a}^T \qquad \mathbf{M} \in \mathcal{R}^{m\times m}, \quad \mathbf{a}, \mathbf{b} \in \mathcal{R}^m \ . \tag{126}$$

If $\mathbf{B}$ and $\mathbf{C}$ independent of $\mathbf{M}$
$$\frac{\partial}{\partial \mathbf{M}}\mathrm{tr}[\mathbf{BM}^T\mathbf{C}] = \mathbf{CB}. \tag{127}$$

If $\mathbf{A}, \mathbf{B}$ are independent of $\mathbf{M}$,
$$\frac{\partial}{\partial \mathbf{M}}\mathrm{tr}\ [\mathbf{AMB}] = \mathbf{A}^T\mathbf{B}^T. \tag{128}$$

If $\mathbf{a}, \mathbf{B}, \mathbf{b}$ are independent of $\mathbf{M}$
$$\frac{\partial}{\partial \mathbf{M}}(\mathbf{a}^T\mathbf{MBM}^T\mathbf{b}) = \mathbf{ab}^T\mathbf{MB}^T + \mathbf{ba}^T\mathbf{MB}\ . \tag{129}$$

If $\mathbf{a}$ is independent of $\mathbf{M}$
$$\frac{\partial}{\partial \mathbf{M}}\mathbf{a}^T\mathbf{M}^{-1}\mathbf{a} = -(\mathbf{M}^{-1})^T\mathbf{aa}^T(\mathbf{M}^{-1})^T. \tag{130}$$

$$n \geq 2: \quad \frac{\partial}{\partial \mathbf{M}}\mathrm{tr}\ [\mathbf{M}^n] = n(\mathbf{M}^{n-1})^T, \tag{131}$$

$$\frac{\partial \mathrm{tr}[\mathbf{M}]}{\partial \mathbf{M}} = \mathbf{I}\ , \tag{132}$$

$$\frac{\partial ||\mathbf{M}||_F^2}{\partial M_{ij}} = \mathrm{tr}[\mathbf{E}_{ij}^T\mathbf{M} + \mathbf{M}^T\mathbf{E}_{ij}] \overset{(12)}{=} 2\,M_{ij} \tag{133}$$

$$\frac{\partial ||\mathbf{M}||_F^2}{\partial \mathbf{M}} = \frac{\partial}{\partial \mathbf{M}}\mathrm{tr}[\mathbf{MM}^T] = 2\,\mathbf{M}, \tag{134}$$

$$\frac{\partial}{\partial \mathbf{M}}||\mathbf{M}||_F = \frac{\partial}{\partial \mathbf{M}}\sqrt{\mathrm{tr}[\mathbf{MM}^T]} = \frac{\mathbf{M}}{||\mathbf{M}||_F}\ , \tag{135}$$

$$\frac{\partial}{\partial \mathbf{M}}\mathrm{tr}\ [e^{\mathbf{M}}] = e^{(\mathbf{M}^T)} = (e^{\mathbf{M}})^T\ . \tag{136}$$

Derivative of the $(2p)$-th power of the norm:

$$\frac{\partial}{\partial \mathbf{K}}||\mathbf{K}||_F^{2p} = \frac{\partial}{\partial \mathbf{K}}(||\mathbf{K}||_F^2)^p \tag{137}$$

$$= p(||\mathbf{K}||_F^2)^{p-1} \cdot \frac{\partial}{\partial \mathbf{K}}||\mathbf{K}||_F^2 = 2p||\mathbf{K}||_F^{2(p-1)}\ \mathbf{K}\ . \tag{138}$$

If $\mathbf{A}$ independent of $\mathbf{M}$ $$\frac{\partial}{\partial \mathbf{M}} \mathrm{tr}\ [\mathbf{A}\mathbf{M}^m] = [\sum_{i=0}^{m-1} \mathbf{M}^i \mathbf{A} \mathbf{M}^{m-i-1}\ ]^T, \tag{139}$$

if $\mathbf{A}$ and $\mathbf{B}$ independent of $\mathbf{A}$ $$\frac{\partial}{\partial \mathbf{M}} \mathrm{tr}\ [\mathbf{AMBM}] = \mathbf{A}^T\mathbf{M}^T\mathbf{B}^T + \mathbf{B}^T\mathbf{M}^T\mathbf{A}^T, \tag{140}$$

$$\frac{\partial}{\partial \mathbf{M}} \mathrm{tr}\ [\mathbf{AMBM}^T] = \frac{\partial}{\partial \mathbf{M}} \mathrm{tr}\ [\mathbf{BM}^T\mathbf{AM}] = \mathbf{A}^T\mathbf{MB}^T + \mathbf{AMB}, \tag{141}$$

$$\frac{\partial}{\partial \mathbf{M}} \mathrm{tr}\ [\mathbf{AM}^{-1}\mathbf{B}] = -(\mathbf{M}^{-1}\mathbf{BAM}^{-1})^T\ . \tag{142}$$

For some $\Delta\omega_0 \triangleq \mathrm{tr}[\mathbf{B}\ \Delta\mathbf{A}]$

$$\frac{\Delta\omega_0}{\Delta\mathbf{A}} \doteq \frac{\partial\omega_0}{\partial\Delta\mathbf{A}} = \mathbf{B}^T \quad \rightsquigarrow \quad \Delta\omega_0 = \mathrm{tr}[(\frac{\partial\omega_0}{\partial\mathbf{A}})^T\Delta\mathbf{A}]\ . \tag{143}$$

For holistic controllers one finds for the trace of the inverse of the closed-loop coefficient matrix (*Weinmann, A., 2001a*)

$$\frac{\partial}{\partial \mathbf{K}} \mathrm{tr}\ [(\mathbf{A}+\mathbf{BK})^{-1}] = -\mathbf{B}^T(\mathbf{A}+\mathbf{BK})^{-2T}\ . \tag{144}$$

Furthermore,

$$\frac{\partial}{\partial \mathbf{M}}||\mathbf{A}+\mathbf{BMC}||_F^2 = \frac{\partial}{\partial \mathbf{M}}\mathrm{tr}[(\mathbf{A}+\mathbf{BMC})^T(\mathbf{A}+\mathbf{BMC})] = 2\ \mathbf{B}^T(\mathbf{A}+\mathbf{BMC})\mathbf{C}^T. \tag{145}$$

Example 1:

$$||\mathbf{KB}-\mathbf{BA}||_F \to \min_{\mathbf{K}} \quad \rightsquigarrow \quad \mathbf{K} = \mathbf{BAB}^T(\mathbf{BB}^T)^{-1}\ .\ \square \tag{146}$$

$$\frac{\partial}{\partial \mathbf{M}}\mathrm{tr}[(\mathbf{A}+\mathbf{BMC})^T(\mathbf{A}+\mathbf{BMC})\mathbf{U}] = \mathbf{B}^T(\mathbf{A}+\mathbf{BMC})(\mathbf{U}+\mathbf{U}^T)\mathbf{C}^T\ . \tag{147}$$

$$\frac{\partial}{\partial \mathbf{A}}\ \mathrm{tr}[\mathbf{A}^T\mathbf{B}(\mathbf{A})] = \mathbf{B}(\mathbf{A}) + \mathbf{matrix}_{jk}\{\mathrm{tr}[\mathbf{A}^T\frac{\partial \mathbf{B}(\mathbf{A})}{\partial A_{jk}}]\}\ . \tag{148}$$

Two equivalent derivations of the trace with respect to a matrix. First

$$\frac{\partial I}{\partial K_{ij}} = \mathbf{x}_o^T\mathbf{M}^T\mathbf{E}_{ij}\mathbf{x}_o = \mathrm{tr}[\mathbf{x}_o\mathbf{x}_o^T\mathbf{M}^T\mathbf{E}_{ij}] = (\mathbf{M}\mathbf{x}_o\mathbf{x}_o^T)_{ij} \ \rightsquigarrow\ \frac{\partial I}{\partial \mathbf{K}} = \mathbf{M}\mathbf{x}_o\mathbf{x}_o^T. \tag{149}$$

Second, using $\mathbf{E}_{ij} = \mathbf{e}_i\mathbf{e}_j^T$,

$$\frac{\partial I}{\partial K_{ij}} = \mathbf{x}_o^T\mathbf{M}^T\mathbf{e}_i\mathbf{e}_j^T\mathbf{x}_o = \mathbf{e}_j^T\mathbf{x}_o\mathbf{x}_o^T\mathbf{M}^T\mathbf{e}_i = \mathbf{e}_i^T\mathbf{M}\mathbf{x}_o\mathbf{x}_o^T\mathbf{e}_j \tag{150}$$

yields the same result.

Based on Eq.(83),

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial A_{ij}} = -\mathrm{tr}[\mathbf{A}^{-2}\mathbf{E}_{ij}] \stackrel{(12)}{=} -(\mathbf{A}^{-2})_{ji} \tag{151}$$

$$\frac{\partial \mathrm{tr}[\mathbf{A}^{-1}]}{\partial \mathbf{A}} = -(\mathbf{A}^{-2})^T \tag{152}$$

$$\Re e \ \frac{\partial \mathrm{tr}[(\mathbf{C}+\Delta\mathbf{A})^{-1}]}{\partial \Delta\mathbf{A}} = -\Re e \ (\mathbf{C}^{-2})^T, \quad \mathbf{C} \in \mathcal{C}^{n\times n} \ . \tag{153}$$

$$\text{For} \quad \mathbf{A}_{cl} \stackrel{\triangle}{=} \mathbf{A} + \mathbf{B}\mathbf{K}_y\mathbf{C} \tag{154}$$

$$\frac{\partial}{\partial \mathbf{K}_y}\mathrm{tr}[\mathbf{A}_{cl}^{-1}] = -[\mathbf{C}(\mathbf{A}_{cl})^{-2}\mathbf{B}]^T \tag{155}$$

$$\frac{\partial}{\partial \mathbf{K}_y}\frac{1/\det \mathbf{A}_{cl}}{\mathrm{tr}[\mathbf{A}_{cl}^{-1}]} = -\frac{[\mathbf{C}\{\mathbf{A}_{cl}^{-2} + \mathbf{A}_{cl}^{-1}\mathrm{tr}[\mathbf{A}_{cl}^{-1}]\}\mathbf{B}]^T}{(\mathrm{tr}[\mathbf{A}_{cl}^{-1}])^2 \det \mathbf{A}_{cl}} \tag{156}$$

$$\frac{\partial}{\partial \mathbf{K}_y}\frac{1/\det \mathbf{A}_{cl}}{\mathrm{tr}[\mathbf{A}_{cl}]} = -\frac{[\mathbf{C}\{\mathbf{I}_n + \mathbf{A}_{cl}^{-1}\mathrm{tr}[\mathbf{A}_{cl}]\}\mathbf{B}]^T}{(\mathrm{tr}[\mathbf{A}_{cl}])^2 \det \mathbf{A}_{cl}} \tag{157}$$

$$\frac{\partial}{\partial \mathbf{K}_y}\frac{1}{\mathrm{tr}[\mathbf{A}_{cl}]} = -\frac{[\mathbf{C}\mathbf{B}]^T}{(\mathrm{tr}[\mathbf{A}_{cl}])^2} \tag{158}$$

(*Weinmann, A., 2005*). Derivative of the power of the trace:

$$\begin{aligned}\frac{\partial[-\{\mathrm{tr}[\mathbf{A}_{cl}^{-1}]\}^p]}{\partial \mathbf{K}_y} &\stackrel{(229)}{=} -p\Big(\mathrm{tr}[\mathbf{A}_{cl}^{-1}]\Big)^{p-1}\frac{\partial \mathrm{tr}[\mathbf{A}_{cl}^{-1}]}{\partial \mathbf{K}_y} \\ &\stackrel{(155)}{=} p\Big(\mathrm{tr}[\mathbf{A}_{cl}^{-1}]\Big)^{p-1}[\mathbf{C}\mathbf{A}_{cl}^{-2}\mathbf{B}]^T \ .\end{aligned} \tag{159}$$

$$\mathbf{M}, \ \Delta\mathbf{M} \in \mathcal{R}^{n\times n} \qquad \frac{\partial \mathrm{tr}[\mathbf{I}_n \otimes \mathbf{M}]}{\partial M_{ij}} = n\delta_{ij} \ , \quad \frac{\partial \mathrm{tr}[\mathbf{I}_n \otimes \mathbf{M}]}{\partial \mathbf{M}} = n\mathbf{I}_n \tag{160}$$

$$\mathrm{tr}[\mathbf{M}^T \otimes \Delta\mathbf{M}] = \mathrm{tr}[\Delta\mathbf{M} \otimes \mathbf{M}^T] = (\mathrm{tr}\mathbf{M})(\mathrm{tr}\Delta\mathbf{M}) \tag{161}$$

$$\frac{\partial \mathrm{tr}[\mathbf{M}^T \otimes \Delta\mathbf{M}]}{\partial \Delta\mathbf{M}} = (\mathrm{tr}\mathbf{M}) \ \mathbf{I}_n \tag{162}$$

$$\mathrm{tr}[\mathbf{I}_n \otimes (\mathbf{M}\cdot\Delta\mathbf{M}^T)] = \mathrm{tr}[\mathbf{I}_n \otimes (\Delta\mathbf{M}\cdot\mathbf{M}^T)] = n\cdot\mathrm{tr}[\mathbf{M}\cdot\Delta\mathbf{M}^T] \tag{163}$$

$$\frac{\partial \mathrm{tr}[\mathbf{I}_n \otimes (\mathbf{M}\cdot\Delta\mathbf{M}^T)]}{\partial \Delta\mathbf{M}} = n\mathbf{M} \ . \tag{164}$$

$$\text{For} \quad \check{\mathbf{M}} \stackrel{\triangle}{=} \mathbf{I}_n \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}_n = \mathbf{M} \oplus \mathbf{M}, \tag{165}$$

$$\frac{\partial ||\check{\mathbf{M}}||_F^2}{\partial \mathbf{M}} = \frac{\partial}{\partial \mathbf{M}}||\mathbf{I}_n \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}_n||_F^2 = 4(n\mathbf{M} + \mathbf{I}_n\mathrm{tr}\mathbf{M}) \tag{166}$$

$$\frac{\partial ||\check{\mathbf{M}}||_F^2}{\partial M_{ij}} = \frac{\partial}{\partial M_{ij}}||\mathbf{I}_n \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}_n||_F^2 = 4(nM_{ij} + \delta_{ij}\mathrm{tr}\mathbf{M}) \ . \tag{167}$$

7 Derivatives of a Column with Respect to a Row or Column

For $\mathbf{A}$ and $\mathbf{C}$ independent of p,

$$\frac{\partial \mathrm{col}\,[\mathbf{ABC}]}{\partial p} = \mathrm{col}(\mathbf{A}\frac{\partial \mathbf{B}}{\partial p}\mathbf{C}). \tag{168}$$

Derivative of a set of functions in the vector $\mathbf{r}(\mathbf{x}) \in \mathcal{R}^m$ (where $\mathbf{x} \in \mathcal{R}^n$) with respect to the transposed vector $\mathbf{x}^T \in \mathcal{R}^{1\times n}$, i.e., the Jacobi matrix $\frac{\partial \mathbf{r}(\mathbf{x})}{\partial \mathbf{x}^T}$

$$\mathbf{J}_{rx} \triangleq \frac{\partial \mathbf{r}(\mathbf{x})}{\partial \mathbf{x}^T} = \begin{pmatrix} \frac{\partial r_1}{\partial \mathbf{x}^T} \\ \frac{\partial r_2}{\partial \mathbf{x}^T} \\ \vdots \\ \frac{\partial r_m}{\partial \mathbf{x}^T} \end{pmatrix} = \Big(\frac{\partial \mathbf{r}(\mathbf{x})}{\partial x_1} \vdots \frac{\partial \mathbf{r}(\mathbf{x})}{\partial x_2} \dots \frac{\partial \mathbf{r}(\mathbf{x})}{\partial x_r}\Big) = \Big(\frac{\partial \mathbf{r}^T}{\partial \mathbf{x}}\Big)^T = \tag{169}$$

$$= \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \dots & \frac{\partial r_1}{\partial x_n} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r_n}{\partial x_1} & \dots & \dots & \frac{\partial r_n}{\partial x_n} \end{pmatrix} \in \mathcal{R}^{m\times n} . \tag{170}$$

(Inconsistently, in the literatur often $\frac{\partial \mathbf{r}(\mathbf{x})}{\partial \mathbf{x}}$ is written.)

Derivative of a scalar product with respect to vector $\mathbf{f}$, $\mathbf{r}(\mathbf{x}) \in \mathcal{R}^m$, $\mathbf{x} \in \mathcal{R}^n$, $\mathbf{y} \in \mathcal{R}^p$, $\mathbf{z} \in \mathcal{R}^s$,

$$\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})^T\mathbf{r}(\mathbf{x}) = \left(\frac{\partial \mathbf{f}^T(\mathbf{x})}{\partial \mathbf{x}}\right)^{[n\times m]}\mathbf{r}(\mathbf{x}) + \left(\frac{\partial \mathbf{r}^T(\mathbf{x})}{\partial \mathbf{x}}\right)^{[n\times m]}\mathbf{f}(\mathbf{x}) \quad \in \mathcal{R}^n. \tag{171}$$

Scalar/vector product with respect to a transposed vector

$$\frac{\partial [f(\mathbf{x})\mathbf{r}(\mathbf{x})]}{\partial \mathbf{x}^T} = f(\mathbf{x})\left(\frac{\partial \mathbf{r}(\mathbf{x})}{\partial \mathbf{x}^T}\right)^{[m\times n]} + \mathbf{r}(\mathbf{x})\left(\frac{\partial f}{\partial \mathbf{x}^T}\right)^{[1\times n]} \quad \in \mathcal{R}^{m\times n} . \tag{172}$$

Frequently, the Jacobi matrix is used for linear approximation (small-scale linearization)

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x},\mathbf{u}) \ , \quad \mathbf{x}(0^+) = \mathbf{x}_o \qquad \rightsquigarrow \qquad \dot{\mathbf{x}} + \Delta\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}+\Delta\mathbf{x},\ \mathbf{u}+\Delta\mathbf{u}) \tag{173}$$

$$\frac{d}{dt}\Delta x_i(t) = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}\Big|_{x_{jo}}\Delta x_j(t) + \frac{\partial g_i}{\partial u_j}\Big|_{u_{jo}}\Delta u_j(t) \qquad \forall i = 1, 2...n \tag{174}$$

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{J}_{gx}^T\Delta\mathbf{x} + \mathbf{J}_{gu}^T\Delta\mathbf{u} \ . \tag{175}$$

8 Derivatives of a Matrix

8.1 Derivations of Matrix-Valued Functions with Respect to a Scalar

$$\text{Kronecker matrix } \frac{\partial \mathbf{K}_y}{\partial K_{y,ij}} = \mathbf{E}_{ij} \triangleq \mathbf{e}_i \mathbf{e}_j^T \text{ with unit vector } \mathbf{e}_i \ . \tag{176}$$

$$\frac{\partial}{\partial p}\mathbf{A}(p)\mathbf{B}(p)\mathbf{C}(p) = \frac{\partial \mathbf{A}}{\partial p}\mathbf{BC} + \mathbf{A}\frac{\partial \mathbf{B}}{\partial p}\mathbf{C} + \mathbf{AB}\frac{\partial \mathbf{C}}{\partial p} \tag{177}$$

$$\text{or} \qquad d(\mathbf{ABC}) = (d\mathbf{A})\mathbf{BC} + \mathbf{A}(d\mathbf{B})\mathbf{C} + \mathbf{AB}(d\mathbf{C}) \ , \tag{178}$$

$$\text{e.g.,} \quad \mathbf{A}^T\mathbf{X}+\mathbf{XA} = \text{constant} \quad \rightsquigarrow \quad (d\mathbf{A}^T)\mathbf{X}+\mathbf{A}^T d\mathbf{X}+(d\mathbf{X})\mathbf{A}+\mathbf{X}d\mathbf{A} = \mathbf{0} \ . \tag{179}$$

$$\frac{\partial \mathbf{A}^m}{\partial p} = \frac{\partial \mathbf{A}}{\partial p}\mathbf{A}^{m-1} + \mathbf{A}\frac{\partial \mathbf{A}}{\partial p}\mathbf{A}^{m-2} + \ldots + \mathbf{A}^{m-1}\frac{\partial \mathbf{A}}{\partial p} \ . \tag{180}$$

From *Doyle, J.C. et al., 1996,*

$$\frac{\partial \mathbf{A}^{-1}}{\partial p} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial p}\mathbf{A}^{-1} \ . \tag{181}$$

$$\frac{\partial}{\partial \Delta P_{ij}}\mathbf{Y}(\mathbf{N}+\Delta\mathbf{P}\cdot\mathbf{W})^{-1}\mathbf{Z} = -\mathbf{Y}\mathbf{N}^{-1}\mathbf{E}_{ij}\mathbf{W}\mathbf{N}^{-1}\mathbf{Z} \tag{182}$$

for $\Delta\mathbf{P}\cdot\mathbf{W} \ll \mathbf{N}$ small in norm sense, or

$$\frac{\partial}{\partial e}\mathbf{Y}(\mathbf{N}+\Delta\mathbf{P}\cdot\mathbf{W})^{-1}\mathbf{Z} = -\mathbf{Y}\mathbf{N}^{-1}\frac{\partial \Delta\mathbf{P}}{\partial e}\mathbf{W}\mathbf{N}^{-1}\mathbf{Z} \tag{183}$$

where only $\Delta\mathbf{P}$ depends on e.

8.2 Basic Columnized Matrices with Respect to a Matrix

Find the following differential quotients for $\mathbf{K} \in \mathcal{R}^{m\times r}$, $\Psi \in \mathcal{R}^{r\times m}$

$$\frac{\partial}{\partial \mathbf{K}} \text{ col } [\mathbf{\Psi K}] \in \mathcal{R}^{mpr\times r} \tag{184}$$

$$\frac{\partial}{\partial \mathbf{K}} \text{ col } [\mathbf{K}^T\mathbf{\Psi}^T] \in \mathcal{R}^{mpr\times r} \ . \tag{185}$$

For omitted $\mathbf{\Psi}$, the results are given by using the Kronecker product $\otimes$

$$\frac{\partial}{\partial \mathbf{K}} \text{ col } \mathbf{K} = (\mathbf{I}_m \otimes \mathbf{U}_{r,m})[(\text{col } \mathbf{I}_m)\otimes \mathbf{I}_r] \overset{(\text{only for } m=1)}{=} \mathbf{I}_r \tag{186}$$

$$\frac{\partial}{\partial \mathbf{K}} \text{ col } \mathbf{K}^T = (\text{col } \mathbf{I}_m)\otimes \mathbf{I}_r \ . \tag{187}$$

For $\mathbf{K}_y \in \mathcal{R}^{m\times r}$ and using the permuation matrix $\mathbf{U}_{r,m}$, see Eq.(15), one has

$$\frac{\partial}{\partial \mathbf{K}_y} \text{ col } \mathbf{K}_y = (\mathbf{I}_m \otimes \mathbf{U}_{r,m})[(\text{col } \mathbf{I}_m) \otimes \mathbf{I}_r] \tag{188}$$

$$= \sum_{j=1,\ k=1}^{m,\ r} \mathbf{E}_{jk} \otimes \text{col } \mathbf{E}_{jk} \in \mathcal{R}^{m^2 r\times r} \tag{189}$$

and

$$\frac{\partial}{\partial \mathbf{K}_y} \text{ col } \mathbf{K}_y^T = (\text{col } \mathbf{I}_m) \otimes \mathbf{I}_r \in \mathcal{R}^{m^2 r\times r} \tag{190}$$

$$= \sum_{j=1,\ k=1}^{m,\ r} \mathbf{E}_{jk} \otimes \text{col}\mathbf{E}_{jk}^T \tag{191}$$

$$= \frac{\partial}{\partial \mathbf{K}_y}\mathbf{U}_{m,r}\text{col}\mathbf{K}_y = (\mathbf{I}_m \otimes \mathbf{U}_{m,r})\frac{\partial \text{col}\mathbf{K}_y}{\partial \mathbf{K}_y}\ . \tag{192}$$

Row manipulation row$\mathbf{A} = (\text{col}\mathbf{A}^T)^T$.

Consider the controller matrix $\mathbf{K} \in \mathcal{R}^{m\times r}$ and an arbitrary matrix $\mathbf{\Psi} \in \mathcal{R}^{p\times m}$

$$\frac{\partial}{\partial \mathbf{K}} \text{ col}[\mathbf{K}^T\mathbf{\Psi}^T] \overset{(29)}{=} \frac{\partial}{\partial \mathbf{K}}(\mathbf{\Psi} \otimes \mathbf{I}_r) \text{ col } \mathbf{K}^T \tag{193}$$

$$\overset{(213)}{=} [\mathbf{I}_m \otimes (\mathbf{\Psi} \otimes \mathbf{I}_r)]\frac{\partial}{\partial \mathbf{K}} \text{ col } \mathbf{K}^T \tag{194}$$

$$\overset{(190)}{=} [(\mathbf{I}_m \otimes \mathbf{\Psi}) \otimes \mathbf{I}_r][(\text{col } \mathbf{I}_m) \otimes \mathbf{I}_r] \tag{195}$$

$$\overset{(24)}{=} [(\mathbf{I}_m \otimes \mathbf{\Psi})\text{col } \mathbf{I}_m] \otimes (\mathbf{I}_r\mathbf{I}_r) \tag{196}$$

$$\frac{\partial}{\partial \mathbf{K}} \text{ col}[\mathbf{K}^T\mathbf{\Psi}^T] = \text{col}(\mathbf{\Psi}\mathbf{I}_m\mathbf{I}_m) \otimes \mathbf{I}_r = (\text{col } \mathbf{\Psi}) \otimes \mathbf{I}_r \ \in \mathcal{R}^{mpr\times r}\ . \tag{197}$$

Using the same correspondences once more and Eq.(186)

$$\frac{\partial}{\partial \mathbf{K}} \text{ col } [\mathbf{\Psi}\mathbf{K}] = \frac{\partial}{\partial \mathbf{K}}(\mathbf{I}_r \otimes \mathbf{\Psi}) \text{ col}\mathbf{K} \tag{198}$$

$$= [\mathbf{I}_m \otimes (\mathbf{I}_r \otimes \mathbf{\Psi})]\frac{\partial}{\partial \mathbf{K}} \text{ col } \mathbf{K} \tag{199}$$

$$\frac{\partial}{\partial \mathbf{K}}\text{col}[\mathbf{\Psi}\mathbf{K}] = [\mathbf{I}_m \otimes \mathbf{I}_r \otimes \mathbf{\Psi}](\mathbf{I}_m \otimes \mathbf{U}_{r,m})[(\text{col } \mathbf{I}_m) \otimes \mathbf{I}_r] \tag{200}$$

which can be simplified to

$$\frac{\partial}{\partial \mathbf{K}}\text{col}[\mathbf{\Psi}\mathbf{K}] = \mathbf{U}_{rm,p}[(\text{col}\mathbf{\Psi}^T) \otimes \mathbf{I}_r] \in \mathcal{R}^{mpr\times r} \tag{201}$$

or to

$$\frac{\partial \text{col}(\mathbf{\Psi}\mathbf{K})}{\partial \mathbf{K}} = \begin{pmatrix} \mathbf{I}_r \otimes (\text{col}\mathbf{\Psi})_1 \\ \vdots \\ \mathbf{I}_r \otimes (\text{col}\mathbf{\Psi})_m \end{pmatrix}\ , \tag{202}$$

where $(\mathrm{col}\boldsymbol{\Psi})_m$ is the m-th column of $\boldsymbol{\Psi}$.

For the sake of systematic presentation, note that from Eq.(199)

$$\frac{\partial}{\partial \mathbf{K}} \, \mathrm{col}[\boldsymbol{\Psi}\mathbf{K}] = (\mathbf{I}_m \otimes \mathbf{I}_r \otimes \boldsymbol{\Psi})\frac{\partial}{\partial \mathbf{K}} \, \mathrm{col}\mathbf{K} = (\mathbf{I}_{mr} \otimes \boldsymbol{\Psi})\frac{\partial}{\partial \mathbf{K}} \, \mathrm{col}\mathbf{K} \ , \tag{203}$$

and from Eqs.(197) and (213)

$$\frac{\partial}{\partial \mathbf{K}}\mathrm{col}\ [\mathbf{K}^T\boldsymbol{\Psi}^T] = (\mathbf{I}_m \otimes \boldsymbol{\Psi} \otimes \mathbf{I}_r)\frac{\partial}{\partial \mathbf{K}} \ \mathrm{col}\ \mathbf{K}^T \ . \tag{204}$$

Generalizing with a matrix $\mathbf{C} \in \mathcal{R}^{r\times v}$ yields on the one hand

$$\begin{aligned} \frac{\partial}{\partial \mathbf{K}}\mathrm{col}\ [\boldsymbol{\Psi}\mathbf{K}\mathbf{C})] \quad &= \quad [\mathbf{I}_m \otimes \mathbf{C}^T \otimes \boldsymbol{\Psi}](\mathbf{I}_m \otimes \mathbf{U}_{r,m})[(\mathrm{col}\ \mathbf{I}_m) \otimes \mathbf{I}_r] && (205)\\ &\overset{(201)}{=} \quad \mathbf{U}_{mv,p}[(\mathrm{col}\ \boldsymbol{\Psi}^T) \otimes \mathbf{C}^T] \in \mathcal{R}^{mpr\times v}, && (206)\end{aligned}$$

and on the other hand

$$\begin{aligned} \frac{\partial}{\partial \mathbf{K}} \ \mathrm{col}[\mathbf{C}^T\mathbf{K}^T\boldsymbol{\Psi}^T] \quad &= \quad [\mathbf{I}_m \otimes \boldsymbol{\Psi} \otimes \mathbf{C}^T][(\mathrm{col}\ \mathbf{I}_m) \otimes \mathbf{I}_r] && (207)\\ &= \quad [(\mathbf{I}_m \otimes \boldsymbol{\Psi})\mathrm{col}\ \mathbf{I}_m] \otimes [\mathbf{C}^T\mathbf{I}_r] && (208)\\ \frac{\partial}{\partial \mathbf{K}} \ \mathrm{col}[\mathbf{C}^T\mathbf{K}^T\boldsymbol{\Psi}^T] \quad &= \quad (\mathrm{col}\boldsymbol{\Psi}) \otimes \mathbf{C}^T \in \mathcal{R}^{mpr\times v} \ . && (209)\end{aligned}$$

8.3 Matrix Gradients with Respect to a Matrix

General definition

$$\frac{\partial \mathbf{G}}{\partial \mathbf{M}} \quad \triangleq \quad \sum_{ij} \mathbf{E}_{ij}^{(r\times s)} \otimes \frac{\partial \mathbf{G}}{\partial M_{ij}} = \begin{pmatrix} \frac{\partial \mathbf{G}}{\partial M_{11}} & \frac{\partial \mathbf{G}}{\partial M_{12}} & \cdots & \frac{\partial \mathbf{G}}{\partial M_{1s}} \\ \frac{\partial \mathbf{G}}{\partial M_{21}} & \frac{\partial \mathbf{G}}{\partial M_{22}} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{G}}{\partial M_{r1}} & \frac{\partial \mathbf{G}}{\partial M_{r2}} & \cdots & \frac{\partial \mathbf{G}}{\partial M_{rs}} \end{pmatrix} \tag{210}$$

$$\mathbf{G} \in \mathcal{R}^{n\times m}, \ \mathbf{M} \in \mathcal{R}^{r\times s}, \ \frac{\partial \mathbf{G}}{\partial \mathbf{M}} \in \mathcal{R}^{nr\times ms} \ . \tag{211}$$

The matrix product rule with $\mathbf{A} \in \mathcal{R}^{n\times m}, \mathbf{B} \in \mathcal{R}^{m\times q}, \mathbf{M} \in \mathcal{R}^{r\times s}$, is

$$\frac{\partial}{\partial \mathbf{M}}\mathbf{A}(\mathbf{M})\mathbf{B}(\mathbf{M}) = \frac{\partial \mathbf{A}}{\partial \mathbf{M}}(\mathbf{I}_s \otimes \mathbf{B}) + (\mathbf{I}_r \otimes \mathbf{A})\frac{\partial \mathbf{B}}{\partial \mathbf{M}} \qquad \frac{\partial (\mathbf{A}\mathbf{B})}{\partial \mathbf{M}} \in \mathcal{R}^{nr\times qs} \ . \tag{212}$$

For $\mathbf{A}$ constant,

$$\frac{\partial}{\partial \mathbf{M}}\mathbf{A}\mathbf{B}^{(n\times m)}(\mathbf{M}) = (\mathbf{I}_r \otimes \mathbf{A})\frac{\partial \mathbf{B}}{\partial \mathbf{M}}. \tag{213}$$

$$\frac{\partial}{\partial \mathbf{M}}a(\mathbf{M})\mathbf{B}(\mathbf{M}) = \frac{\partial a}{\partial \mathbf{M}} \otimes \mathbf{B} + \frac{\partial \mathbf{B}}{\partial \mathbf{M}} \otimes a = \frac{\partial a(\mathbf{M})}{\partial \mathbf{M}} \otimes \mathbf{B} + a(\mathbf{M})\frac{\partial \mathbf{B}}{\partial \mathbf{M}} \ . \tag{214}$$

Derivative of the inverse matrix (*Vetter, W.J., 1973*)

$$\frac{\partial \mathbf{A}^{-1}}{\partial \mathbf{M}} = -(\mathbf{I}_r \otimes \mathbf{A}^{-1})\frac{\partial \mathbf{A}}{\partial \mathbf{M}}(\mathbf{I}_s \otimes \mathbf{A}^{-1}) \qquad \mathbf{A} \in \mathcal{R}^{n\times m} \ . \tag{215}$$

$$\text{For } \ \mathbf{X} \in \mathcal{R}^{n\times n} \quad \frac{\partial(\mathbf{X}\mathbf{X}^{-1})}{\partial \mathbf{X}} = \frac{\partial \mathbf{I}_n}{\partial \mathbf{X}} \quad = \quad \mathbf{0} \tag{216}$$

$$\frac{\partial \mathbf{X}}{\partial \mathbf{X}}(\mathbf{I}_n \otimes \mathbf{X}^{-1}) + (\mathbf{I}_n \otimes \mathbf{X})\frac{\partial \mathbf{X}^{-1}}{\partial \mathbf{X}} \stackrel{(212)}{=} \mathbf{0} \tag{217}$$

$$\frac{\partial \mathbf{X}^{-1}}{\partial \mathbf{X}} \stackrel{(22)}{=} -(\mathbf{I}_n \otimes \mathbf{X}^{-1})\bar{\mathbf{U}}_{n,n}(\mathbf{I}_n \otimes \mathbf{X}^{-1}) \tag{218}$$

since

$$(\mathbf{N} \otimes \mathbf{M})^{-1} = \mathbf{N}^{-1} \otimes \mathbf{M}^{-1} \ . \tag{219}$$

The derivative of the Kronecker product of $\mathbf{A} \in \mathcal{R}^{n\times m}$ and $\mathbf{B} \in \mathcal{R}^{k\times l}$ with respect to a matrix $\mathbf{M} \in \mathcal{R}^{r\times s}$ is

$$\frac{\partial(\mathbf{A} \otimes \mathbf{B})}{\partial \mathbf{M}} = \frac{\partial \mathbf{A}}{\partial \mathbf{M}} \otimes \mathbf{B} + (\mathbf{I}_r \otimes \mathbf{U}_{nk})\Big(\frac{\partial \mathbf{B}}{\partial \mathbf{M}} \otimes \mathbf{A}\Big)(\mathbf{I}_s \otimes \mathbf{U}_{lm}) \ . \tag{220}$$

Note that $\mathbf{I}_r, \mathbf{U}_{nk}$ etc. are not conformable for matrix multiplication.

Hadamard product derivatives: Following Eq.(51), and using $\mathbf{1}^{[r\times s]} = \mathtt{ones(n,m)}$.

$$\frac{\partial \mathbf{A}^{[n\times m]} \cdot * \ \mathbf{B}(\mathbf{M})}{\partial \mathbf{M}^{[r\times s]}} = \Big(\mathbf{1}^{[r\times s]} \otimes \mathbf{A}^{[n\times m]}\Big) \cdot * \ (\frac{\partial \mathbf{B}^{[n\times m]}}{\partial \mathbf{M}}) \tag{221}$$

$$\stackrel{(\mathbf{M}=\mathbf{B})}{=} \Big(\mathbf{1}^{[n\times m]} \otimes \mathbf{A}^{[n\times m]}\Big) \cdot * \ \sum_{i,j} \mathbf{E}_{ij}^{[n\times m]} \otimes \frac{\partial \mathbf{B}}{\partial B_{ij}} \tag{222}$$

$$= \Big(\mathbf{1}^{[n\times m]} \otimes \mathbf{A}^{[n\times m]}\Big) \cdot * \ \sum_{i,j} \mathbf{E}_{ij}^{[n\times m]}\mathbf{E}_{ij}^{[n\times m]}. \tag{223}$$

Alternatively,

$$\frac{\partial \mathbf{A}^{[n\times m]} \cdot * \ \mathbf{B}(\mathbf{M})}{\partial \mathbf{M}^{[r\times s]}} = \sum_{i,j}^{r,s} \mathbf{E}_{ij}^{[r\times s]} \otimes \frac{\partial \mathbf{A} \cdot * \ \mathbf{B}(\mathbf{M})}{\partial M_{ij}} \tag{224}$$

$$\stackrel{(\mathbf{M}=\mathbf{B})}{=} \sum_{i,j}^{n,m} \mathbf{E}_{ij} \otimes (A_{ij}\mathbf{E}_{ij}) = \sum_{i,j}^{n,m} A_{ij}\mathbf{E}_{ij} \otimes \mathbf{E}_{ij} \ . \tag{225}$$

$$\frac{\partial \mathrm{adj}[\mathbf{X}]}{\partial \mathbf{X}} = \frac{\partial \mathbf{X}^{-1} \det \mathbf{X}}{\partial \mathbf{X}} \stackrel{(214)}{=} \frac{\partial \det \mathbf{X}}{\partial \mathbf{X}} \otimes \mathbf{X}^{-1} + (\det \mathbf{X})\frac{\partial \mathbf{X}^{-1}}{\partial \mathbf{X}} \tag{226}$$

$$\stackrel{(117),(218),(22)}{=} [\mathrm{adj}^T(\mathbf{X})] \otimes \mathbf{X}^{-1} - (\det \mathbf{X}) \cdot (\mathbf{I}_n \otimes \mathbf{X}^{-1}) \cdot \bar{\mathbf{U}}_{n,n} \cdot (\mathbf{I}_n \otimes \mathbf{X}^{-1}) \ . \tag{227}$$

8.4 Chain Rules

The chain rule for scalar functions $\mathbf{M} \in \mathcal{R}^{r\times s}$ is (*Brewer, J.W., 1977; 1977a; 1978a*)

$$\frac{\partial a[b(\mathbf{M})]}{\partial \mathbf{M}} = \left(\mathbf{I}_r \otimes \frac{\partial a}{\partial b}\right)\left(\frac{\partial b}{\partial \mathbf{M}} \otimes \mathbf{I}_m\right) \tag{228}$$

$$= \left(\frac{\partial a}{\partial b}\mathbf{I}_r\right)\left(\frac{\partial b}{\partial \mathbf{M}} \otimes 1\right) = \frac{\partial a}{\partial b}\mathbf{I}_r \frac{\partial b}{\partial \mathbf{M}} = \frac{\partial a}{\partial b}\frac{\partial b}{\partial \mathbf{M}} \; . \tag{229}$$

8.5 Chain Rule for Vector Functions

$$\frac{\partial f[\mathbf{r}(\mathbf{x})]}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{r}^T(\mathbf{x})}{\partial \mathbf{x}}\right)^{[n\times m]}\left(\frac{\partial f}{\partial \mathbf{r}}\right)^{[m\times 1]} = \mathbf{J}_{rx}^T \frac{\partial f}{\partial \mathbf{r}} \quad \in \mathcal{R}^n \tag{230}$$

8.6 Jacobi Matrix

From Eq.(170),

$$\text{Jacobi matrix} \quad \mathbf{J}_{rx} \triangleq \frac{\partial \mathbf{r}}{\partial \mathbf{x}^T} \quad \text{from source } \mathbf{x} \text{ to target } \mathbf{r}, \tag{231}$$

where in the Jacobi matrix $\mathbf{J}_{rx} \in \mathcal{R}^{\dim(\mathbf{r})\times\dim(\mathbf{x})} = \mathcal{R}^{m\times n}$ the rows are oriented at $\mathbf{r} \in \mathcal{R}^m$ and the columns at $\mathbf{x} \in \mathcal{R}^n$. For the special case $\mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{x}$, $\mathbf{Q} \in \mathcal{R}^{m\times n}$, one has the Jacobi matrix $\frac{\partial \mathbf{y}}{\partial \mathbf{x}^T} = \mathbf{Q}$. A partition $\frac{\partial \mathbf{y}}{\partial x_i}$ is the ith column of $\mathbf{Q}$ or the ith column of $\frac{\partial \mathbf{y}}{\partial \mathbf{x}^T}$. In transposed form $\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}} = \mathbf{Q}^T$.

$$\text{Scalar } f: \quad \frac{\partial f[\mathbf{y}(\mathbf{x})]}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}}\right)^{[n\times p]}\left(\frac{\partial f}{\partial \mathbf{y}}\right)^{[p\times 1]} = \mathbf{J}_{yx}^T \frac{\partial f}{\partial \mathbf{y}} \quad \in \mathcal{R}^n \tag{232}$$

$$\frac{\partial f[\mathbf{z}(\mathbf{y}(\mathbf{x}))]}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}}\right)^{[n\times p]}\left(\frac{\partial \mathbf{z}^T}{\partial \mathbf{y}}\right)^{[p\times s]}\left(\frac{\partial f}{\partial \mathbf{z}}\right)^{[s\times 1]} = \mathbf{J}_{yx}^T \mathbf{J}_{zy}^T \frac{\partial f}{\partial \mathbf{z}} \quad \in \mathcal{R}^n \tag{233}$$

$$\text{Vector } \mathbf{r}: \quad \frac{\partial \mathbf{r}[\mathbf{z}(\mathbf{y}(\mathbf{x}))]}{\partial \mathbf{x}^T} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{z}^T}\right)^{[m\times s]}\left(\frac{\partial \mathbf{z}}{\partial \mathbf{y}^T}\right)^{[s\times p]}\left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}^T}\right)^{[p\times n]} = \mathbf{J}_{rz}\mathbf{J}_{zy}\mathbf{J}_{yx} \quad \in \mathcal{R}^{m\times n} \; . \tag{234}$$

Comparing Eq.(234) with Eq.(230) looks as some irregularity at a first glance, as far as the order and the transpositions are concerned. However, all is true because

$$\frac{\partial f[\mathbf{r}(\mathbf{x})]}{\partial \mathbf{x}} = \frac{\partial \mathbf{r}^T}{\partial \mathbf{x}}\frac{\partial f}{\partial \mathbf{r}} = \mathbf{J}_{rx}^T \frac{\partial f}{\partial \mathbf{r}} \; . \tag{235}$$

From Eq.(234)

$$\frac{\partial \mathbf{f}[\mathbf{r}(\mathbf{x})]}{\partial \mathbf{x}^T} = \mathbf{J}_{fr}\mathbf{J}_{rx} \tag{236}$$

$$\text{transposing} \quad \rightsquigarrow \quad \frac{\partial \mathbf{f}^T[\mathbf{r}(\mathbf{x})]}{\partial \mathbf{x}} = \mathbf{J}_{rx}^T \mathbf{J}_{fr}^T = \mathbf{J}_{rx}^T \frac{\partial \mathbf{f}^T}{\partial \mathbf{r}} \; . \tag{237}$$

Comparing Eq.(235) and Eq.(237): Eq.(237) comprises a complete row $\mathbf{f}^T$, Eq.(235) is only one element of this row.

8.7 Second Derivative (Hesse Matrix)

$$\frac{\partial}{\partial \mathbf{x}^T}\left(\frac{\partial f}{\partial \mathbf{x}}\right) = \begin{pmatrix} \frac{\partial}{\partial x_1}(\frac{\partial f}{\partial x_1}) & \frac{\partial}{\partial x_2}(\frac{\partial f}{\partial x_1}) & \frac{\partial}{\partial x_3}(\frac{\partial f}{\partial x_1}) & \cdots & \frac{\partial}{\partial x_n}(\frac{\partial f}{\partial x_1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n}(\frac{\partial f}{\partial x_1}) & \frac{\partial}{\partial x_n}(\frac{\partial f}{\partial x_2}) & \frac{\partial}{\partial x_n}(\frac{\partial f}{\partial x_3}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}) \end{pmatrix} \in \mathcal{R}^{n\times n}\ . \tag{238}$$

(The Hesse matrix is often written as $\frac{\partial^2 f}{\partial \mathbf{x}^2}$.)

8.8 Amplitude Scaling

Scaling of variables provides a numerically different (but physically identic) system of variables the range of which is smaller and yields less computational errors. Consider an arbitrary index of performance $f(\mathbf{x})$. Apply a similarity transform $\mathbf{y} = \mathbf{Qx}$ with an identic index $f(\mathbf{x}) \equiv h(\mathbf{y})$.

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial h(\mathbf{y})}{\partial \mathbf{x}} \overset{(230)}{=} \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}^T}\right)^T \frac{\partial h}{\partial \mathbf{y}} = \mathbf{Q}^T \frac{\partial h}{\partial \mathbf{y}} \tag{239}$$

$$\frac{\partial f}{\partial \mathbf{x}^T} = \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}^T}\mathbf{Q} \tag{240}$$

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial f}{\partial \mathbf{x}^T}\right) \overset{(239)}{=} \mathbf{Q}^T \frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial f}{\partial \mathbf{x}^T}\right) \overset{(240)}{=} \mathbf{Q}^T \frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial h}{\partial \mathbf{y}^T}\mathbf{Q}\right) = \mathbf{Q}^T \frac{\partial^2 h}{\partial \mathbf{y}\partial \mathbf{y}^T}\mathbf{Q} \tag{241}$$

$$\frac{\partial^2 h}{\partial \mathbf{y}\partial \mathbf{y}^T} = \mathbf{Q}^{-T}\frac{\partial^2 f}{\partial \mathbf{x}\partial \mathbf{x}^T}\mathbf{Q}^{-1}\ . \tag{242}$$

The right-hand side of Eq.(242) should be close to the identity matrix to optimize the condition.

E.g., $f(\mathbf{x}) = a^2x_1^2 + b^2x_2^2 \equiv h(\mathbf{y})$, where $a \gg b$. For f =constant there result ellipses in the (x_1, x_2)-plane.

$$\frac{\partial^2 f}{\partial \mathbf{x}\partial \mathbf{x}^T} = \frac{\partial}{\partial \mathbf{x}^T}\frac{\partial}{\partial \mathbf{x}}f = \frac{\partial}{\partial \mathbf{x}^T}\begin{pmatrix} 2a^2x_1 & 0 \\ 0 & 2b^2x_2 \end{pmatrix} = \begin{pmatrix} 2a^2 & 0 \\ 0 & 2b^2 \end{pmatrix}\ . \tag{243}$$

Selecting $\mathbf{Q} = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, from Eq.(242)

$$\frac{\partial^2 h}{\partial \mathbf{y}\partial \mathbf{y}^T} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\frac{\partial^2 f}{\partial \mathbf{x}\partial \mathbf{x}^T}\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} = 2\,\mathbf{I}_2 \tag{244}$$

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}^T\mathbf{Q}\mathbf{x} = \mathbf{x}^T\mathbf{Q}^2\mathbf{x} = \mathbf{y}^T\mathbf{Q}^{-T}\mathbf{Q}^2\mathbf{Q}^{-1}\mathbf{y} = \mathbf{y}^T\mathbf{y} \equiv h(\mathbf{y}) \ . \tag{245}$$

In $\mathbf{y}$-space, $h(\mathbf{y}) =$ constant is circle shaped (*Franklin, G.F., et al., 2002; Papageorgiou, M., 1991*).

9 Complex Closed-Loop Transfer Matrices and Their Derivatives

Argument and logarithmic expression for the following elementary correspondences $v = a + jb \equiv |v|e^{\arg(v)}$, $|v| = \sqrt{a^2+b^2}$, $\arg(v) = \arctan(b/a)$

$$\begin{aligned} \arg F &= \Im m\ [\ln F] & (246)\\ \frac{\partial \arg F}{\partial \omega} &= \Im m\ [\frac{1}{F}\frac{\partial F}{\partial \omega}] & (247)\\ \Delta \arg F &= \Im m\ [\frac{1}{F}\Delta F] \ \text{ for } \ \Delta F \ \text{ small in norm sense.} & (248) \end{aligned}$$

Evolving the symmetric expression

$$\begin{aligned} K^*\frac{\partial K}{\partial \mathbf{p}} + K\frac{\partial K^*}{\partial \mathbf{p}} &= (\Re e\ K - j\Im m\ K)\frac{\partial \Re e\ K + j\Im m\ K}{\partial \mathbf{p}} + \ \ldots & (249)\\ &= 2\ \Re e\ K\frac{\partial \Re e\ K}{\partial \mathbf{p}} + 2\ \Im m\ K\frac{\partial \Im m\ K}{\partial \mathbf{p}}\ . & (250) \end{aligned}$$

Since

$$K\frac{\partial K^*}{\partial \mathbf{p}} = \left(K^*\frac{\partial K}{\partial \mathbf{p}}\right)^* \tag{251}$$

from Eq.(249), left-hand side, one finds

$$\Re e\ [K^*\frac{\partial K}{\partial \mathbf{p}}] = \Re e\ K\frac{\partial \Re e\ K}{\partial \mathbf{p}} + \Im m\ K\frac{\partial \Im m\ K}{\partial \mathbf{p}}\ . \tag{252}$$

For complex $\mathbf{G}$, the derivatives of $\mathrm{tr}[\mathbf{G}^H\mathbf{G}]$ and $||\mathbf{G}||_F$ with respect to the real and imaginary part obey

$$\frac{\partial\ \mathrm{tr}\ \mathbf{G}^H\mathbf{G}}{\partial(\Re e\ \mathbf{G})} = 2\ \Re e\ \mathbf{G} \qquad \frac{\partial\ \mathrm{tr}\ \mathbf{G}^H\mathbf{G}}{\partial(\Im m\ \mathbf{G})} = 2\ \Im m\ \mathbf{G} \tag{253}$$

$$\frac{\partial||\mathbf{G}||_F}{\partial(\Re e\ \mathbf{G})} = \frac{\Re e\ \mathbf{G}}{||\mathbf{G}||_F} \qquad \frac{\partial||\mathbf{G}||_F}{\partial(\Im m\ \mathbf{G})} = \frac{\Im m\ \mathbf{G}}{||\mathbf{G}||_F}\ . \tag{254}$$

Generalizing Eq.(145) for $\mathbf{B}, \mathbf{C}$ complex and $\mathbf{A}, \mathbf{M}$ real

$$\frac{\partial}{\partial \mathbf{M}}||\mathbf{A} + \mathbf{BMC}||_F^2 = 2\Re e\ \mathbf{B}^H(\mathbf{A} + \mathbf{BMC})\mathbf{C}^H \tag{255}$$

since

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{M}}\|\mathbf{A}+\mathbf{BMC}\|_F^2 &= \frac{\partial}{\partial \mathbf{M}}\mathrm{tr}[(\mathbf{A}+\mathbf{BMC})^H(\mathbf{A}+\mathbf{BMC})] && (256)\\
&= \frac{\partial}{\partial \mathbf{M}}\mathrm{tr}\ [\mathbf{A}^H\mathbf{A}+\mathbf{C}^H\mathbf{M}^H\mathbf{B}^H\mathbf{A}+\mathbf{A}^H\mathbf{BMC}+\mathbf{C}^H\mathbf{M}^H\mathbf{B}^H\mathbf{BMC}] && (257)\\
&= \frac{\partial}{\partial \mathbf{M}}\mathrm{tr}\ [\mathbf{A}^T\mathbf{B}^*\mathbf{MC}^*+\mathbf{A}^T\mathbf{BMC}+\mathbf{B}^H\mathbf{BMCC}^H\mathbf{M}^T] && (258)\\
&= \mathbf{B}^H\mathbf{AC}^H+\mathbf{B}^T\mathbf{AC}+\mathbf{B}^T\mathbf{B}^*\mathbf{MCC}^T+\mathbf{B}^H\mathbf{BMCC}^H && (259)\\
&= 2\Re e\ (\mathbf{B}^H\mathbf{AC}^H+\mathbf{B}^H\mathbf{BMCC}^H)=2\Re e\ \mathbf{B}^H(\mathbf{A}+\mathbf{BMC})\mathbf{C}^H\ . && (260)
\end{aligned}$$

The combined expression leads to

$$\begin{aligned}
&\frac{\partial}{\partial \mathbf{M}}\|\mathbf{N}_1\mathbf{MN}_2+\mathbf{N}_3\mathbf{MN}_4+\mathbf{N}_5\|_F = \\
&\qquad = 2\Re e\ \{\mathbf{N}_3^H(\mathbf{N}_1\mathbf{MN}_2+\mathbf{N}_3\mathbf{MN}_4+\mathbf{N}_5)\mathbf{N}_4^H \\
&\qquad\qquad +\mathbf{N}_1^H(\mathbf{N}_1\mathbf{MN}_2+\mathbf{N}_3\mathbf{MN}_4+\mathbf{N}_5)\mathbf{N}_2^H\}\ , \qquad (261)
\end{aligned}$$

where $\mathbf{N}_1....\mathbf{N}_5$ are complex and $\mathbf{M}$ is real. To continue this result

$$\frac{\partial}{\partial \mathbf{K}}\|\mathbf{F}+\mathbf{LKR}+\mathbf{R}^H\mathbf{K}^T\mathbf{L}^H\|_F^2 = 4\Re e\ \{\mathbf{L}^H(\mathbf{F}+\mathbf{LKR}+\mathbf{R}^H\mathbf{K}^T\mathbf{L}^H)\mathbf{R}^H\}\ . \qquad (262)$$

For the closed-loop matrix-valued transfer function

$$T_{cl} \stackrel{\triangle}{=} \|\mathbf{T}_{ref}(s)-\mathbf{C}(s\mathbf{I}-\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}\|_F^2\ , \qquad (263)$$

using $\mathbf{U} \stackrel{\triangle}{=} s\mathbf{I}-\mathbf{A}-\mathbf{BK}=\mathbf{\Phi}_{cl}^{-1}(s)$ leads to (*Weinmann, A., 2001*)

$$\begin{aligned}
\frac{\partial T_{cl}}{\partial \mathbf{K}} &= \frac{\partial\|\mathbf{T}_{ref}(s)-\mathbf{CU}^{-1}\mathbf{B}\|_F^2}{\partial \mathbf{K}} && (264)\\
&= -2\Re e\ \{\mathbf{B}^H\mathbf{U}^{-H}\mathbf{C}^H(\mathbf{T}_{ref}-\mathbf{CU}^{-1}\mathbf{B})\mathbf{B}^H\mathbf{U}^{-H}\}\ . && (265)
\end{aligned}$$

Consider the transfer matrix $\mathbf{F}_{St}(s)$ (or a related type) of the closed-loop system

$$\mathbf{F}_{St}(s)=\mathbf{F}_1(s\mathbf{I}-\mathbf{A}-\mathbf{BK})^{-1}\mathbf{F}_2\ \in\mathcal{C}^{r\times p} \qquad \mathbf{y}=\mathbf{F}_{St}\mathbf{w}_d \qquad (266)$$

$$\mathbf{B}\in\mathcal{R}^{n\times m}\ ,\quad \mathbf{K}\in\mathcal{R}^{m\times n}\ ,\quad \mathbf{A}\in\mathcal{R}^{n\times n}\ ,\ \mathbf{w}_d\in\mathcal{C}^p\ ,\ \mathbf{y}\in\mathcal{C}^r\ ,\ \mathbf{F}_2\in\mathcal{C}^{n\times p}\quad \mathbf{F}_1\in\mathcal{C}^{r\times n}\ . \qquad (267)$$

Abbreviating $\mathbf{F}_3=\mathbf{F}_1^H\mathbf{F}_1,\ \mathbf{F}_4=\mathbf{F}_2\mathbf{F}_2^H,\ \mathbf{U}=s\mathbf{I}-\mathbf{A}-\mathbf{BK}=\mathbf{\Phi}_{cl}^{-1}(s)$,

$$\frac{\partial\|\mathbf{F}_{St}\|_F^2}{\partial \mathbf{K}} = 2\mathbf{B}^T\Re e\ \{\mathbf{U}^{-H}\mathbf{F}_3\mathbf{U}^{-1}\mathbf{F}_4\mathbf{R}^{-H}\}\ . \qquad (268)$$

The derivative of the prefilter $\mathbf{V}=-[\mathbf{C}(\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}]^{-1}$ with respect $\mathbf{K}$ leads to the differential quotient as

$$\frac{\partial\|[\mathbf{C}(\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}]^{-1}\|_F^2}{\partial \mathbf{K}} = 2\mathbf{T}_1^{-1}\mathbf{T}_1^{-T}\mathbf{B}^T(\mathbf{A}+\mathbf{BK})^{-T}\ , \qquad (269)$$

where $\mathbf{T}_1 \stackrel{\triangle}{=} \mathbf{C}(\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}$.

10 Derivatives with Respect to Time

Total derivative if c depends on t, additionally,

$$\frac{d}{dt}c[\mathbf{x}(t),t] = \frac{d\mathbf{x}^T(t)}{dt}\frac{\partial c}{\partial \mathbf{x}} + \frac{\partial c}{\partial t}\ . \tag{270}$$

Vector (column) function $\mathbf{z}$ dependent on $\mathbf{x}(t)$ and t: $\mathbf{z} \in \mathcal{R}^m$, $\mathbf{x}(t) \in \mathcal{R}^q$, $\mathbf{M} := t$ (*Vetter, W.J., 1970*)

$$\frac{d\mathbf{z}[\mathbf{x}(t)]}{dt} = \Big(\frac{d\mathbf{x}^T}{dt} \otimes \mathbf{I}_m\Big)\Big(1 \otimes \frac{\partial \mathbf{z}}{\partial \mathbf{x}}\Big) = \Big(\frac{d\mathbf{x}^T}{dt} \otimes \mathbf{I}_m\Big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \tag{271}$$

$$\frac{d\mathbf{z}[\mathbf{x}(t),t]}{dt} = \Big(\frac{d\mathbf{x}^T}{dt} \otimes \mathbf{I}_m\Big)\frac{\partial \mathbf{z}}{\partial \mathbf{x}} + \frac{\partial \mathbf{z}}{\partial t}\ . \tag{272}$$

Row function $\mathbf{y}^T$ dependent on $\mathbf{x}(t)$ and t with $\mathbf{y} \in \mathcal{R}^n$

$$\frac{d\mathbf{y}^T[\mathbf{x}(t),t]}{dt} = \Big(\frac{d\mathbf{x}^T}{dt} \otimes \mathbf{I}_1\Big)\Big(\mathbf{I}_1 \otimes \frac{\partial \mathbf{y}^T}{\partial \mathbf{x}}\Big) = \frac{d\mathbf{x}^T}{dt}\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}}\ . \tag{273}$$

Matrix-valued function $\mathbf{A}$ dependent on the vector $\mathbf{x}(t) \in \mathcal{R}^q, \mathbf{A} \in \mathcal{R}^{n\times m}$,

$$\frac{d\mathbf{A}[\mathbf{x}(t),t]}{dt} = \Big(\frac{d\mathbf{x}^T}{dt} \otimes \mathbf{I}_n\Big)\Big(\mathbf{I}_1 \otimes \frac{\partial \mathbf{A}}{\partial \mathbf{x}}\Big) + \frac{\partial \mathbf{A}}{\partial t} = \Big(\frac{d\mathbf{x}^T}{dt} \otimes \mathbf{I}_n\Big)\frac{\partial \mathbf{A}}{\partial \mathbf{x}} + \frac{\partial \mathbf{A}}{\partial t}\ . \tag{274}$$

Example 2: For $\mathbf{y} \in \mathcal{R}^n$, $\mathbf{z} \in \mathcal{R}^m$, $\mathbf{x} \in \mathcal{R}^q$, $\mathbf{A} \in \mathcal{R}^{n\times m}$ derivatives on the product $c = \mathbf{y}^T[\mathbf{x}(t),t]\ \mathbf{A}[\mathbf{x}(t),t]\ \mathbf{z}[\mathbf{x}(t),t]$ (*Vetter, W.J., 1970*)

$$\frac{d}{dt}\ \mathbf{y}^T[\mathbf{x}(t),t]\ \mathbf{A}[\mathbf{x}(t),t]\ \mathbf{z}[\mathbf{x}(t),t] = \frac{d\mathbf{y}^T}{dt}\mathbf{A}\mathbf{z} + \mathbf{y}^T\frac{d\mathbf{A}}{dt}\mathbf{z} + \mathbf{y}^T\mathbf{A}\frac{d\mathbf{z}}{dt}\ . \tag{275}$$

$$\frac{\partial c}{\partial \mathbf{y}} = \mathbf{A}\mathbf{z} \qquad \frac{\partial c}{\partial \mathbf{z}} = (\mathbf{y}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{y} \tag{276}$$

$$\frac{\partial c}{\partial \mathbf{A}} = \frac{\partial}{\partial \mathbf{A}}\mathrm{tr}\ (\mathbf{y}^T\mathbf{A}\mathbf{z}) = \frac{\partial}{\partial \mathbf{A}}\mathrm{tr}\ (\mathbf{z}\mathbf{y}^T\mathbf{A}) = (\mathbf{z}\mathbf{y}^T)^T = \mathbf{y}\mathbf{z}^T\ .\ \square \tag{277}$$

Example 3:

$$\frac{\partial}{\partial \mathbf{x}^T}\ \mathbf{x}^T\mathbf{Q}(\mathbf{x})\ \mathbf{x} = \mathbf{x}^T[\mathbf{Q}^T(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) + \frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}^T}(\mathbf{I}_q \otimes \mathbf{x})] \qquad \mathbf{x} \in \mathcal{R}^q\ .\ \square \tag{278}$$

Example 4:

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \qquad \text{and} \qquad \int_0^t e^{\mathbf{A}\tau}d\tau = \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\ .\ \square \tag{279}$$

11 Per-Unit Square Index of Performance

Find the maximum of a per-unit energy expression versus $\mathbf{x}$, i.e.,

$$\sup_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{R} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \ . \tag{280}$$

For $\mathbf{R} = \mathbf{R}^H$, the result is given by the Rayleigh Principle. Comparison of Eq. 280 with the definition Eq.(281) of the singular value

$$\sigma_{\max}[\mathbf{M}] = \sup_{\mathbf{x}} \frac{\|\mathbf{M}\mathbf{x}\|_F}{\|\mathbf{x}\|_F} = \sup_{\mathbf{x}} \sqrt{\frac{\mathbf{x}^T \mathbf{M}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}} \tag{281}$$

yields the correspondences $\mathbf{R} = \mathbf{M}^T \mathbf{M} \in \mathcal{R}^{n \times n}$ and

$$\sup_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{R} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \sigma^2_{\max}[\mathbf{M}] = \sigma^2_{\max}[\sqrt{\mathbf{R}}] \ . \tag{282}$$

The worst initial condition $\mathbf{x}_0$ which yields the maximum per-unit index of performance I_{pu} results from

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \frac{\mathbf{x}^T \mathbf{M}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} &= \mathbf{0} && (283) \\ \mathbf{M}^T \mathbf{M} \mathbf{x} &= \frac{\mathbf{x}^T \mathbf{M}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \mathbf{x} && (284) \\ \mathbf{M}^T \mathbf{M} \mathbf{x} &= \sigma^2_{\max}[\mathbf{M}] \ \cdot \ \mathbf{x} \ . && (285) \end{aligned}$$

The last but one line equals the correspondence of eigenvalues and eigenvectors $\mathbf{A}\mathbf{a} = \lambda(\mathbf{A}) \cdot \mathbf{a}$, in detail

$$(\mathbf{M}^T \mathbf{M}) \cdot \mathrm{eigv}[\mathbf{M}^T \mathbf{M}] = \mathrm{eig}_{\max}[\mathbf{M}^T \mathbf{M}] \cdot \mathrm{eigv}[\mathbf{M}^T \mathbf{M}] = \sigma^2_{\max}[\mathbf{M}] \cdot \mathrm{eigv}[\mathbf{M}^T \mathbf{M}] \ . \tag{286}$$

The largest eigenvalue is selected in order to find the largest I_{pu}. The largest eigenvalue of $\mathbf{M}^T \mathbf{M}$ is the squared singular value $\sigma_{\max}[\mathbf{M}]$. The worst vector $\mathbf{x}$ is $\mathbf{x}_0 = \mathrm{eigv}[\mathbf{R}]$ and has the manifold of the eigenvector. Only the direction is determined, not the length.

Referring to Eq.(282) and anticipating Eq.(332) (*Weinmann, A., 2003; 2004a*), an upper bound for the maximum per-unit energy of the actuating variable is

$$I_{um} = \sigma^2_{\max}[\sqrt{\mathbf{P}_{Ry}}] \leq \|\sqrt{\mathbf{P}_{Ry}}\|^2_F = \mathrm{tr}[(\sqrt{\mathbf{P}_{Ry}})^T (\sqrt{\mathbf{P}_{Ry}})] = \mathrm{tr}[\mathbf{P}_{Ry}] \tag{287}$$

and the worst initial condition is $\mathbf{x}_0 = \mathrm{eigv}[\mathbf{P}_{Ry}]$.

12 Derivatives of the Matrix Pencil

With the definition of the inverse resolvent matrix (equivalent to the so-called matrix pencil, *Laub, A.J., 2005*)

$$\mathbf{H} \triangleq s\mathbf{I} - \mathbf{A} - \mathbf{BK} \ , \tag{288}$$

the first and the higher derivatives with respect to s are presented (*Weinmann, A., 2004*). We presuppose $\mathbf{H}$ nonsingular.

12.1 First Derivative

$$\frac{\partial \det \mathbf{H}}{\partial s} = \mathrm{tr}[\frac{\partial \mathbf{H}}{\partial s}\mathbf{adj\ H}] = \mathrm{tr}[\mathbf{I\ adj\ H}] = \mathrm{tr}[\mathbf{adj\ H}] = \ \mathrm{tr}[\mathbf{H}^{-1}\det \mathbf{H}] \ . \tag{289}$$

With the definition $e_i \triangleq \mathrm{tr}\mathbf{H}^{-i}(s)$ and the definition of Q_i, one finds

$$\frac{\partial \det \mathbf{H}}{\partial s} = \det \mathbf{H} \cdot \mathrm{tr}\mathbf{H}^{-1} = \det \mathbf{H} \cdot e_1 \triangleq \det \mathbf{H} \cdot Q_1 \ . \tag{290}$$

For $\mathbf{H}$ singular, see *Weinmann, A., 2004, Section 6*. Furthermore, utilize the relations and abbreviations

$$\mathbf{L}(s) \triangleq \mathbf{adj\ H}(s) \tag{291}$$

$$\frac{\partial \mathrm{tr}\mathbf{H}^{-1}}{\partial s} = -\mathrm{tr}\mathbf{H}^{-2} \tag{292}$$

$$\frac{\partial \mathrm{tr}\mathbf{H}^{-i}}{\partial s} = -i\ \mathrm{tr}\mathbf{H}^{-i-1} \quad \text{or} \quad \frac{\partial e_i}{\partial s} = -i\ e_{i+1} \tag{293}$$

$$\frac{\partial^n}{\partial s^n}\mathrm{tr}\mathbf{H}^{-1} = (-1)^n\ n!\ \mathrm{tr}\mathbf{H}^{-n-1} \ . \tag{294}$$

In general,

$$e_i^k = (\mathrm{tr}\mathbf{H}^{-i})^k = \frac{(\mathrm{tr}\mathbf{L}^i)^k}{(\det \mathbf{H})^{ik}} \ . \tag{295}$$

12.2 Second Derivative

Continuing the derivations using differentiation of the product

$$\begin{aligned}
\frac{\partial^2 \det \mathbf{H}}{\partial s^2} &= \frac{\partial}{\partial s}\frac{\partial \det \mathbf{H}}{\partial s} = \frac{\partial}{\partial s}\ \det \mathbf{H} \cdot \mathrm{tr}\mathbf{H}^{-1} && (296)\\
&= \frac{\partial \det \mathbf{H}}{\partial s}\mathrm{tr}\mathbf{H}^{-1} + \det \mathbf{H}\frac{\partial \mathrm{tr}\mathbf{H}^{-1}}{\partial s} && (297)\\
&= \det \mathbf{H} \cdot (\mathrm{tr}\mathbf{H}^{-1})^2 + \det \mathbf{H} \cdot \mathrm{tr}[-\mathbf{H}^{-2}] && (298)\\
&= \det \mathbf{H} \cdot [e_1^2 - e_2] \triangleq \det \mathbf{H} \cdot Q_2 && (299)\\
&= (\mathrm{tr\ adj\ }\mathbf{H})(\mathrm{tr}\mathbf{H}^{-1} - \frac{\mathrm{tr}\mathbf{H}^{-2}}{\mathrm{tr}\mathbf{H}^{-1}}) && (300)
\end{aligned}$$

$$= \operatorname{tr} \operatorname{adj} \mathbf{H} \cdot \operatorname{tr}\mathbf{H}^{-1} - \frac{1}{\det \mathbf{H}} \operatorname{tr}(\operatorname{adj} \mathbf{H})^2 \tag{301}$$

$$= \frac{(\operatorname{tr} \operatorname{adj} \mathbf{H})^2 - \operatorname{tr} (\operatorname{adj} \mathbf{H})^2}{\det \mathbf{H}} \,. \tag{302}$$

12.3 i-th Derivative

For the i-th derivative one has

$$\frac{\partial^i \det \mathbf{H}}{\partial s^i} = Q_i \det \mathbf{H} \,. \tag{303}$$

The list of factors Q_i of $\det \mathbf{H}$ for its i-th derivative is

$$Q_0 = 1 \tag{304}$$

$$Q_1 = e_1 \tag{305}$$

$$Q_2 = e_1^2 - e_2 \tag{306}$$

$$Q_3 = e_1^3 - 3e_1e_2 + 2e_3 \tag{307}$$

$$Q_4 = e_1^4 - 6e_1^2e_2 + 8e_1e_3 + 3e_2^2 - 6e_4 \tag{308}$$

$$Q_5 = e_1^5 - 10e_1^3e_2 + 20e_1^2e_3 + 15e_1e_2^2 - 30e_1e_4 - 20e_2e_3 + 24e_5 \tag{309}$$

$$\begin{aligned} Q_6 = {} & e_1^6 - 15e_1^4e_2 + 40e_1^3e_3 + 45e_1^2e_2^2 - 90e_1^2e_4 \\ & -80e_1e_2e_3 + 144e_1e_5 + 120e_1e_2e_4 - 15e_2^3 \\ & +90e_2e_4 + 40e_3^2 - 120e_6 \,. \end{aligned} \tag{310}$$

For numerical consideration concerning the inverse of $\mathbf{H}(s)$ and its powers see Section 6 of *Weinmann, A., 2004*.

13 Derivative of Square Performance Expressions

13.1 Energy of the State Variable

Consider the weighted energy of the transient $\mathbf{x}(t) = e^{\mathbf{A}_{cl}t}\mathbf{x}_0$ in a closed-loop control system with $\mathbf{A}_{cl} \stackrel{\triangle}{=} \mathbf{A}+\mathbf{BK}$ where $\mathbf{A} \in \mathcal{R}^{n\times n}$, $\mathbf{B} \in \mathcal{R}^{n\times m}$, $\mathbf{K} \in \mathcal{R}^{m\times n}$, and the index of performance I_x based on the state variable $\mathbf{x}(t)$

$$I_x = \int_0^\infty \mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t)dt = \int_0^\infty \mathbf{x}_0^T e^{\mathbf{A}_{cl}^T t}\mathbf{Q}e^{\mathbf{A}_{cl}t}\mathbf{x}_0 dt \stackrel{\triangle}{=} \mathbf{x}_0^T\mathbf{P}_Q\mathbf{x}_0 \,, \tag{311}$$

where $\mathbf{A}_{cl}^T\mathbf{P}_Q + \mathbf{P}_Q\mathbf{A}_{cl} = -\mathbf{Q} \in \mathcal{R}^{n\times n}$ and $\mathbf{P}_Q$ is symmetric (e.g., *Weinmann, A., 1995*). The integral

$$\mathbf{P}_Q = \int_0^\infty e^{\mathbf{A}_{cl}^T t}\mathbf{Q}e^{\mathbf{A}_{cl}t}dt \in \mathcal{R}^{n\times n} \tag{312}$$

cannot be utilized for $\frac{\partial \mathbf{P}_Q}{\partial \mathbf{K}}$ or $\frac{\partial I_x}{\partial \mathbf{K}}$ because $e^{\mathbf{A}+\mathbf{BK}} \neq e^{\mathbf{A}}e^{\mathbf{BK}}$, even for $\mathbf{K}+\Delta\mathbf{K}$ and $\Delta\mathbf{K}$ small in norm sense. However, resolving

$$\mathbf{A}_{cl}^T\mathbf{P}_Q + \mathbf{P}_Q\mathbf{A}_{cl} + \mathbf{Q} = \mathbf{0} \tag{313}$$

with respect to $\mathbf{P}_Q$, its derivation with respect to $\mathbf{K}$ can be obtained employing the first-order Taylor expansion

$$(\mathbf{A}_{cl}^T + \Delta\mathbf{K}^T\mathbf{B}^T)(\mathbf{P}_Q + \Delta\mathbf{P}_Q) + (\mathbf{P}_Q + \Delta\mathbf{P}_Q)(\mathbf{A}_{cl} + \mathbf{B}\Delta\mathbf{K}) + \mathbf{Q} = \mathbf{0}\ . \tag{314}$$

$$\mathbf{E}_a \ \triangleq\ (\mathbf{A}^T \oplus \mathbf{A}^T)^{-1} = (\mathbf{I}_n \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}_n)^{-1} \in \mathcal{R}^{n^2\times n^2}\ . \tag{315}$$

With the additional abbreviation

$$\mathbf{Z}_0 \ \triangleq\ (\mathbf{P}_Q\mathbf{B}) \otimes \mathbf{I}_n + [\mathbf{I}_n \otimes (\mathbf{P}_Q\mathbf{B})]\mathbf{U}_{nm} \ \in \mathcal{R}^{n^2\times mn} \tag{316}$$

one finds the derivative

$$\begin{aligned} \frac{\partial I_x}{\partial \mathbf{K}} &= -\{\mathbf{I}_m \otimes [(\mathbf{x}_0^T \otimes \mathbf{x}_0^T)\ \mathbf{E}_a\mathbf{Z}_0]\}[(\mathrm{col}\ \mathbf{I}_m) \otimes \mathbf{I}_n] & (317)\\ &\overset{(41)}{=} \{\mathrm{loc}_n[\mathbf{Z}_0^T\mathbf{E}_a^T(\mathbf{x}_0 \otimes \mathbf{x}_0)]\}^T\ . & (318) \end{aligned}$$

13.2 Time-Weighted Energy of the State Variable

Now, based on Lyapunov equations, the sensitivity of time-weighted and time-to-powers-weighted energy associated with the state variable is derived for an arbitrary power i

$$\mathbf{P}_{Qi} \triangleq \int_0^\infty e^{\mathbf{A}^Tt}\mathbf{Q}e^{\mathbf{A}t}\ \ t^i\ \ dt \qquad I_{xi} = \mathbf{x}_o^T\mathbf{P}_{Qi}\mathbf{x}_o\ . \tag{319}$$

$$\mathbf{A}_{cl}^T\mathbf{P}_{Qi} + \mathbf{P}_{Qi}\mathbf{A}_{cl} + i\ \mathbf{P}_{Q,i-1} = \mathbf{0}\ , \tag{320}$$

where $\mathbf{P}_{Qo} = \mathbf{P}_Q$. Replacing $\mathbf{Z}_0$ by $\mathbf{Z}_i$ and $\mathbf{P}_Q$ by $\mathbf{P}_{Qi}$ in Eq.(316), one has new matrices $\mathbf{S}_i \in \mathcal{R}^{n^2\times mn}$

$$\begin{aligned} \mathbf{S}_0 &= -\mathbf{E}_a\mathbf{Z}_0 & (321)\\ \mathbf{S}_1 &= -\mathbf{E}_a\mathbf{Z}_1 + \mathbf{E}_a^2\mathbf{Z}_0 & (322)\\ \mathbf{S}_2 &= -\mathbf{E}_a\mathbf{Z}_2 + 2\mathbf{E}_a^2\mathbf{Z}_1 - 2\mathbf{E}_a^3\mathbf{Z}_0 & (323)\\ \mathbf{S}_3 &= -\mathbf{E}_a\mathbf{Z}_3 + 3\mathbf{E}_a^2\mathbf{Z}_2 - 6\mathbf{E}_a^3\mathbf{Z}_1 + 6\mathbf{E}_a^4\mathbf{Z}_0 & (324)\\ \mathbf{S}_4 &= -\mathbf{E}_a\mathbf{Z}_4 + 4\mathbf{E}_a^2\mathbf{Z}_3 - 12\mathbf{E}_a^3\mathbf{Z}_2 + 24\mathbf{E}_a^4\mathbf{Z}_1 - 24\mathbf{E}_a^5\mathbf{Z}_0 & (325)\\ &\vdots & (326)\\ \mathbf{S}_i &= -\mathbf{E}_a\mathbf{Z}_i - i\ \mathbf{E}_a\mathbf{S}_{i-1}\ . & (327) \end{aligned}$$

Using Eq.(327), the increment ΔI_{xi} weighted with the i-th power of t is (*Weinmann, A., 2003*)

$$\Delta I_{xi} = (\mathbf{x}_0^T \otimes \mathbf{x}_0^T)\mathrm{col}\Delta\mathbf{P}_{Qi} \ . \tag{328}$$

The matricial gradient results as

$$\frac{\partial I_{xi}}{\partial \mathbf{K}} = \Big\{\mathbf{I}_m \otimes [(\mathbf{x}_0^T \otimes \mathbf{x}_0^T)\mathbf{S}_i]\Big\}[(\mathrm{col}\ \mathbf{I}_m) \otimes \mathbf{I}_n] \tag{329}$$

$$\stackrel{(41)}{=} \{\mathrm{loc}_n[\mathbf{S}_i^T(\mathbf{x}_0 \otimes \mathbf{x}_0)]\}^T \ . \tag{330}$$

13.3 Actuating Energy

From *Weinmann, A., 2003*, with $\mathbf{K}_y \in \mathcal{R}^{m\times r}$, $\mathbf{C} \in \mathcal{R}^{r\times n}$

$$I_u \triangleq \int_0^\infty \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)dt = \int_0^\infty \mathbf{x}^T(t)\mathbf{C}^T\mathbf{K}_y^T\mathbf{R}\mathbf{K}_y\mathbf{C}\mathbf{x}(t)dt \tag{331}$$

$$= \mathbf{x}_0^T \int_0^\infty e^{\mathbf{A}_{cl}^T t}\mathbf{C}^T\mathbf{K}_y^T\mathbf{R}\mathbf{K}_y\mathbf{C}e^{\mathbf{A}_{cl}t}dt\ \mathbf{x}_0 \triangleq \mathbf{x}_0^T\mathbf{P}_{Ry}\mathbf{x}_0 \ , \tag{332}$$

where

$$\mathbf{A}_{cl}^T\mathbf{P}_{Ry} + \mathbf{P}_{Ry}\mathbf{A}_{cl} + \mathbf{C}^T\mathbf{K}_y^T\mathbf{R}\mathbf{K}_y\mathbf{C} = \mathbf{0} \ . \tag{333}$$

Defining

$$\mathbf{E}_1 \triangleq (\mathbf{I}_n \otimes \mathbf{A}_{cl}^T + \mathbf{A}_{cl}^T \otimes \mathbf{I}_n)^{-1} \in \mathcal{R}^{n^2\times n^2} \tag{334}$$

$$\mathbf{E}_2 \triangleq \mathbf{P}_{Ry}\mathbf{B} + \mathbf{C}^T\mathbf{K}_y^T\mathbf{R} \in \mathcal{R}^{n\times m} \tag{335}$$

$$\frac{\partial I_u}{\partial \mathbf{K}_y} = -\mathbf{E}_2^T\{\mathrm{loc}_n[\mathrm{col}\mathbf{X} + \mathbf{U}_{nn}\mathrm{col}\mathbf{X}]\}^T\mathbf{C}^T = -2\mathbf{E}_2^T\mathbf{X}\mathbf{C}^T \ , \tag{336}$$

where

$$\mathbf{X} \triangleq \mathtt{lyap(Acl, Acl', -x0 * x0')} \tag{337}$$

$$\mathbf{A}_{cl}\mathbf{X} + \mathbf{X}\mathbf{A}_{cl}^T = \mathbf{x}_0\mathbf{x}_0^T \tag{338}$$

$$(\mathbf{I}_n \otimes \mathbf{A}_{cl} + \mathbf{A}_{cl} \otimes \mathbf{I}_n)\mathrm{col}\,\mathbf{X} = \mathrm{col}(\mathbf{x}_0\mathbf{x}_0^T) \tag{339}$$

$$\mathrm{col}\,\mathbf{X} = (\mathbf{A}_{cl} \oplus \mathbf{A}_{cl})^{-1}(\mathbf{x}_0 \otimes \mathbf{x}_0) = \mathbf{E}_1^T(\mathbf{x}_0 \otimes \mathbf{x}_0) \tag{340}$$

$$\mathrm{loc}_r(\mathbf{v}^{[rm\times 1]}) \triangleq \mathbf{V}^{[r\times m]} \tag{341}$$

and r indicates the number of rows produced by loc_r.

Remark 1: There is an equivalent result although it looks quite different, hence the corresponding derivation is carried out: First, suppose

$$\mathbf{y} \triangleq [\mathbf{I}_{n^2} + \mathbf{U}_{nn}]\mathbf{E}_1^T(\mathbf{x}_0 \otimes \mathbf{x}_0) \quad \in \mathcal{R}^{n^2\times 1} \tag{342}$$

$$\mathbf{Y} \stackrel{\triangle}{=} \mathrm{loc}_n[\mathbf{y}]\ , \qquad \mathbf{y} = \mathrm{col}\mathbf{Y} \in \mathcal{R}^{n^2\times 1} \tag{343}$$

$$\{\mathrm{loc}_r[\mathrm{col}\,\mathbf{M}^{[r\times m]}\}^T = \mathbf{M}^T\ . \tag{344}$$

Second, from Eq.(333) $\Delta\mathbf{K}_y \mapsto \Delta\mathbf{P}_{Ry}$, from Eq.(332) $\Delta\mathbf{P}_{Ry} \mapsto \Delta I_u$, together $\Delta\mathbf{K}_y \mapsto \Delta I_u$. Third, using Eq.(212) etc. $\rightsquigarrow$ $\frac{\partial I_u}{\partial \mathbf{K}_y}$

$$\frac{\partial I_u}{\partial \mathbf{K}_y} = -\{\mathbf{I}_m \otimes [(\mathbf{x}_0 \otimes \mathbf{x}_0)^T\ \mathbf{E}_1(\mathbf{I}_{n^2} + \mathbf{U}_{nn})(\mathbf{E}_2 \otimes \mathbf{C}^T)]\}[(\mathrm{col}\mathbf{I}_m) \otimes \mathbf{I}_r] \tag{345}$$

$$\stackrel{(41)}{\equiv} -\Big(\mathrm{loc}_r[(\mathbf{E}_2^T \otimes \mathbf{C})(\mathbf{I}_{n^2} + \mathbf{U}_{nn})\mathbf{E}_1^T(\mathbf{x}_0 \otimes \mathbf{x}_0)]\Big)^T \tag{346}$$

$$= -\{\mathrm{loc}_r[(\mathbf{E}_2^T \otimes \mathbf{C})\mathbf{y}]\}^T = -\{\mathrm{loc}_r[(\mathbf{E}_2^T \otimes \mathbf{C})\mathrm{col}\mathbf{Y}]\}^T \tag{347}$$

$$= -\{\mathrm{loc}_r[\mathrm{col}(\mathbf{C}\mathbf{Y}\mathbf{E}_2)]\}^T \stackrel{(344)}{=} -(\mathbf{C}\mathbf{Y}\mathbf{E}_2)^T \quad [\equiv (351),\ (336)] \tag{348}$$

$$\equiv -\mathbf{E}_2^T\{\mathrm{loc}_n[(\mathbf{I}_{n^2} + \mathbf{U}_{nn})\mathbf{E}_1^T(\mathbf{x}_0 \otimes \mathbf{x}_0)]\}^T\mathbf{C}^T \tag{349}$$

$$\stackrel{(342)}{=} -\mathbf{E}_2^T\{\mathrm{loc}_n[\mathbf{y}]\}^T\mathbf{C}^T = -\mathbf{E}_2^T\mathbf{Y}^T\mathbf{C}^T \tag{350}$$

$$\stackrel{(340)}{=} -\mathbf{E}_2^T\{\mathrm{loc}_n[\mathrm{col}\,\mathbf{X} + \mathbf{U}_{nn}\mathrm{col}\,\mathbf{X}]\}^T\mathbf{C}^T = -2\mathbf{E}_2^T\mathbf{X}\mathbf{C}^T\ . \tag{351}$$

(Note that $\mathbf{Y} = 2\mathbf{X}$. The matrix $\mathbf{X}$ is symmetric, hence $\mathbf{U}_{nn}\mathrm{col}\mathbf{X} = \mathrm{col}\mathbf{X}$. But note: For $\mathbf{B} \in \mathcal{R}^{n\times m}$ one has $\mathrm{loc}_m(\mathrm{col}\mathbf{B}) \neq \mathbf{B}^T$.)

Remark 2: The Lyapunov equation

$$\mathbf{A}^{(n\times n)}\mathbf{X}^{(n\times m)} + \mathbf{X}\mathbf{B}^{(m\times m)} = \mathbf{C}^{(n\times m)} \tag{352}$$

$$\underbrace{(\mathbf{I}_m \otimes \mathbf{A} + \mathbf{B}^T \otimes \mathbf{I}_n)}_{\mathbf{R}}\,\mathrm{col}\mathbf{X} = \mathrm{col}\ \mathbf{C} \tag{353}$$

is only solvable if $\mathbf{R}$ is nonsingular, i.e., iff $\mathbf{R}$ has no zero eigenvalues. The eigenvalues are $\lambda_i + \mu_j$ where $\lambda_i \in \Lambda(\mathbf{A})$ and $\mu_j \in \Lambda(\mathbf{B})$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, i.e., $\mathbf{A}$ and $-\mathbf{B}$ must not have eigenvalues in common. If $\mathbf{A}$ and $\mathbf{B}$ are stable the solution is also given by

$$\mathbf{X} = -\int_0^\infty e^{\mathbf{A}t}\mathbf{C}e^{\mathbf{B}t}dt\ . \tag{354}$$

For eliminating double solutions in Lyapunov equations see *Weinmann, A., 2007d.*

Closed loop versus output controller matrix

$$\frac{\partial(\mathbf{A} + \mathbf{B}\mathbf{K}_y\mathbf{C})}{\partial \mathbf{K}_y^{(m\times r)}} = (\mathbf{I}_m \otimes \mathbf{B}^{(n\times m)})\bar{\mathbf{U}}_{mr}(\mathbf{I}_r \otimes \mathbf{C}^{(r\times n)}) \quad \in \mathcal{R}^{mn\times nr}\ . \tag{355}$$

14 Stability Margin, Continuous-Time Systems, State Space

14.1 Minimum Hodograph Distance

Presupposing a stable control system, the closest point of the closed-loop hodograph (Mikhailow hodograph) to the origin follows from *Weinmann, A., 2005a*

$$\min_\omega |p_{cl}(j\omega)|^2 = \min_\omega p_{cl}^*(j\omega)p_{cl}(j\omega) \stackrel{\triangle}{=} h_0^2\ . \tag{356}$$

Using

$$p_{cl}(j\omega) = \det(j\omega \mathbf{I}_n - \mathbf{A}_{cl}) \tag{357}$$

and differentiating (*Brewer, J.W., 1978*), the optimum frequency ω_0 is determined by

$$f(\omega_0, \mathbf{A}_{cl}) = \mathrm{tr}[\mathrm{adj}(\omega_0^2 \mathbf{I}_n + \mathbf{A}_{cl}^2)] = 0 \tag{358}$$

and the minimum distance h_0 squared is

$$h_0^2 = \det(\omega_0^2 \mathbf{I}_n + \mathbf{A}_{cl}^2)\ . \tag{359}$$

The differential sensitivity of h_0^2 with respect to $\mathbf{K}_y$ results as

$$\frac{\partial \det(\omega_0^2 \mathbf{I}_n + \mathbf{A}_{cl}^2)}{\partial \mathbf{K}_y} = \frac{\partial (h_0^2)}{\partial \mathbf{K}_y} \overset{(12)}{=} 2[\mathbf{C}\mathbf{A}_{cl}(\mathrm{adj}\mathbf{U})\mathbf{B}]^T\ . \tag{360}$$

In a reasonable range the minimum h_0 is obtained for $\omega_0 = 0$ in Eq.(358). Then, from Eq.(360)

$$\frac{\partial (h_0^2)}{\partial \mathbf{K}_y} = 2[\mathbf{C}\mathbf{A}_{cl}^{-1}\mathbf{B}]^T \det(\mathbf{A}_{cl}^2)\ . \tag{361}$$

This result shows a similarity to the gradient of holistic controllers, which minimize $\mathrm{tr}[\mathbf{A}_{cl}^{-1}]$. Their gradient is $\partial \mathrm{tr}[\mathbf{A}_{cl}^{-1}]/\partial \mathbf{K}_y = [\mathbf{C}\mathbf{A}_{cl}^{-2}\mathbf{B}]^T$ (*Weinmann, A., 2005*) .

14.2 Resonant Frequency ω_0 with Respect to the Output State Controller

We evaluate the increment of ω_0 when increasing $\mathbf{K}_y$ with an increment $\Delta \mathbf{K}_y$. From Eq.(358) the total differential expression is (*Weinmann, A., 2005a*)

$$d\,f = \frac{\partial f}{\partial \omega_0} d\omega_0 + \sum_{ij} \frac{\partial f}{\partial K_{y,ij}} dK_{y,ij} = \frac{\partial f}{\partial \omega_0} d\omega_0 + \mathrm{tr}[(\frac{\partial f}{\partial \mathbf{K}_y})^T d\mathbf{K}_y] = 0 \tag{362}$$

$$\mathbf{U} \triangleq \omega_0^2 \mathbf{I}_n + \mathbf{A}_{cl}^2 \overset{(288)}{=} \mathbf{H}(j\omega)\mathbf{H}^\star(j\omega), \tag{363}$$

$$\frac{d\omega_0}{d\mathbf{K}_y} = -\frac{\frac{\partial f}{\partial \mathbf{K}_y}}{\frac{\partial f}{\partial \omega_0}} = \frac{\{\mathbf{C}[\mathbf{U}^{-1} - \mathbf{I}_n \mathrm{tr}\mathbf{U}^{-1}]\mathbf{U}^{-1}\mathbf{A}_{cl}\mathbf{B}\}^T}{\omega_0\{(\mathrm{tr}[\mathbf{U}^{-1}])^2 - \mathrm{tr}[\mathbf{U}^{-2}]\}}\ . \tag{364}$$

14.3 Descriptor Systems

Often the process equations are composed not only of a linear combination of state variables but also a linear combination of the first derivatives. Then,

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{365}$$

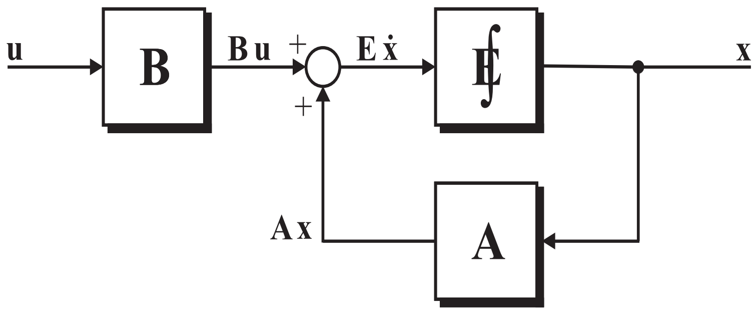

Figure 2: Block diagramm for a descriptor system

is a generalization of $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$. A block diagram can be set up transmitting the linear combination $\mathbf{E}\dot{\mathbf{x}}$ to $\mathbf{x}$ by $\not{E}\!\!\oint$, see Fig. 14.3.

A corresponding coefficient matrix $\mathbf{E}^{-1}\mathbf{A}$ cannot be used since $\mathbf{E}$ is singular in general. Defining

$$\mathbf{Av} = \lambda_g \mathbf{Ev} \tag{366}$$

constitutes the generalized eigenvalue λ_g. Then, $\mathbf{A} - \lambda_g \mathbf{E}$ is termed the matrix pencil of the matrices $\mathbf{A}$ and $\mathbf{E}$ (*Laub, A.J., 2005*). The characteristic polynomial of the matrix pair $(\mathbf{A}, \mathbf{E})$ is $p(s) = \det(s\mathbf{E} - \mathbf{A})$. If, e.g., one of the equations of (365) is purely algebraic, one of the eigenvalues λ_g is $-\infty$, thus expressing the lost dynamics in this equation. From $\mathbf{Aa}_g = \lambda_g \mathbf{Ea}_g$, $\mathbf{a}_g$ is the right generalized eigenvector.

15 Stability Margin, Discrete-Time Systems, State Space

Consider the nth order discrete-time plant with sampling time T_s

$$\mathbf{x}(k+1) = \mathbf{\Phi}(T_s)\mathbf{x}(k) + \mathbf{\Psi u}(k) \quad \in \mathcal{R}^n \tag{367}$$

$$\mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k), \quad \mathbf{D} = \mathbf{0}\ , \tag{368}$$

and a controller $\mathbf{u}(k) = \mathbf{K}_y \mathbf{y}(k) = \mathbf{K}_y \mathbf{Cx}(k)$. The input matrix $\mathbf{\Psi}$ is set constant although there is a slight dependence on T_s.

Then the characteristic equation of the closed-loop system for $z = e^{j\omega T_s}$ is

$$p_{cl}(z)|_{z=e^{j\omega T_s}} = h(z)|_{z=e^{j\omega T_s}} \stackrel{\triangle}{=} \det(e^{j\omega T_s}\mathbf{I}_n - \mathbf{\Phi}_{cl}) = \det(e^{j\omega T_s}\mathbf{I}_n - \mathbf{\Phi} - \mathbf{\Psi K}_y \mathbf{C}) = 0\ . \tag{369}$$

Selecting the closed-loop polynomial and replacing $z = e^{j\omega T_s}$, the hodograph (Mikhailov plot) is obtained (*Weinmann, A., 2006b*). The closest point to the origin at frequency ω_0 results from

$$|\det(e^{j\omega T_s}\mathbf{I}_n - \mathbf{\Phi}_{cl})| \to \min_{\omega}\ , \tag{370}$$

where $\mathbf{\Phi}_{cl} = \mathbf{\Phi}(T_s) + \mathbf{\Psi}\mathbf{K}_y\mathbf{C}$. The minimum distance squared is

$$\begin{aligned} h_0^2 &\triangleq \min_\omega \det(e^{j\omega T_s}\mathbf{I}_n - \mathbf{\Phi}_{cl}) \det(e^{j\omega T_s}\mathbf{I}_n - \mathbf{\Phi}_{cl})^* && (371)\\ &= \min_\omega \det(\mathbf{I}_n - 2\,\mathbf{\Phi}_{cl}\cos\omega T_s + \mathbf{\Phi}_{cl}^2)\ . && (372)\end{aligned}$$

Differentiating Eq.(372) with respect to ω yields the solution ω_0

$$\mathrm{tr}[\mathbf{\Phi}_{cl}\mathrm{adj}(\mathbf{I}_n - 2\mathbf{\Phi}_{cl}\cos\omega_0 T_s + \mathbf{\Phi}_{cl}^2)] = 0 \tag{373}$$

$$\frac{\partial(h_o^2)}{\partial \mathbf{K}_y} = 2\mathbf{\Psi}\mathbf{U}_{d0}^{-T}[\mathbf{\Phi}_{cl}^T - \mathbf{I}_n\cos\omega_0 T_s]\mathbf{C}^T \det\mathbf{U}_{d0}\ , \tag{374}$$

$$\text{where} \quad \mathbf{U}_{d0} \triangleq \mathbf{I}_n - 2\mathbf{\Phi}_{cl}\cos\omega_0 T_s + \mathbf{\Phi}_{cl}^2. \tag{375}$$

Maximizing the *integral* of the stability margin in a discrete-time system

$$I = \int_0^{\omega_T} |p_{cl}(j\omega)|^2 d\omega = \int_0^{\omega_T} (\det\mathbf{U}_d)\ d\omega \to \max_{\mathbf{K}_y}\ , \tag{376}$$

where Eqs.(369) and $\mathbf{U}_d = \mathbf{I}_n - 2\mathbf{\Phi}_{cl}\cos\omega T_s + \mathbf{\Phi}_{cl}^2$ are used. For $\omega_T = 2\pi/T_s$, one finds

$$\frac{\partial I}{\partial \mathbf{K}_y} = 2\mathbf{\Psi}^T \int_0^{\omega_T/2} (\mathbf{\Phi}_{cl}^T - \mathbf{I}_n\cos\omega T_s)(\mathrm{adj}^T\mathbf{U}_d)d\omega\cdot\mathbf{C}^T\ . \tag{377}$$

16 Stability Margin, Continuous-Time Transfer Functions

16.1 Single-Input Single-Output Sytems

The closed-loop characteristic polynomial results from $1 + G(s)K(s)$ as

$$h(s) = (\mathbf{n}_k^T\ \mathbf{s})(\mathbf{n}_G^T\ \mathbf{s}) + (\mathbf{z}_k^T\ \mathbf{s})(\mathbf{z}_G^T\ \mathbf{s})\ , \tag{378}$$

where

$$\mathbf{s} \triangleq (1\quad s\quad s^2\ \dots\ s^n)^T \stackrel{(4)}{=} \sum_0^n s^i \mathbf{e}_{i+1}^{[n+1]} \quad \in \mathcal{C}^{n+1}. \tag{379}$$

By replacing $s = j\omega$, the distance (squared) from the origin to the so-called Mikhailov plot is (*Weinmann, A., 2006*)

$$\begin{aligned} |h(j\omega)|^2 &= [n_K(s)n_G(s) + z_K(s)z_G(s)]\\ &\quad \times[n_K(s)\ n_G(s) + z_K(s)\ z_G(s)]^*\Big|_{s=j\omega} && (380)\\ &= \mathbf{n}_K^T\ \mathbf{s}\cdot\mathbf{n}_K^T\ \mathbf{s}^*\cdot\mathbf{n}_G^T\ \mathbf{s}\cdot\mathbf{n}_G^T\ \mathbf{s}^* + \dots\Big|_{s=j\omega} && (381)\\ &= \mathbf{n}_K^T\mathbf{S}\mathbf{n}_K\cdot\mathbf{n}_G^T\mathbf{S}\mathbf{n}_G + \dots\Big|_{s=j\omega}\ , && (382)\end{aligned}$$

where

$$\mathbf{S} \triangleq \mathbf{s}\mathbf{s}^H \in \mathcal{C}^{(n+1)\times(n+1)} , \qquad \mathbf{s} \triangleq (1 \quad s \quad s^2 \quad \ldots \quad s^n)^T , \tag{383}$$

is the dyadic product and $\mathbf{s}^H \triangleq (\mathbf{s}^*)^T$ is the "Hermite" conjugate transpose.

The coefficients of the polynomials $\mathbf{z}_K$ in the numerator are collected in a vector $\mathbf{z}_K$, which is the first half of the parameter vector $\mathbf{p}_{ij}$; the denominator polynomial is $n_K(s)$ corresponding to the second half of the parameter vector $\mathbf{p}_{ij}$. Hence, one has

$$\mathbf{p}_K \triangleq \begin{pmatrix} \mathbf{z}_K \\ \mathbf{n}_K \end{pmatrix}, \quad \mathbf{z}_K = \mathbf{P}_z \mathbf{p}_K, \quad \mathbf{n}_K = \mathbf{P}_n \mathbf{p}_K, \tag{384}$$

$$\mathbf{p}_G \triangleq \begin{pmatrix} \mathbf{z}_G \\ \mathbf{n}_G \end{pmatrix}, \quad \mathbf{z}_G = \mathbf{P}_z \mathbf{p}_G, \quad \mathbf{n}_G = \mathbf{P}_n \mathbf{p}_G, \tag{385}$$

$$\text{where } \mathbf{P}_z \triangleq (\mathbf{I}_n \ \vdots \ \mathbf{0}_n), \qquad \mathbf{P}_n \triangleq (\mathbf{0}_n \ \vdots \ \mathbf{I}_n) \in \mathcal{R}^{n\times 2n} . \tag{386}$$

The derivative of the squared absolute value $|v|^2$ with respect to a real-valued parameter is

$$\frac{\partial |v|^2}{\partial \mathbf{p}} = \frac{\partial (vv^*)}{\partial \mathbf{p}} = \frac{\partial v}{\partial \mathbf{p}} v^* + v \frac{\partial v^*}{\partial \mathbf{p}} \tag{387}$$

$$\begin{aligned} &= \frac{\partial(\Re e\ v + j\Im m\ v)}{\partial \mathbf{p}}(\ \Re e\ v - j\Im m\ v) \\ &\quad +(\Re e\ v + j\Im m\ v)\frac{\partial(\Re e\ v - j\Im m\ v)}{\partial \mathbf{p}} \end{aligned} \tag{388}$$

$$\frac{\partial |v|^2}{\partial \mathbf{p}} = 2\left[(\Re e\ v)\ \frac{\partial(\Re e\ v)}{\partial \mathbf{p}} + (\Im m\ v)\frac{\partial(\Im m\ v)}{\partial \mathbf{p}}\right] = 2\ \Re e\ [v^* \frac{\partial v}{\partial \mathbf{p}}] . \tag{389}$$

Continuing from Eq.(382), one has

$$|h(j\omega)|^2 = [n_K n_K^* n_G n_G^* + z_K n_K^* z_G n_G^* + n_K z_K^* n_G z_G^* + z_K z_K^* z_G z_G^*]\Big|_{s=j\omega} \tag{390}$$

$$\begin{aligned} &\overset{(383)}{=} [\mathbf{p}_K^T \mathbf{P}_n^T \mathbf{S} \mathbf{P}_n \mathbf{p}_K \mathbf{n}_G^T \mathbf{S} \mathbf{n}_G + \mathbf{p}_K^T \mathbf{P}_z^T \mathbf{S} \mathbf{P}_n \mathbf{p}_K \mathbf{z}_G^T \mathbf{S} \mathbf{n}_G \\ &\quad + \mathbf{p}_K^T \mathbf{P}_n^T \mathbf{S} \mathbf{P}_z \mathbf{p}_K \mathbf{n}_G^T \mathbf{S} \mathbf{z}_G + \mathbf{p}_K^T \mathbf{P}_z^T \mathbf{S} \mathbf{P}_z \mathbf{p}_K \mathbf{z}_G^T \mathbf{S} \mathbf{z}_G]\Big|_{s=j\omega} . \end{aligned} \tag{391}$$

Because $\mathbf{n}_G$ is fixed it is kept unchanged in Eq.(392) in what follows; $\mathbf{n}_K$ and $\mathbf{z}_K$ are separated from $\mathbf{P}_K$ in order to be operated by the gradient method. Using the definitions

$$\mathbf{A}_G(s) \triangleq \begin{pmatrix} \mathbf{n}_G^T \\ \mathbf{z}_G^T \end{pmatrix} \mathbf{S}(\mathbf{n}_G \ \vdots \ \mathbf{z}_G) = \begin{pmatrix} \mathbf{n}_G^T \mathbf{S} \mathbf{n}_G & \mathbf{n}_G^T \mathbf{S} \mathbf{z}_G \\ \mathbf{z}_G^T \mathbf{S} \mathbf{n}_G & \mathbf{z}_G^T \mathbf{S} \mathbf{z}_G \end{pmatrix} \tag{392}$$

$$\mathbf{B}_K(s) \triangleq \begin{pmatrix} \mathbf{P}_n^T \\ \mathbf{P}_z^T \end{pmatrix} \mathbf{S}(\mathbf{P}_n \vdots \mathbf{P}_z) = \begin{pmatrix} \mathbf{P}_n^T \mathbf{S} \mathbf{P}_n & \mathbf{P}_n^T \mathbf{S} \mathbf{P}_z \\ \mathbf{P}_z^T \mathbf{S} \mathbf{P}_n & \mathbf{P}_z^T \mathbf{S} \mathbf{P}_z \end{pmatrix} \tag{393}$$

and an elementwise (Hadamard) product of matrices

$$\begin{aligned}
\mathbf{F} &\triangleq A_{G11}(s)\mathbf{B}_{K11}(s) + A_{G21}(s)\mathbf{B}_{K21}(s) + A_{G12}(s)\mathbf{B}_{K12}(s) + A_{G22}(s)\mathbf{B}_{K22}(s) \\
&\triangleq \mathbf{A}_G(s) \ .* \ \mathbf{B}_K(s) = \mathrm{btr}[\mathbf{A}_G^T(s)\mathbf{B}_K(s)], \qquad (394)
\end{aligned}$$

where btr$[\cdot]$ is a blocktrace. Applying block transposition in $\mathbf{F}$ does not influence the result. $\mathbf{S}$ is Hermite, $\mathbf{S} = \mathbf{S}^H$. Hence, one can state $\mathbf{F} = \mathbf{F}^H$ in this SISO case. Then,

$$|h(j\omega)|^2 = \mathbf{p}_K^T \mathbf{F}\Big|_{s=j\omega} \mathbf{p}_K \ . \qquad (395)$$

The differential quotient is

$$\begin{aligned}
\frac{\partial |h(j\omega)|^2}{\partial \mathbf{p}_K} &= 2\Big(\mathbf{P}_n^T\mathbf{S}\mathbf{P}_n\mathbf{n}_G^T\mathbf{S}\mathbf{n}_G + \mathbf{P}_z^T\mathbf{S}\mathbf{P}_n\mathbf{z}_G^T\mathbf{S}\mathbf{n}_G \\
&\quad + \mathbf{P}_n^T\mathbf{S}\mathbf{P}_z\mathbf{n}_G^T\mathbf{S}\mathbf{z}_G + \mathbf{P}_z^T\mathbf{S}\mathbf{P}_z\mathbf{z}_G^T\mathbf{S}\mathbf{z}_G\Big)\Big|_{s=j\omega}\mathbf{p}_K \qquad (396) \\
&= \frac{\partial \mathbf{p}_K^T\mathbf{F}\mathbf{p}_K}{\partial \mathbf{p}} = (\mathbf{F} + \mathbf{F}^T)\mathbf{p} \qquad (397) \\
\frac{\partial |h(j\omega)|^2}{\partial \mathbf{p}_K}\Big|_{\omega=\omega_0} &= 2\Re e\, \mathbf{F}\Big|_{s=j\omega_0}\mathbf{p}_K = 2\Re e \sum_{i=1,j=1}^{i=2,j=2} [A_{Gij}\mathbf{B}_{Kij}]\Big|_{j\omega_0}\mathbf{p}_K \ . \qquad (398)
\end{aligned}$$

16.2 Derivative of a Transfer Function

Suppose

$$K_{ij} = \frac{\mathbf{z}^T\mathbf{s}}{\mathbf{n}^T\mathbf{s}} = \frac{\mathbf{s}^T\mathbf{P}_z\mathbf{p}}{\mathbf{s}^T\mathbf{P}_n\mathbf{p}} \ . \qquad (399)$$

Differentiating with respect to the νth element of $\mathbf{p}$ yields (with $\mathbf{s}$ from Eq.(379) and $\mathbf{S} \triangleq \mathbf{s}\mathbf{s}^H$)

$$\begin{aligned}
\frac{\partial K_{ij}(s)}{\partial p_\nu} &= \frac{1}{\mathbf{s}^T\mathbf{P}_n\mathbf{p}}\mathbf{s}^T\mathbf{P}_z\mathbf{e}_\nu + \frac{\mathbf{s}^T\mathbf{P}_z\mathbf{p}}{(\mathbf{s}^T\mathbf{P}_n\mathbf{p})^2}(-1)\mathbf{s}^T\mathbf{P}_n\mathbf{e}_\nu \qquad (400) \\
&= \frac{1}{(\mathbf{s}^T\mathbf{P}_n\mathbf{p})^2}\mathbf{e}_\nu^T[\mathbf{P}_z^T\mathbf{S}\mathbf{P}_n - \mathbf{P}_n^T\mathbf{S}\mathbf{P}_z]\mathbf{p} \qquad (401) \\
\frac{\partial K_{ij}(s)}{\partial \mathbf{p}} &= \frac{1}{(\mathbf{s}^T\mathbf{P}_n\mathbf{p})^2}[\mathbf{P}_z^T\mathbf{S}\mathbf{P}_n - \mathbf{P}_n^T\mathbf{S}\mathbf{P}_z]\mathbf{p} \ . \qquad (402)
\end{aligned}$$

17 Stability Margin

17.1 Stability Margin, Polynomial Matrices

Consider a multiple-input multiple-output controller and plant given by polynomial matrices $\in \mathcal{C}^{n\times n}$

$$\mathbf{K}(s) = \mathbf{N}_K^{-1}(s)\mathbf{Z}_K(s) \ , \qquad \mathbf{G}(s) = \mathbf{Z}_G(s)\mathbf{N}_G^{-1}(s). \qquad (403)$$

Then, the characteristic equation of the closed-loop system is

$$\det \mathbf{Q} \triangleq \det[\mathbf{Z}_K(s)\mathbf{Z}_G(s) + \mathbf{N}_K(s)\mathbf{N}_G(s)] = 0. \tag{404}$$

The Mikhailov hodograph results from the characteristic polynomial by replacing s by $j\omega$ and the results is (*Weinmann, A., 2006*)

$$\det[\mathbf{Z}_K(j\omega)\mathbf{Z}_G(j\omega) + \mathbf{N}_K(j\omega)\mathbf{N}_G(j\omega)] = \det \mathbf{Q} \tag{405}$$

$$\begin{aligned} \frac{\partial h_0^2}{\partial p_{ijk}} = \frac{\partial |\det \mathbf{Q}|^2}{\partial p_{ijk}} = 2|\det \mathbf{Q}|^2 \quad & \Big[\Re e\,(\mathbf{Z}_G\mathbf{Q}^{-1})_{ji}\frac{\partial \Re e\, Z_K(i,j)}{\partial p_{ijk}} \\ & -\Im m\,(\mathbf{Z}_G\mathbf{Q}^{-1})_{ji}\frac{\partial \Im m\, Z_K(i,j)}{\partial p_{ijk}} \\ & +\Re e\,(\mathbf{N}_G\mathbf{Q}^{-1})_{ji}\frac{\partial \Re e\, N_K(i,j)}{\partial p_{ijk}} \\ & -\Im m\,(\mathbf{N}_G\mathbf{Q}^{-1})_{ji}\frac{\partial \Re e\, N_K(i,j)}{\partial p_{ijk}}\Big]. \end{aligned} \tag{406}$$

17.2 Inverse sensitivity

The maximum absolute value of the sensitivity function $S(j\omega)$ yields the worst steady-state oscillation and its corresponding frequency, the closed-loop system reacts with the maximum deviation. Referring to

$$\omega_0 = \arg\max_{\omega} |S(j\omega)| = \arg\min_{\omega} |S^{-1}(j\omega)| \tag{407}$$

and

$$f^2 \triangleq |\det(\mathbf{I} + \mathbf{G}\mathbf{K})|^2 = \det(\mathbf{I} + \mathbf{G}\mathbf{K})\det(\mathbf{I} + \mathbf{G}^*\mathbf{K}^*) \tag{408}$$

find

$$f_0 \triangleq \min_{\omega} |S^{-1}(j\omega)| = \min_{\omega} |\det(\mathbf{I} + \mathbf{G}\mathbf{K})| \ . \tag{409}$$

The gradient of f_0^2 with respect to the matrix elements K_{ij} and its determining parameter $\mathbf{p}_{ij}$ is

$$\begin{aligned} \frac{\partial f_0^2}{\partial \mathbf{p}_{ij}} \ = \ & 2|\det(\mathbf{I} + \mathbf{G}\mathbf{K})|^2 \Big\{[\Re e\,(\mathbf{I} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}]_{ji}\frac{\partial \Re e\, K_{ij}}{\partial \mathbf{p}_{ij}} \\ & - [\Im m\,(\mathbf{I} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}]_{ji}\frac{\partial \Im m\, K_{ij}}{\partial \mathbf{p}_{ij}}\Big\} \ . \end{aligned} \tag{410}$$

17.3 Stability Margin, Power-Oriented Matrices

Suppose now, the polynomial matrix $\mathbf{N}(s)$ is split into separate coefficient matrices $\mathbf{N}_i$

$$\mathbf{N}(s) = \sum_0^n \mathbf{N}_i s^i \;\in \mathcal{C}^{m\times m}. \tag{411}$$

Concatenating these coefficient matrices in a hyper parameter matrix $\mathbf{P}_N$

$$\mathbf{P}_N \triangleq (\mathbf{N}_0 \vdots \mathbf{N}_1 \vdots \ldots \vdots \mathbf{N}_i \vdots \ldots \vdots \mathbf{N}_n) \in \mathcal{R}^{m\times m(n+1)}\ . \tag{412}$$

Consider both the plant and controller given by fractions of polynomial matrices $\mathbf{G}(s) \triangleq \mathbf{Z}_G \mathbf{N}_G^{-1}$ and $\mathbf{K}(s) \triangleq \mathbf{N}_K^{-1} \mathbf{Z}_K$, respectively; in this specific order. Moreover,

$$\mathbf{N} \triangleq \mathbf{N}_K \mathbf{N}_G, \quad \mathbf{Z} \triangleq \mathbf{Z}_K \mathbf{Z}_G \tag{413}$$

$$\mathbf{N}_G \triangleq \mathbf{P}_{NG}\mathbf{S}_\sigma, \quad \mathbf{N}_K \triangleq \mathbf{P}_{NK}\mathbf{S}_\sigma, \quad \mathbf{Z}_G \triangleq \mathbf{P}_{ZG}\mathbf{S}_\sigma, \quad \mathbf{Z}_K \triangleq \mathbf{P}_{ZK}\mathbf{S}_\sigma, \tag{414}$$

where $\mathbf{P}_{ZK},\ \mathbf{P}_{NK},\ \mathbf{P}_{NG},\ \mathbf{P}_{ZG}\ \in \mathcal{R}^{m\times m(n+1)}$ and

$$\mathbf{S}_\sigma \triangleq \begin{pmatrix} \mathbf{I}_m \\ \mathbf{I}_m s \\ \vdots \\ \mathbf{I}_m s^i \\ \vdots \\ \mathbf{I}_m s^n \end{pmatrix} \overset{(379)}{\equiv} \mathbf{s} \otimes \mathbf{I}_m \quad \in \mathcal{C}^{m(n+1)\times m}, \quad \begin{matrix}\text{e.g., for} \\ m=3 \\ n=2:\end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \\ s^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s^2 \end{pmatrix}. \tag{415}$$

Then the open-loop transfer function of the multivariable system is $\mathbf{N}^{-1}\mathbf{Z}$. From the return difference $\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s) = \mathbf{I}+\mathbf{N}_K^{-1}\mathbf{Z}_K\mathbf{Z}_G\mathbf{N}_G^{-1}$, by pre- and postmultiplying there results $\mathbf{N} + \mathbf{Z}$. Keeping plant and controller separated, the characteristic polynomial of the closed-loop system is

$$h(s) = \det[\mathbf{N}(s) + \mathbf{Z}(s)] = \det[\mathbf{N}_K(s)\mathbf{N}_G(s) + \mathbf{Z}_K(s)\mathbf{Z}_G(s)]\ . \tag{416}$$

From the frequency hodograph (Mikhailov hodograph) the stability margin is defined $h_0 \triangleq \min_\omega |h(j\omega)|^2$. Changing $h_0^2 = |h(j\omega_0)|^2$ stepwise with respect to $\mathbf{P}_{ZK}$ and $\mathbf{P}_{NK}$, one finds (*Weinmann, A., 2007a*)

$$0.5\frac{\partial(h_0^2)}{\partial\binom{\mathbf{P}_{ZK}}{\mathbf{P}_{NK}}} = |\det(\mathbf{N}+\mathbf{Z})|^2\ \Re e\ [\mathbf{S}_\sigma(\mathbf{Z}_G \vdots \mathbf{N}_G)\mathrm{bd}\{(\mathbf{N}+\mathbf{Z})^{-1}\}]^T\ . \tag{417}$$

The operator "bd" terms block diagonal, i.e., $\mathrm{bd}(\mathbf{J}) \triangleq \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{pmatrix}$. The problem subject to the condition $||\Delta\mathbf{P}_K||_F = n_{Ko}$, see *Weinmann, A., 2007a*.

18 Nyquist Encirclements, Transfer Functions

The encirclements of $1 + F_o(j\omega)$ around the origin, corresponding to the Nyquist criterion as derived from the open-loop transfer function (not coinciding with the encirclements of the Mikhailov hodograph), are determined by (*Weinmann, A., 2007b*)

$$1 + F_o(j\omega) \stackrel{\triangle}{=} \frac{z(s)}{p(s)}|_{s=j\omega} \stackrel{\triangle}{=} \frac{\mathbf{z}^T\mathbf{s}}{\mathbf{n}^T\mathbf{s}}|_{s=j\omega} \quad = \quad f\ e^{j\beta} \tag{418}$$

$$\Big[\ln(\mathbf{z}^T\mathbf{s}) - \ln(\mathbf{n}^T\mathbf{s})\Big]_{s=j\omega} \quad \stackrel{\triangle}{=} \quad \ln f + j\beta. \tag{419}$$

The abbreviations $\mathbf{s} \stackrel{\triangle}{=} (1 \quad s^1 \quad s^2 \quad \dots \quad s^n)$ and $\frac{\partial \mathbf{s}}{\partial s} = (0 \quad 1 \quad 2s \quad \dots \quad ns^{n-1})$ are used. In addition, $\frac{\partial^2 \mathbf{s}}{\partial s^2} = (0 \ \vdots \ 0 \ \vdots \ 2 \ \vdots \ 6s \ \vdots \ 12s^2 \ \vdots \dots \vdots \ n(n-1)s^{n-2})^T$. Then,

$$\beta \quad = \quad [-j\ln(\mathbf{z}^T\mathbf{s}) + j\ln(\mathbf{n}^T\mathbf{s}) + \ln f]_{s=j\omega} = \Im m\ [\ln(\mathbf{z}^T\mathbf{s}) - \ln(\mathbf{n}^T\mathbf{s})]_{s=j\omega} \tag{420}$$

$$\frac{\partial \beta}{\partial \omega} \quad = \quad \frac{\partial \beta}{\partial s}\frac{\partial s}{\partial \omega} = \frac{\partial \beta}{\partial s} \cdot j = \left[(\frac{1}{\mathbf{z}^T\mathbf{s}}\mathbf{z}^T - \frac{1}{\mathbf{n}^T\mathbf{s}}\mathbf{n}^T)\frac{\partial \mathbf{s}}{\partial s}\right]_{s=j\omega} + j\frac{\partial f}{\partial \omega} \tag{421}$$

$$\frac{\partial \beta}{\partial \omega} \quad = \quad \Re e\ \left[(\frac{\mathbf{z}^T}{\mathbf{z}^T\mathbf{s}} - \frac{\mathbf{n}^T}{\mathbf{n}^T\mathbf{s}})\frac{\partial \mathbf{s}}{\partial s}\right]_{s=j\omega} \tag{422}$$

$$\frac{\partial^2 \beta}{\partial \omega^2} \quad = \quad \Im m\ \left[\frac{1}{(\mathbf{z}^T\mathbf{s})^2}(\mathbf{z}^T\frac{\partial \mathbf{s}}{\partial s})^2 - \frac{1}{(\mathbf{n}^T\mathbf{s})^2}(\mathbf{n}^T\frac{\partial \mathbf{s}}{\partial s})^2 - (\frac{\mathbf{z}^T}{\mathbf{z}^T\mathbf{s}} - \frac{\mathbf{n}^T}{\mathbf{n}^T\mathbf{s}})\frac{\partial^2 \mathbf{s}}{\partial s^2}\right]_{s=j\omega}. \tag{423}$$

18.1 Derivative of the Closed-Loop Encirclements

For a linear continuous-time dynamic system with the coefficient matrix $\mathbf{A}_{cl}$, stability requires that arg $h(j\omega)$ increases monotonically with rising ω, where

$$h(j\omega) \stackrel{\triangle}{=} \det(j\omega\mathbf{I}_n - \mathbf{A}_{cl}) \tag{424}$$

$$\mathbf{A}_{cl} \stackrel{\triangle}{=} \mathbf{A} + \mathbf{B}\mathbf{K}_y\mathbf{C} \tag{425}$$

and $\mathbf{A} \in \mathcal{R}^{n\times n}$; $\mathbf{B} \in \mathcal{R}^{n\times m}$; $\mathbf{C} \in \mathcal{R}^{r\times n}$, $\mathbf{K}_y \in \mathcal{R}^{m\times r}$. The open-loop coefficient matrix is $\mathbf{A}$, the closed-loop one $\mathbf{A}_{cl}$; $\mathbf{K}_y$ is the output state controller matrix. Focus is put on the phase of $h(j\omega) = h_0 e^{j\alpha(\omega)}$ and its differential quotients with respect to ω (*Weinmann, A., 2007b*)

$$\frac{\partial \alpha(\omega)}{\partial \omega} \quad = \quad \Re e\ \mathrm{tr}[(j\omega\mathbf{I}_n - \mathbf{A}_{cl})^{-1}] \tag{426}$$

$$\frac{\partial^2 \alpha(\omega)}{\partial \omega^2} \quad \stackrel{(426)}{=} \quad \Re e\ \mathrm{tr}[(j\omega\mathbf{I}_n - \mathbf{A}_{cl})^{-2} \cdot j] = -\Im m\ \mathrm{tr}[(j\omega\mathbf{I}_n - \mathbf{A}_{cl})^{-2}]\ . \tag{427}$$

19 Phase and Gain Margin Gradient

Consider a nominal plant $G(s) = \frac{z_G(s)}{n_G(s)}$ and a nominal controller $K(s) = \frac{z_K(s)}{p_K(s)} = \frac{\mathbf{p}_{Kz}\mathbf{s}}{\mathbf{p}_{Kn}\mathbf{s}}$ where $\mathbf{s} \triangleq (1 \;\; s \;\; s^2 \;\; \dots \;\; s^n)^T \;\; \in \mathcal{C}^{n+1}$ from Eq.(379). The order of the plant and the controller is n_G and n_K, respectively. The coefficients of the polynomials $z_K(s)$ in the numerator are collected in a vector $\mathbf{p}_{Kz}$, which is the first half of the parameter vector $\mathbf{p}_K$; the denominator polynomial is $p_K(s)$ corresponding to the second half of the parameter vector $\mathbf{p}_K$. Hence, one has

$$\mathbf{p}_K \;\; \triangleq \;\; \begin{pmatrix} \mathbf{p}_{Kz} \\ \mathbf{p}_{Kn} \end{pmatrix}, \quad \mathbf{p}_{Kz} = \mathbf{P}_z \mathbf{p}_K, \quad \mathbf{p}_{Kn} = \mathbf{P}_n \mathbf{p}_K \quad \in \mathcal{R}^{n_K+1}, \tag{428}$$

$$\mathbf{p}_G \;\; \triangleq \;\; \begin{pmatrix} \mathbf{p}_{Gz} \\ \mathbf{p}_{Gn} \end{pmatrix}, \quad \mathbf{p}_{Gz} = \mathbf{P}_z \mathbf{p}_G, \quad \mathbf{p}_{Gn} = \mathbf{P}_n \mathbf{p}_G \quad \in \mathcal{R}^{n_G+1}, \tag{429}$$

$$\text{where } \mathbf{P}_z \triangleq (\mathbf{I}_{n+1} \vdots \mathbf{0}_{n+1}), \qquad \mathbf{P}_n \triangleq (\mathbf{0}_{n+1} \vdots \mathbf{I}_{n+1}) \in \mathcal{R}^{(n+1)\times 2(n+1)} \tag{430}$$

for $n = n_K$ or $n = n_G$, respectively. The open-loop transfer function is

$$F_0(s) = G(s)K(s) = G(s)\frac{\mathbf{p}_{Kz}\mathbf{s}}{\mathbf{p}_{Kn}\mathbf{s}}\Big|_{s=j\omega} \;\; \triangleq \;\; f_0 e^{j\beta_0} \tag{431}$$

$$\ln G + \ln(\mathbf{p}_{Kz}\mathbf{s}) - \ln(\mathbf{p}_{Kn}\mathbf{s}) \;\; = \;\; \ln f_0 + j\beta_0 \;\; . \tag{432}$$

$$\beta_0 = \Im m \; [\ln G(s) + \ln(\mathbf{p}_{Kz}\mathbf{s}) - \ln(\mathbf{p}_{Kn}\mathbf{s})]_{s=j\omega_D} = \beta_0(\omega_D) \; . \tag{433}$$

The classical phase margin α_R results from

$$\alpha_R = \pi + \arg G(s)K(s)\Big|_{s=j\omega_D} = \pi + \beta_0(\omega_D) \; , \qquad |G(j\omega_D)K(j\omega_D)| = 1 \; , \tag{434}$$

$$\frac{\partial \alpha_R}{\partial \mathbf{p}_K} = \Im m \; [(\frac{\mathbf{P}_z^T}{\mathbf{p}_K^T \mathbf{P}_z^T \mathbf{s}} - \frac{\mathbf{P}_n^T}{\mathbf{p}_K^T \mathbf{P}_n^T \mathbf{s}})\mathbf{s}]_{s=j\omega_D} \tag{435}$$

and in state space

$$\frac{\partial \alpha_R}{\partial \mathbf{k}} = \Im m \; [\frac{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}}{\mathbf{k}^T(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}}\Big|_{s=j\omega_D} \;] \; . \tag{436}$$

The gain margin gradient is

$$\frac{\partial \ln 1/A_R}{\partial \mathbf{p}_K} = -\frac{\partial \ln A_R}{\partial \mathbf{p}_K} = \frac{\partial f_0}{\partial \mathbf{p}_K} = \Re e \; [\frac{1}{\mathbf{p}_K^T \mathbf{P}_z^T \mathbf{s}} \mathbf{P}_z^T - \frac{1}{\mathbf{p}_K^T \mathbf{P}_n^T \mathbf{s}} \mathbf{P}_n^T]_{j\omega_R} \; , \tag{437}$$

where $\arg F_0(j\omega_R) = -\pi$.

Based on frequency domain functions, for the crossover frequency gradient one finds

$$\frac{\partial \omega_D}{\partial \mathbf{p}_K} = -\frac{|G|^2 \; \Re e \; [K^* \frac{\partial K}{\partial \mathbf{p}_K}]}{|K|^2 \; \Re e \; [G^* \frac{\partial G}{\partial \omega_D}] + |G|^2 \; \Re e \; [K^* \frac{\partial K}{\partial \omega_D}]} \; . \tag{438}$$

For the plant $(\mathbf{A}, \mathbf{b})$ and the state space controller $\mathbf{k}$

$$\frac{\partial \omega_D}{\partial \mathbf{k}} = \frac{\Re e \; [\mathbf{k}^T \mathbf{R}(-j\omega)\mathbf{b} \cdot \mathbf{R}(j\omega)\mathbf{b}]}{\Im m \; [\mathbf{k}^T \mathbf{R}(-j\omega)\mathbf{b} \cdot \mathbf{k}^T \mathbf{R}^2(j\omega)\mathbf{b}]}, \quad \text{where } \; \mathbf{R}(j\omega) \triangleq j\omega \mathbf{I} - \mathbf{A} \tag{439}$$

Weinmann, A., 2008.

20 Limit Cycle Frequency

Consider a single-loop system with a series interconnection of a nonlinear element N, the plant G and a linear controller K. Presuppose the existence of a limit cycle with control error e_r and frequency ω_r. Then, for the gradient of the limit cycle error amplitude and frequency, we utilize $1 + N(e_r)G(j\omega_r)K(j\omega_r) = 0$. Defining

$$a \triangleq \frac{\partial N}{\partial e_r} GK, \quad b \triangleq N(\frac{\partial G}{\partial \omega_r} K, \quad c \triangleq NG, \tag{440}$$

$$\begin{pmatrix} \frac{\partial e_r}{\partial \mathbf{p}_K} \\ \frac{\partial \omega_r}{\partial \mathbf{p}_K} \end{pmatrix} = - \begin{pmatrix} \Re e[a]\ \mathbf{I} & \Re e[b]\ \mathbf{I} \\ \Im m[a]\ \mathbf{I} & \Im m[b]\ \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} \Re e[c]\ \mathbf{I} & -\Im m[c]\ \mathbf{I} \\ \Im m[c]\ \mathbf{I} & \Re e[c]\ \mathbf{I} \end{pmatrix} \begin{pmatrix} \Re e\ \frac{\partial K}{\partial \mathbf{p}_K} \\ \Im m\ \frac{\partial K}{\partial \mathbf{p}_K} \end{pmatrix} \tag{441}$$

is achieved (*Weinmann, A., 2008a*).

21 Interaction Energy Gradient, Continuous Time

For the closed-loop system of Fig. 3 with output controller $\mathbf{K}_y$ and prefilter $\mathbf{V}_F$, one has

$$\mathbf{y}(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C})^{-1}\mathbf{B}\mathbf{V}_F\mathbf{y}_{ref}(s)\ . \tag{442}$$

From the input $y_{ref,k} = 1$ to the output $y_i(s)$, the impulse response is

$$y_i(s) = \mathbf{c}_i^T(s\mathbf{I}_n - \mathbf{A}_{cl})^{-1}\mathbf{w}_k \tag{443}$$

where $\mathbf{w}_k$ is the corresponding column of $\mathbf{W} \triangleq \mathbf{B}\mathbf{V}_F$.

The controllability Gramian $\mathbf{L}_{ck}$ for the open-loop single input system with input matrix $\mathbf{b}_k$ results from

$$\mathbf{A}\mathbf{L}_{ck} + \mathbf{L}_{ck}\mathbf{A}^T + \mathbf{b}_k\mathbf{b}_k^T = \mathbf{0}\ . \tag{444}$$

In the Lyapunov Eq.(444), $\mathbf{A}$ has to be replaced by $\mathbf{A}_{cl} \triangleq \mathbf{A} + \mathbf{B}\mathbf{K}_y\mathbf{C}$ for the closed-loop system of Fig. 3, and $\mathbf{b}_k$ by $\mathbf{w}_k$.

To design a system with low interaction, the output y_i should be influenced as low as possible by $y_{ref,k}$ for $k \neq i$. Hence, for unequally enumerated reference and output signals the entire energy associated with the interaction should obey to

$$\sum_{i=1,\ k=1;\ i\neq k}^{r} \mathbf{c}_i^T\mathbf{L}_{ck}\mathbf{c}_i \quad \rightarrow \quad \min_{\mathbf{K}_y}\ . \tag{445}$$

The controller $\mathbf{K}_y$ is used to drive the system to a low interaction level.

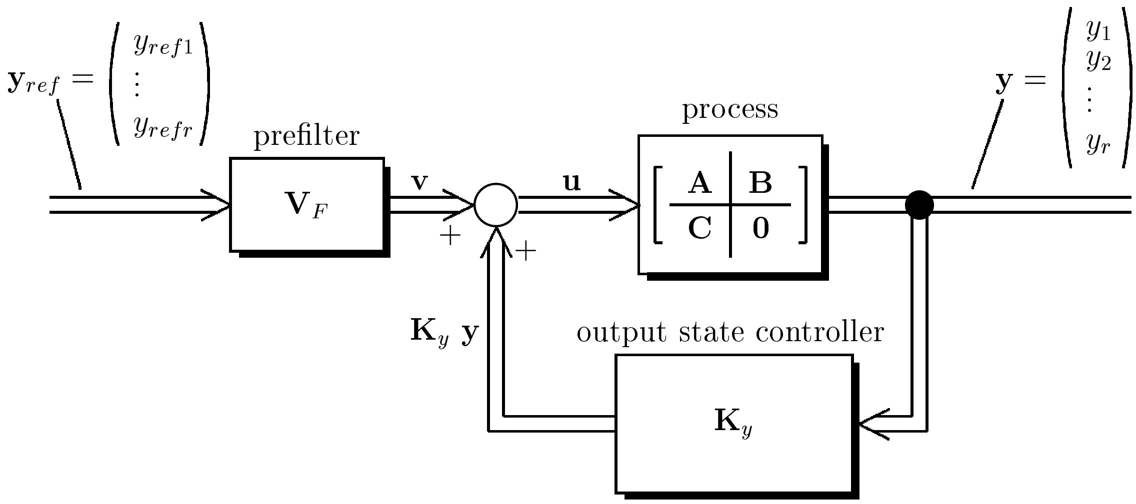

Figure 3: Output state controller

Investigating the increments (*Weinmann, A., 2005b*) $\Delta\mathbf{L}_{ck}$ caused by $\Delta\mathbf{K}_y$ when referring to the Lyapunov equation Eq.(444) and abbreviating

$$\mathbf{E}_1 \stackrel{\triangle}{=} (\mathbf{I}_n \otimes \mathbf{A}_{cl} + \mathbf{A}_{cl} \otimes \mathbf{I}_n)^{-1} \ \in \mathcal{R}^{n^2 \times n^2} \tag{446}$$

one finds

$$\frac{\partial I_{G,ik}}{\partial \mathbf{K}_y} \stackrel{(212),(186)}{=} -\{\mathbf{I}_m \otimes [(\mathbf{c}_i^T \otimes \mathbf{c}_i^T)\mathbf{E}_1(\mathbf{I}_{n^2} + \mathbf{U}_{nn})[\mathbf{B} \otimes (\mathbf{CL}_{ck})^T]\} \times \\ \times[(\mathrm{col}\,\mathbf{I}_m) \otimes \mathbf{I}_r] \ \in \mathcal{R}^{m \times r} \tag{447}$$

$$\frac{\partial I_{G,ik}}{\partial \mathbf{K}_y} \stackrel{(41)}{=} -\{\mathrm{loc}_r[\mathbf{B}^T \otimes (\mathbf{CL}_{ck})](\mathbf{I}_{n^2} + \mathbf{U}_{nn})\mathbf{E}_1^T(\mathbf{c}_i \otimes \mathbf{c}_i)\}^T \ , \tag{448}$$

where loc_r is a command for decolumnizing a column such that a matrix with r rows is produced.

Defining $\mathbf{X}_1$ via

$$\mathbf{E}_1^T(\mathbf{c}_i \otimes \mathbf{c}_i) \stackrel{\triangle}{=} \mathrm{col}\ \mathbf{X}_1 \tag{449}$$

$$(\mathbf{I}_n \otimes \mathbf{A}_{cl}^T + \mathbf{A}_{cl}^T \otimes \mathbf{I}_n)^{-1}(\mathbf{c}_i \otimes \mathbf{c}_i) = \mathrm{col}\ \mathbf{X}_1 \tag{450}$$

$$(\mathbf{c}_i \otimes \mathbf{c}_i) = \mathrm{col}(\mathbf{c}_i\mathbf{c}_i^T) = (\mathbf{I}_n \otimes \mathbf{A}_{cl}^T + \mathbf{A}_{cl}^T \otimes \mathbf{I}_n)\mathrm{col}\ \mathbf{X}_1 \tag{451}$$

$$\mathbf{A}_{cl}^T\mathbf{X}_1 + \mathbf{X}_1\mathbf{A}_{cl} = \mathbf{c}_i\mathbf{c}_i^T \ , \tag{452}$$

one finally has

$$\frac{\partial I_{G,ik}}{\partial \mathbf{K}_y} = -\{\mathrm{loc}_r[\mathbf{B}^T \otimes (\mathbf{CL}_{ck})]\ 2\ \mathrm{col}\,\mathbf{X}_1\}^T \tag{453}$$

$$= -2\{\mathrm{loc}_r\mathrm{col}(\mathbf{CL}_{ck})\mathbf{X}_1\mathbf{B}\}^T = -2\mathbf{B}^T\mathbf{X}_1\mathbf{L}_{ck}\mathbf{C}^T \ , \tag{454}$$

where $\mathbf{X}_1$ and $\mathbf{L}_{ck}$ result from Eqs.(452) and (444), respectively.

22 Interaction Energy Gradient, Discrete Time

Based on the continuous-time single-input single-output system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \ , \qquad y(t) = \mathbf{c}^T\mathbf{x}(t) \ , \tag{455}$$

employing a sampler with sampling time T_s and a zero-order hold, one has a discrete-time system

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \boldsymbol{\psi}u(k) \ , \qquad y(k) = \mathbf{c}^T\mathbf{x}(k) \ , \tag{456}$$

where

$$\mathbf{\Phi} = \mathbf{\Phi}(T_s) = e^{\mathbf{A}T_s} \ , \qquad \boldsymbol{\psi}(T_s) = \mathbf{A}^{-1}[\mathbf{\Phi}(T_s) - \mathbf{I}]\mathbf{b} \tag{457}$$

(*Kuo, B.C., 1992*).

If the stable discrete-time single-input single-output system is excited by a single input $u(0) = 1, u(k) = 0 \ \forall \ k \geq 1$, $\mathbf{x}(0) = 0$, then

$$\begin{aligned} y(k+1) &= \mathbf{c}^T\mathbf{x}(k+1) = \mathbf{c}^T\mathbf{\Phi}^k\boldsymbol{\psi} && (458)\\ \sum_{k=0}^{\infty} y^2(k+1) &= \mathbf{c}^T(\sum_{k=0}^{\infty}\mathbf{\Phi}^k\boldsymbol{\psi}\boldsymbol{\psi}^T\mathbf{\Phi}^{k,T})\mathbf{c} \ \stackrel{\triangle}{=} \ \mathbf{c}^T\mathbf{L}_c\mathbf{c} \ . && (459)\end{aligned}$$

Simply splitting the controllability Gramian $\mathbf{L}_c$ leads to

$$\begin{aligned} \mathbf{L}_c &\stackrel{\triangle}{=} \sum_{k=0}^{\infty}\mathbf{\Phi}^k\boldsymbol{\psi}\boldsymbol{\psi}^T\mathbf{\Phi}^{k,T} = \boldsymbol{\psi}\boldsymbol{\psi}^T + \sum_{k=1}^{\infty}\mathbf{\Phi}^k\boldsymbol{\psi}\boldsymbol{\psi}^T\mathbf{\Phi}^{k,T} && (460)\\ \mathbf{L}_c &= \boldsymbol{\psi}\boldsymbol{\psi}^T + \sum_{k=0}^{\infty}\mathbf{\Phi}^{k+1}\boldsymbol{\psi}\boldsymbol{\psi}^T\mathbf{\Phi}^{k+1,T} = \boldsymbol{\psi}\boldsymbol{\psi}^T + \mathbf{\Phi}\mathbf{L}_c\mathbf{\Phi}^T && (461)\\ \mathrm{col}\mathbf{L}_c &= (\mathbf{I} - \mathbf{\Phi}\otimes\mathbf{\Phi})^{-1}\mathrm{col}(\boldsymbol{\psi}\boldsymbol{\psi}^T) \ , && (462)\end{aligned}$$

where “col” is an operator columnizing the matrix. Eq.(461) is the appropriate Lyapunov equation.

Consider the output energy of a discrete-time single-input single-output system excited by a single input $u(0) = 1$. Referring to Eq.(459) and $\mathbf{\Phi}_{cl} \stackrel{\triangle}{=} \mathbf{\Phi} + \mathbf{\Psi}\mathbf{K}_y\mathbf{C}$, the output energy is

$$I_y = \sum_{0}^{\infty} y^2(k+1) = \mathbf{c}^T(\sum_{k=0}^{\infty}\mathbf{\Phi}^k\boldsymbol{\psi}\boldsymbol{\psi}^T\mathbf{\Phi}^{k,T})\mathbf{c} \ = \ \mathbf{c}^T\mathbf{L}_c\mathbf{c} \ , \tag{463}$$

where $\mathbf{L}_c$ follows from Eq.(461). The changes in $\Delta\mathbf{L}_c$ in terms of the increments $\Delta\mathbf{K}_y$ result from

$$\mathrm{col}\Delta\mathbf{L}_c = \mathbf{E}_3[(\mathbf{E}_4\otimes\mathbf{\Psi})\mathrm{col}\Delta\mathbf{K}_y + (\mathbf{\Psi}\otimes\mathbf{E}_4)\mathrm{col}\Delta\mathbf{K}_y^T], \tag{464}$$

where $\mathbf{E}_4 \ \stackrel{\triangle}{=} \ \mathbf{\Phi}_{cl}\mathbf{L}_c\mathbf{C}^T$ and $\mathbf{E}_3 \stackrel{\triangle}{=} (\mathbf{I} - \mathbf{\Phi}_{cl}\otimes\mathbf{\Phi}_{cl})^{-1}$.

Stepping forward to multiple-input multiple-output systems (*Weinmann, A., 2006a*), denote single-input single-output partitions from input k to output i. Referring to Eq.(459), the index of performance for y_i is

$$I_{yi} = (\mathbf{c}_i^T \otimes \mathbf{c}_i^T)\mathrm{col}\,\mathbf{L}_{ck} \ , \tag{465}$$

where $\mathbf{c}_i$ belongs to y_i and $\mathbf{L}_{ck}$ follows from Eq.(461) with columns $\boldsymbol{\psi} = \boldsymbol{\psi}_k$. The index I_{yi} should decrease to reduce the interaction. (When a prefilter $\mathbf{V}$ is switched in front of the system, then the kth column of $\mathbf{\Psi V}$ is used.)

The MATLAB command `dlyap` is introduced. Then, $(\mathbf{\Psi}^T \otimes \mathbf{E}_4^T)\mathrm{col}\mathbf{X}_3$ is rephrased as $\mathrm{col}[\mathbf{E}_4^T\mathbf{X}_3\mathbf{\Psi}] = \mathrm{col}[\mathbf{C}\mathbf{L}_c\mathbf{\Phi}_{cl}^T\mathbf{X}_3\mathbf{\Psi}]$.

$$\frac{\partial I_{yi}}{\partial \mathbf{K}_y} = 2\mathbf{\Psi}^T\mathbf{X}_3\mathbf{\Phi}_{cl}\mathbf{L}_{ck}\mathbf{C}^T \ , \tag{466}$$

$$\text{where} \qquad \mathbf{X}_3 = \mathtt{dlyap}(\mathbf{\Phi}_{cl}^T,\ \mathbf{c}_i\mathbf{c}_i^T) \tag{467}$$

$$\mathbf{L}_{ck} = \mathtt{dlyap}(\mathbf{\Phi}_{cl}, \boldsymbol{\psi}_k\boldsymbol{\psi}_k^T) \tag{468}$$

$$\boldsymbol{\psi}_k = k\text{-th column of}\mathbf{\Psi V} \ . \tag{469}$$

The influence of uncertainties in the plant is carried out in *Weinmann, A., 2007*.

23 Minimizing Controller Matrix Norm vs Pole Assignment. Desensitization

An important problem is minimizing the norm of the state controller $\mathbf{K}$

$$\|\mathbf{K}\| \to \min_{\mathbf{K}} \tag{470}$$

$$\text{subject to} \ \ \lambda_i[\mathbf{A} + \mathbf{BK}] = \lambda_{di} \quad \forall i = \{1, 2 \ldots n\} \ . \tag{471}$$

The condition of desired closed-loop eigenvalues can be stated as a vector-valued condition

$$\mathbf{f} \ \stackrel{\triangle}{=} \ \mathbf{vec}_i[\det(\lambda_{di}\mathbf{I}_n - \mathbf{A} - \mathbf{BK})] = \mathbf{0} \ . \tag{472}$$

Starting from Eq.(472) and using Eq.(212), one has

$$2\mathbf{K} + (\mathbf{I}_m \otimes \boldsymbol{\mu}^T)\frac{\partial \mathbf{f}}{\partial \mathbf{K}} = \mathbf{0} \in \mathcal{R}^{m\times n} \ , \tag{473}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{K}} = \begin{pmatrix} \frac{\partial \mathbf{f}}{\partial K_{11}} & \frac{\partial \mathbf{f}}{\partial K_{12}} & \cdots & \frac{\partial \mathbf{f}}{\partial K_{1n}} \\ \vdots & \ddots & & \vdots \\ \frac{\partial \mathbf{f}}{\partial K_{m1}} & \cdots & & \frac{\partial \mathbf{f}}{\partial K_{mn}} \end{pmatrix} \in \mathcal{R}^{nm\times n} \ . \tag{474}$$

Applying

$$\frac{\partial}{\partial \mathbf{K}} \det(s_i \mathbf{I}_n - \mathbf{A} - \mathbf{B}\mathbf{K}) = -2\mathbf{B}^T \mathbf{adj}^T(s_i \mathbf{I}_n - \mathbf{A} - \mathbf{B}\mathbf{K}) \ , \tag{475}$$

a matrix is defined in what follows

$$\mathbf{P}_P \triangleq \begin{pmatrix} -2\mathbf{B}^T \mathbf{adj}^T(\lambda_{d1} \mathbf{I}_n - \mathbf{A} - \mathbf{B}\mathbf{K}) \\ -2\mathbf{B}^T \mathbf{adj}^T(\lambda_{d2} \mathbf{I}_n - \mathbf{A} - \mathbf{B}\mathbf{K}) \\ \vdots \\ -2\mathbf{B}^T \mathbf{adj}^T(\lambda_{dn} \mathbf{I}_n - \mathbf{A} - \mathbf{B}\mathbf{K}) \end{pmatrix} \in \mathcal{R}^{nm \times n} \ . \tag{476}$$

Afterwards a fixed-valued permutation matrix $\mathbf{M}_P \in \mathcal{R}^{mn \times mn}$ is used to change the row order and to precisely achieve the order of Eq.(474), where $\mathbf{M}_P = \mathbf{U}_{k,l} = \mathbf{U}_{m,n}$, see Eq.(15).

Finally, the entire result is given by

$$\begin{aligned} 2\mathbf{K} + (\mathbf{I}_m \otimes \boldsymbol{\mu}^T) \ \mathbf{M}_P \ \mathbf{P}_P(\mathbf{K}) &= \mathbf{0} \quad \in \mathcal{R}^{m \times n} && (477) \\ \mathbf{f}(\mathbf{K}) &= 0 \quad \in \mathcal{R}^n \ . && (478) \end{aligned}$$

This is a nonlinear system of mn unknowns in the matrix $\mathbf{K}$ and n unknowns in the Lagrange multiplier $\boldsymbol{\mu}$.

Including desensitization, the problem is

$$\|\mathbf{K}\|_F + \alpha \|\frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{K}}\|_F \to \min_{\mathbf{K}} \tag{479}$$

$$\text{subject to} \ \ \lambda_i[\mathbf{A} + \mathbf{B}\mathbf{K}] = \lambda_{di} \ \ \forall i = \{1, 2 \ldots n\} \ . \tag{480}$$

The gradient of λ_i with respect to $\mathbf{K}$ is

$$\frac{\partial \lambda_i[\mathbf{A}_{cl}]}{\partial \mathbf{K}} = (\mathbf{I}_m \otimes \mathbf{v}_i^{\triangleleft * T}) \frac{\partial \mathbf{A}_{cl}}{\partial \mathbf{K}} (\mathbf{I}_n \otimes \mathbf{v}_i) \ , \tag{481}$$

where $\mathbf{v}_i$ is the right eigenvector of the matrix $\mathbf{A}_{cl} \triangleq \mathbf{A} + \mathbf{B}\mathbf{K}$ and $\mathbf{v}_i^{\triangleleft * T}$ denotes the conjugate transpose of the left eigenvector.

Eq.(473) turns out as

$$2 \ \mathbf{K} + \alpha \frac{\partial}{\partial \mathbf{K}} \|\frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{K}}\|_F + (\mathbf{I}_m \otimes \boldsymbol{\mu}^T) \frac{\partial \mathbf{f}}{\partial \mathbf{K}} = \mathbf{0} \in \mathcal{R}^{m \times n} \ . \tag{482}$$

For examples see *Weinmann, A., 2003a.*

24 Incremental Notation, Fréchet Derivatives

The incremental notation is addressed as advantageously applicable in control theory. The increment of a matrix $\mathbf{A}$ is termed $\Delta\mathbf{A}$.

For $\mathbf{A}, \mathbf{M} \in \mathcal{R}^{n\times m}$ and $f(\mathbf{M})$ given as

$$f(\mathbf{M}) \triangleq \operatorname{tr}[\mathbf{A}^T\mathbf{M}] \tag{483}$$

$$\Delta f(\mathbf{M}, \Delta\mathbf{M}) \triangleq f(\mathbf{M}+\Delta\mathbf{M}) - f(\mathbf{M}) = \operatorname{tr}[\mathbf{A}^T\Delta\mathbf{M}] = \operatorname{tr}[\Delta\mathbf{M}^T \cdot \mathbf{A}] \tag{484}$$

and

$$\frac{\partial f}{\partial \mathbf{M}} = \mathbf{A} \ . \tag{485}$$

The factor of the increment $\Delta\mathbf{M}$, having been transposed, equals the derivative.

24.1 Increment of the Frobenius Norm

Suppose real matrices $\mathbf{K}$, $\mathbf{L}$, $\mathbf{R}$ with appropriate dimension, the increment $\Delta\mathbf{K}$ of $\mathbf{K}$ causes an increment (*Vetter, W.J., 1971*)

$$\begin{aligned} \Delta\|\mathbf{LKR}\|_F^2(\mathbf{K}, \Delta\mathbf{K}) &\triangleq \|\mathbf{L}(\mathbf{K}+\Delta\mathbf{K})\mathbf{R}\|_F^2 - \|\mathbf{LKR}\|_F^2 \\ &= \operatorname{tr}\left\{2\mathbf{L}^T\mathbf{LKRR}^T\Delta\mathbf{K}^T\right\} \ . \end{aligned} \tag{486}$$

Hence,

$$\frac{\partial}{\partial \mathbf{K}}\|\mathbf{LKR}\|_F^2 = 2\mathbf{L}^T\mathbf{LKRR}^T \ . \tag{487}$$

The prefactor matrix in Eq.(486) corresponds to the differential quotient with respect to $\mathbf{K}$.

The differential or increment of $\|\mathbf{A}-\mathbf{KC}\|_F^2$ with respect to $\mathbf{A}$ is

$$\Delta\|\mathbf{A}-\mathbf{KC}\|_F^2(\mathbf{A}, \Delta\mathbf{A}) \triangleq \|\mathbf{A}+\Delta\mathbf{A}-\mathbf{KC}\|_F^2 - \|\mathbf{A}-\mathbf{KC}\|_F^2 = \tag{488}$$

$$= 2\operatorname{tr}\left\{[\mathbf{A}^T - (\mathbf{KC})^T]\Delta\mathbf{A}\right\}. \tag{489}$$

24.2 Taylor Expansion of the Determinant

$$\det(\mathbf{A}+\mathbf{V}\Delta p)\Big|_{\Delta p\ll 1} \doteq \det\mathbf{A} + \frac{\partial \det(\mathbf{A}+\mathbf{V}\Delta p)}{\partial \Delta p}\Delta p \tag{490}$$

$$\doteq \det\mathbf{A} + \operatorname{tr}[\mathbf{V}\operatorname{adj}\mathbf{A}]\Delta p \tag{491}$$

$$\doteq \det\mathbf{A} + \operatorname{tr}[\mathbf{V}\mathbf{A}^{-1}] \cdot \det\mathbf{A} \cdot \Delta p \ . \tag{492}$$

$$\begin{aligned}
\det(\mathbf{A}+\mathbf{E}_{ij}\Delta p)\Big|_{\Delta p\ll 1} &\doteq \det\mathbf{A}+\frac{\partial\det(\mathbf{A}+\mathbf{E}_{ij}\Delta p)}{\partial\Delta p}\Delta p && (493)\\
&\doteq \det\mathbf{A}+\mathrm{tr}[\mathbf{E}_{ij}\mathrm{adj}\mathbf{A}]\Delta p && (494)\\
&\doteq \det\mathbf{A}+(\mathrm{adj}\mathbf{A})_{ji}\cdot\Delta p\ . && (495)
\end{aligned}$$

$$\begin{aligned}
\det(\mathbf{A}+\Delta\mathbf{A})\Big|_{\|\Delta\mathbf{A}\|_F\ll 1} &\doteq \det\mathbf{A}+\sum_{ij}(\mathrm{adj}\mathbf{A})_{ji}\Delta A_{ij} && (496)\\
&\doteq \det\mathbf{A}+\sum_{ij}[(\mathrm{adj}\mathbf{A})^T\cdot*\ \Delta\mathbf{A}]_{ij} && (497)\\
&\doteq \det\mathbf{A}+[\mathrm{col}(\mathrm{adj}\mathbf{A}^T)]^T\cdot\mathrm{col}\Delta\mathbf{A}\ . && (498)
\end{aligned}$$

24.3 Increment of the Inverse

$$(\mathbf{X}+\Delta\mathbf{X})^{-1}\doteq\mathbf{X}^{-1}-\mathbf{X}^{-1}\Delta\mathbf{X}\cdot\mathbf{X}^{-1}+\mathbf{X}^{-1}\Delta\mathbf{X}\cdot\mathbf{X}^{-1}\Delta\mathbf{X}\cdot\mathbf{X}^{-1}-+\dots \qquad (499)$$

$$\Delta(\mathbf{X}^{-1})\doteq-\mathbf{X}^{-1}(\Delta\mathbf{X})\mathbf{X}^{-1}\ . \qquad (500)$$

If $\mathbf{A}=\mathbf{G}\mathbf{X}^{-1}$ and both $\mathbf{G}$ and $\mathbf{X}$ are variable

$$\begin{aligned}
\Delta\mathbf{A} &= (\Delta\mathbf{G})\mathbf{X}^{-1}+\mathbf{G}\ \Delta(\mathbf{X}^{-1})=(\Delta\mathbf{G})\mathbf{X}^{-1}-\mathbf{G}\mathbf{X}^{-1}(\Delta\mathbf{X})\mathbf{X}^{-1} && (501)\\
&= (\Delta\mathbf{G})\mathbf{X}^{-1}-\mathbf{A}(\Delta\mathbf{X})\mathbf{X}^{-1}=(\Delta\mathbf{G}-\mathbf{A}\ \Delta\mathbf{X})\mathbf{X}^{-1}\ , && (502)
\end{aligned}$$

leading to

$$\Delta\|\mathbf{A}-\mathbf{K}\mathbf{C}\|_F^2(\mathbf{A},\Delta\mathbf{X})=2\mathrm{tr}\ \{[\mathbf{A}^T-(\mathbf{K}\mathbf{C})^T](\Delta\mathbf{G}-\mathbf{A}\ \Delta\mathbf{X})\mathbf{X}^{-1}\}\ . \qquad (503)$$

24.4 Increment of the Adjoint Matrix and the Matrix Exponential

$$\begin{aligned}
\mathrm{adj}(\mathbf{X}^{[n\times n]}+\Delta\mathbf{X}) &= (\mathbf{X}+\Delta\mathbf{X})^{-1}\det(\mathbf{X}+\Delta\mathbf{X}) && (504)\\
&\overset{(499),(492)}{=} (\mathbf{X}^{-1}-\mathbf{X}^{-1}\Delta\mathbf{X}\cdot\mathbf{X}^{-1})[\det\mathbf{X}+\mathrm{tr}(\Delta\mathbf{X}\cdot\mathbf{X}^{-1})\det\mathbf{X}]\\
&= \mathrm{adj}(\mathbf{X})\cdot[\mathbf{I}_n-\Delta\mathbf{X}\cdot\mathbf{X}^{-1}+\mathbf{I}_n\mathrm{tr}(\mathbf{X}^{-1}\Delta\mathbf{X})] && (505)\\
\Delta\mathrm{adj}(\mathbf{X})(\mathbf{X},\Delta\mathbf{X}) &= \mathbf{I}_n\mathrm{tr}(\mathbf{X}^{-1}\Delta\mathbf{X})-\Delta\mathbf{X}\cdot\mathbf{X}^{-1}\ . && (506)
\end{aligned}$$

Furthermore for the increments of the matrix exponential

$$g(\mathbf{M})\overset{\triangle}{=}\mathrm{tr}[e^{\mathbf{M}}] \qquad (507)$$

$$\Delta g(\mathbf{M},\Delta\mathbf{M})=\mathrm{tr}[e^{\mathbf{M}}\Delta\mathbf{M}] \qquad (508)$$

$$\frac{\partial g(\mathbf{M})}{\partial \mathbf{M}} = e^{\mathbf{M}^T} \ . \tag{509}$$

Finally (*Levine, W.S., and Athans,M., 1970; Bellman, R., 1960*)

$$\mathbf{h}(\mathbf{M}) \triangleq e^{\mathbf{BM}t} \tag{510}$$

$$\Delta\mathbf{h}(\mathbf{M},\Delta\mathbf{M}) = \int_0^t e^{\mathbf{BM}(t-\sigma)}\mathbf{B}\Delta\mathbf{M}e^{\mathbf{BM}\sigma}d\sigma \ . \tag{511}$$

25 Matrix Element Interdependencies

For specific matrix interdependencies, e.g., matrix $\mathbf{M}$ symmetric or skew-symmetric (*Geering, H.P., 1976*) there are different correspondences, different from Eq.(176),

$$\frac{\partial \mathbf{M}}{\partial M_{ik}} = \mathbf{E}_{ik} + \mathbf{E}_{ki} - \mathbf{E}_{ii}\delta_{ik} \ , \quad \mathbf{M} \text{ symmetric} \tag{512}$$

$$\frac{\partial \mathbf{M}}{\partial M_{ik}} = \mathbf{E}_{ik} - \mathbf{E}_{ki} \ , \quad \mathbf{M} \text{ skew symmetric} \ , \tag{513}$$

where $\mathbf{E}_{ik} = \mathbf{e}_i\mathbf{e}_k^T$ is the Kronecker matrix. E.g., Eq.(117) is not true if the matrix is symmetric. For further examples, see *Geering, H.P., 1976; Weinmann, A., 2001b*. For symmetric matrices, there is the general relation

$$\frac{\partial \det \mathbf{M}}{\partial \mathbf{M}} = (2\mathbf{M}^{-1} - \mathbf{diag}\ \{\mathbf{M}^{-1}\}) \det \mathbf{M} \ , \tag{514}$$

where $\mathbf{diag}\ \{\mathbf{M}^{-1}\}$ is a diagonal matrix made of main-diagonal elements of $\mathbf{M}^{-1}$.

For $f(\mathbf{M}) \triangleq \operatorname{tr}[\mathbf{LM}]$

$$\frac{\partial f(\mathbf{M})}{\partial \mathbf{M}} = \begin{cases} (\mathbf{L}^T + \mathbf{L} - \operatorname{diag} \mathbf{L}) & \text{for } \mathbf{M} \text{ symmetric} \\ (\mathbf{L}^T - \mathbf{L}) & \text{for } \mathbf{M} \text{ skew symmetric} \ . \end{cases} \tag{515}$$

For $\mathbf{M}$ symmetric the linear relation $\mathbf{L}_1\mathbf{M}\mathbf{L}_2$ yields (*Brewer, J.W., 1977*)

$$\frac{\partial}{\partial \mathbf{M}} \operatorname{tr}[\mathbf{L}_1\mathbf{M}\mathbf{L}_2] = \mathbf{L}_1^T\mathbf{L}_2^T + \mathbf{L}_2\mathbf{L}_1 - \operatorname{diag}\{\mathbf{L}_2\mathbf{L}_1\} \tag{516}$$

and for the relation of second order

$$\frac{\partial}{\partial \mathbf{M}} \operatorname{tr}[\mathbf{L}_1\mathbf{M}\mathbf{L}_2\mathbf{M}] = (\mathbf{L}_2\mathbf{M}\mathbf{L}_1)^T + \mathbf{L}_1\mathbf{M}\mathbf{L}_2 - \mathbf{diag}\{\mathbf{L}_2\mathbf{M}\mathbf{L}_1 + \mathbf{L}_1\mathbf{M}\mathbf{L}_2\} \ . \tag{517}$$

The reason for the strong differences in the differential quotients for occasionally existing matrix interdependencies is the fact that the partial derivative conditions are not fulfilled.

26 Concluding Remarks

Gradients and matricial gradients of several scalar functions of vectors and matrices have been presented as a collection.

Numerically checking the presented results or new ones can be carried out as follows. For $\frac{\partial f}{\partial \mathbf{M}}$, the (i, j) position in the matrix differential is $\frac{\partial f}{\partial M_{ij}}$. It is approximated by

$$\frac{\partial f}{\partial M_{ij}} = \lim_{\Delta M_{ij} \to 0} \frac{\Delta f(\mathbf{M}, \Delta M_{ij})}{\Delta M_{ij}} = \lim_{\Delta M_{ij} \to 0} \frac{f(\mathbf{M}) - f(\mathbf{M} + \mathbf{E}_{ij} \cdot \Delta M_{ij})}{\Delta M_{ij}} . \qquad (518)$$

Gradients are very useful for designing controllers in a stepwise version; optimizing a given criterion (or a combination of criteria) while observing some other conditions, e.g., for the actuating signal amplitude.

In the presence of spherical uncertainties (*Weinmann, A., 2007e*) their influence on the control system performance can easily be included into the design process.

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26. International Conference on Computer Safety, Reliability and Security (SAFECOMP)

18. – 21. September 2007
Universität Erlangen-Nürnberg

Die seit 1979 jährlich vom European Workshop on Industrial Computersystems, Technisches Komitee „Safety, Reliability and Security“ (EWICS TC7), veranstalte Internationl Conference on Computer Safety, Reliability and Security (SAFECOMP) wurde vom 18. bis 21. September 2007 an der Universität Erlangen-Nürnberg in Nürnberg abgehalten.

SAFECOMP zeigt nicht nur den Stand der Technik, die Erfahrungen und den Trend auf dem Gebiet kritischer Computeranwendungen, sondern bildet auch eine Plattform des Wissensaustausches zwischen Wissenschaft, Industrie, Forschungsinstitutionen, Anwendern und Zulassungsbehörden. Lagen am Anfang die Schwerpunkte in den Bereichen Automation, Eisenbahnen Kraftwerke, chemische Industrie und Avionik, so kamen inzwischen Kraftfahrzeuge und Medizintechnik dazu.

EWICS TC7 berichtete über den Entwicklungsstand der Richtlinien über „ Safety of Medical Devices“, „Security of Safety-Critical Computer Systems“, „Maintenance and Modification of Diverse Systems“ und „Education and Training in Dependable Systems“. Die heutige Norm IEC 61508 (Functional Safety) entstand aus einer der EWICS TC7-Richtlinien.

Sicherheits-Fallstudien wurden für ein Kraftfahrzeug-System und medizinische Einrichtungen vorgestellt. In der Sitzung „Impact of Security on Safety“ fand das französische Modell über die Abhängigkeiten zwischen Elektroversorgung und Informations-Infrastruktur Beachtung. Der Beitrag über „Learning from your Elders“ wäre lehrreich für manche im Datenschutz tätige Manager. Der Bericht über „Safety Demonstration and Software Development“ bei RATP (Régie Autonome des Transports Parisiens) gab die Erfahrung über die Anwendung formaler Methoden wieder. Die Arbeitssitzung von EWICS TC7, Workshop on Software Dependability DECOS/ERCIM Workshop 2007 on Dependable Embedded Systems und das Tutorial on High-Level Modeling Evironments for the Dependability Assessment of Dynamic Fault-Tolerant Systems rundeten SAFECOMP 2007 ab.

Die Proceedings erschienen in der Serie Lecture Notes in Computer Science vom Springer-Verlag als Band 4680 und diesen erhielten die Teilnehmer bereits bei der Tagung.

Bei über hundert Teilnehmern waren fast alle Kontinente vertreten. Es gab hinreichend Zeit für einen angenehmen Informationsaustausch. Es war wieder eine sehr erfolgreiche Tagung.

SAFECOMP 2008 findet vom 22. bis 25 September 2008 in Newcastle-upon-Tyne (UK) an der Newcastle University statt, siehe http://www.safecomp2008.org.

R. Genser

22. Österreichischen Automatisierungstag

24.Oktober 2007
Johannes Kepler Universität Linz

Der schon zur Tradition gewordene Automatisierungstag fand diesmal an der Johannes Kepler Universität in Linz statt. Er stand unter dem Titel „Automatisierungstechnik: Forschung – Industrie" und wurde vom Institut für Regelungstechnik und Prozessautomatisierung der Johannes Kepler Universität bestens organisiert. Die österreichische Industrie und hier insbesondere Klein- und Mittelbetriebe kooperieren zunehmend mit Universitätsinstituten und Forschungseinrichtungen um neueste Automatisierungslösungen in ihre Produkte zu integrieren. Zweck dieses Automatisierungstages war es, der österreichischen Industrie, forschungsfördernden Stellen sowie Vertretern von Technologietransferzentren und Universitäten einen Überblick über mögliche Ansätze zukünftiger Entwicklungen und realisierte Projekte zu geben.

Die Eröffnung in den Repräsentationenräumen der JKU fand durch P. Kopacek, K. Schlacher, T. Sauter, in Vertretung der ÖGMA, und Frau Vizerektorin G. Kotsis der Universität Linz statt.

In seinem Einleitungsreferat „Automatisierungstechnik an der JKU Linz" gab Prof. Dr. K. Schlacher einen Überblick über aktuelle Arbeiten nicht nur seines Institutes, sonder auch fachverwandter Mechatronik Institute. Er stellte zunächst den neuen Bachelor- und Masterstudienplan „Mechatronik" an der Universität Linz vor. Sodann präsentierte er ausgewählte Forschungsarbeiten auf dem Gebiet der Automatisierungstechnik aus dem Fachbereich Mechatronik.

Im Anschluss daran fand die traditionelle Verleihung des Fred Margulies-Preises für herausragende Arbeiten auf dem Gebiet der Automatisierungstechnik unter besonderer Berücksichtigung sozialer und gesellschaftspolitischer Aspekte statt. Der Preis ging dieses Jahr zur Hälfte an Dr. Markus Schöberl sowie an die HTL Leonding, für das Roboterfußballteam Leonding-Micros.

Im Anschluss daran wurde ein neues Verfahren zur Dickeregelung von Blechen in Warmwalzgerüsten vorgestellt. Dieses Projekt wurde im Zusammenhang mit dem Institut für Regelungstechnik und Prozessautomatisierung der JKU realisiert und in einem Walzwerk in Polen implementiert. Es war dies der Vortrag von G. Keintzel (Siemens VAI Metals) „Einsatz aktiver Schwingungsdämpfung mittels virtueller Sensoren in Industrieanlagen".

Über die „Reglerentwicklung im Bereich Spritzgießmaschinen" berichtete G. Grabmair von der Firma Engel Austria GmbH. Es handelt sich dabei um die Anwendung fortgeschrittener Regelalgorithmen zur Erhöhung der Produktivität von Spritzgießmaschinen. Dieses Projekt wurde ebenfalls in Zusammenarbeit mit dem Repro der JKU Linz durchgeführt. Das Konzept wurde äußerst erfolgreich in eine neue Generation von Spritzgussmaschinen implementiert.

Ein neuartiges modulares Automatisierungskonzept für Produktionsmaschinen stellte Herr Kickinger, von der Bernecker+Rainer Industrie-Elektronik Ges.m.b.H,. in seinem Vortrag „Perfection in Automation for Innovative Machines" vor. Dieses Konzept erlaubt die kostengünstige Automatisierung zahlreicher Produktionsmaschinen.

Neuland am Automatisierungstag betrat M. Zauner von der Fachhochschule OÖ in seinem Vortrag „Automatisierung in der Medizintechnik“. Er zeigte, dass Automatisierungstechnik sowohl im medizinischen als auch im Rehabilitationsbereich immer steigendere Bedeutung zukommt. Moderne Automatisierungslösungen in diesen Gebieten tragen wesentlich zur Verbesserung der ärztlichen Versorgung der Bevölkerung bei.

Über den Stand der Robotertechnik in Österreich gab P. Kopacek in seinem Vortrag eine Übersicht über erste Ergebnisse einer Studie im Auftrag des BMVIT. Auf Grund dieser Ergebnisse gibt es in Österreich relativ wenig Roboterproduzenten, zahlreiche Systemintegratoren sowie Hersteller von Baukoppeln von Industrierobotern. Auf dem Gebiet der Forschung liegt der Schwerpunkt auf mobilen intelligenten Robotern, wobei überwiegend die mobilen Plattformen zum Test und Implementierung neuer theoretischer Verfahren dienen.

Als Neuheit bei Automatisierungstagen fand in der Pause und nach Beendigung eine Demonstration „Roboterfußball“ mit den Teams Leonding-Micros, HTL Leonding und Austro, IHRT-TU Wien statt, welche beim Publikum begeisterte Aufnahme fand.

In seiner Zusammenfassung stellte P. Kopacek fest, dass auch dieser überdurchschnittlich gut besuchte Automatisierungstag seinen Zweck voll erfüllt hat.

P: Kopacek

6. Österreichisch-Bulgarischer Automatisierungstag

26. Oktober 2007
Sofia, Bulgarien

Die bilateralen Automatisierungstage zwischen Österreich und Bulgarien haben eine langjährige Tradition. Erstmals wurden jedoch auch die Nachbarländer Bulgariens Rumänien und Mazedonien mit einbezogen. Er wurde vom „Zentrallaboratorium Mechatronik und Instrumentierung" der bulgarischen Akademie der Wissenschaften (BAS) auf deren Campus bestens organisiert und von der ASO Österreich finanziell unterstützt. Leider konnten die Kollegen aus Skopje nicht daran teilnehmen.
Die Eröffnung erfolgte durch den Leiter von ASO Sofia M.F. Gajdusek, den Sekretär für Ingenieurwissenschaften der bulgarischen Akademie der Wissenschaften, sowie P. Kopacek (IHRT/VUT) und R. Zahariev (BAS).
Danach gab P. Kopacek einen Überblick über den derzeitigen Stand und zukünftige Entwicklungstendenzen der Automatisierungstechnik in Österreich. Er ging sowohl auf Trends in der Prozess- sowie in der Fertigungsautomatisierung ein, wobei er bei letzterer erste Ergebnisse einer Studie „Robotertechnik in Österreich" präsentierte. R. Zahariev berichtete in seinem Übersichtsvortrag über neuere Arbeiten am BAS, wobei er ebenfalls die Robotertechnik in den Mittelpunkt stellte.

Über Entwicklungstendenzen der Fertigungsautomatisierung in Rumänien berichtete Th. Borangiou von der Polytechnischen Universität Bukarest. Er kam zu dem Ergebnis, dass die Fertigungsautomatisierung in Rumänien derzeit von Bedürfnissen der Klein- und Mittelbetriebe bestimmt wird.
Der Vortrag von P.H. Osanna (AUM/VUT) beschäftigte sich mit der industriellen Messtechnik unter dem Einfluss von Nano Technologie und Nano Messtechnik, sowie jener von M.W. Han (IHRT/VUT) mit dem Robotereinsatz für Edutainment (Education by Entertainment). Anschließend berichtete P. Kopacek über ein neues Post Graduales Studium „Mechatronik Management" welches im Rahmen eines EU-Projektes entwickelt und nun an der Universität für Business and Technology (UBT) in Prishtina; Kosovo implementiert wird.
Über Prozessautomatisierung in Rumänien berichtete N. Ivanescu von der Polytechnischen Universität Bukarest. Diese wird derzeit durch intelligente Datenerfassung und verteilte Regelung von komplexen industriellen Prozessen geprägt. Grund dafür ist der Nachholbedarf von Untenehmen, überwiegend aus dem Bereich der Chemie und Petrochemie in Rumänien.
Abgeschlossen wurde dieser Automatisierungstag durch eine Präsentation von V. Vinkov (FESTO Bulgarien) über eine neue Generation von Sensor und Sensorsystemen.

Die mehr als hundert Teilnehmer konnten nicht nur den vorgenannten Vorträgen beiwohnen, sondern auch – erstmals in Bulgarien – Demonstrationen im Roboterfußball des Teams AUSTRO der TU Wien beiwohnen. Diese Demonstrationen fanden beim bulgarischen Fernsehen und Rundfunk regen Widerhall.

R. Zahariev und P. Kopacek stellten in ihrer Zusammenfassung fest, dass dieser multinationale Automatisierungstag seinen Zweck den derzeitigen Stand und zukünftige Entwicklungstendenzen der Automatisierungstechnik in den beteiligten Ländern aufzuzeigen vollkommen erfüllt hat. Die Vorträge werden zu einem späteren Zeitpunkt in der Schriftenreihe der bulgarischen Akademie publiziert.

P. Kopacek

International Conference on Higher Education in Iraq

For the advancement of higher education in Iraq

December 11-13, 2007
Erbil, Kurdistan Region, Iraq

This event was organised by the Ministry of Higher Education and Scientific Research of Kurdistan with collaboration and cooperation of the Ministry of Higher Education and Scientific Research of Federal Government of Iraq and the Iraqi Higher Education Organising Committee - UK. The Conference aims where to improve higher education in Iraq and keeping abreast with international developments in the field of higher education and scientific research, enabling higher education institutions to be active in the community and be responsive to the needs of the country and its development programs.

Therefore the main topics of the conference were: Modern and efficient university management, human resources and capacity building of university higher management, curriculum development and modern teaching methods, quality assurance (assessment and evaluation of the members of academic staff activities), graduates studies programs and scientific research, university and community (role, the labour market), higher education modes (private, open learning, distance education) and university autonomy.

The conference took place in the convention centre (Matyer Saad Centre) in the capital of Kurdistan Erbil. There were 3 plenary sessions with 8 keynote speeches and 27 workshops with over 150 papers. Over 400 academics actively contributed in the 3days conference but at the opening session there were over 900 participants. Contributors were from 23 different countries representing 70 Higher Education and research institutions. Papers represented at he conference were as follows: Modalities of Higher Education – 13 papers, Capacity Building – 14 papers, Curriculum Development – 42 papers, Quality Assurance – 31, Higher Education and Research – 14 papers, Universities and Society – 21 papers, Types of Higher Education (distance and private, etc.) – 16 papers, Decentralisation of Higher Education – 8 papers.
The very well organised conference was under the patronage of the President of Kurdistan Region and was financed at the Prime Minister of the Kurdistan Government.

Austria was represented with one invited paper on “Education in Mechatronics Management” and P. Kopacek chaired two sessions.

Congratulations to the organizers, we are looking forward to the next similar events.

P. Kopacek

IARP/EURON Workshop on Robotics for Risky Interventions and Environmental Surveillance

7. – 8. Jänner 2008
Benacàssim, Spanien

Zentraler Themenschwerpunkt war der Einsatz von Robotern für gefährliche Eingriffe und Aufklärung. Dabei standen jedoch nicht, wie manch einer vermuten könnte, militärische Interessen im Vordergrund. Vielmehr ging es um humanitäre und zivile Problemstellungen, allen voran der Brandbekämpfung.

Mit Hilfe von EU Förderungen wurden vor wenigen Jahren zwei bedeutende Projekte auf diesem Gebiet ins Leben gerufen. Im „Guardians“ sowie „Viewfinder“ Projekt geht es darum Feuerwehrmänner bei ihrer gefährlichen Arbeit zu unterstützen. Ein Schwarm von kooperierenden Robotern soll dabei Informationen über den Zustand eines Gebäudes, Sichtverhältnisse und den Brandherd sammeln, bevor das Gebäude durch die Feuerwehr betreten wird. Dadurch wird eine verbesserte Einsatzplanung und Risikoabschätzung möglich. Außerdem werden die Roboter dazu benutzt die Feuerwehrleute effizient durch rauchgefüllte Räume zu lotsen.

Das IHRT der Technischen Universität Wien beteiligte sich am Workshop mit einer Präsentation des kürzlich entwickelten Minensuchroboters „Humi“. Dieser Roboter ist mittels eines Metalldetektors in der Lage Landminen auch in schwierigem Gelände aufzuspüren. Bei der Entwicklung dieses Roboters wurde penibel darauf geachtet die Kosten so gering als möglich zu halten, nachdem von Landminen betroffene Regionen zu den ärmsten der Welt zählen. Sobald „Humi“ eine potentielle Mine findet markiert er die entsprechende Stelle mit Farbe. Zukünftige Entwicklungen am IHRT werden sich mit dem Ausgraben und dem Abtransport der Mine beschäftigen.

Weitere Präsentationen am IARP/EURON Workshop beinhalteten andere riskante Aufgaben für die Roboter effizient eingesetzt werden können, beispielsweise Inspektionen in Reaktorkernen, Felssicherungen auf Steilhängen und Reparaturen auf Ölbohrinseln. Das Ziel hinter all diesen Bestrebungen ist die Aussetzung des Menschen in gefährliche Situationen zu reduzieren.

L. Silberbauer

Optimization of organizations and correlated management systems including improved knowledge management by using an organizational learning and knowledge system based on constructivist theory.

Dipl.-Ing. Dr.techn. Margareth Stoll

Begutachter: O.Univ.Prof. Dr.techn. Dr.mult.h.c. P. Herbert Osanna

Due to globalization, the increasing customer and market requirements and stronger competition organizations must improve increasingly faster their products, services, organization and technologies according to stakeholder requirements. Thus the largest potential is continual improvement. In this respect an effective holistic management system including knowledge management for organizational learning based on individual learning is central to the success of the organization.

The theoretical approaches for knowledge management consider the knowledge identification, knowledge representation, knowledge communication, knowledge use and knowledge generation, as well as knowledge objectives and knowledge evaluation. In the e-learning approaches external teachers select and prepare normally the content in accordance to media didactical principles. E-collaboration and JIT-E-Learning consider workplace integrated, need-oriented learning and knowledge acquisition to support collaborators to fulfill their job effectively and efficiently. Thereby internal experts elaborate the content. Knowledge management systems and e-learning systems become always more similar and integrated.

More than one million organizations of different sizes and scopes are implementing already since several years management systems, such as quality ISO9001, environmental ISO14001, information security ISO27001, hygienic ISO22000 or others. They are based on international standards with common principles: organization objectives and strategies based on the requirements of interested parties, business processes oriented on organization objectives, resource management and continuously optimization of the organization. These systems are implemented more frequently in a holistic way, whereby are integrated according with the organizational purpose and objectives different aspects, like quality, environment, hygiene, data security, occupational health and safety, as well as personnel development, resource management, IT - management, communication management, controlling, as well as knowledge management. The established management system must be documented, communicated, implemented and continual improved. Although the system documentation contains the entire explicit knowledge, it is almost felt as additional workload with little advantage, sparsely corresponding with lived processes and it is not the basis for individual and organizational learning and continual organizational development.

Based on this problem a holistic management system is established including knowledge management. Therefore the organization policy is elaborated with consistent strategies and objectives considering the needs and expectations of all interested parties and including also knowledge strategies and objectives. The applied processes are analyzed bottom up by interviewing the collaborators involved and by integrating knowledge management and optimized in accordance with the organizational objectives. The necessary resources, tools, instruments, trainings required for achieving the objectives and for improving the management system are identified, provided and continually optimized. Also training and human resource development process are optimized and established. The organization defines and implements also the monitoring, measurement, analysis and optimization processes for continually improvement of the effectiveness of the management system. Thus knowledge management and learning organization are integrated and their improvement is structured and systematically planned. The entire system documentation must correspond with the lived processes.

After the development of the management system documentation, it must be prepared regarding media-pedagogical, motivation-psychological and didactical principles In order to support the collaborators need-oriented workplace integrated access, the way for accessing the single modules should be as short as possible, the content must be practice oriented divided into small modules and functionally well structured.

The requirements for the extension of the learning system are defined on scientific and theoretical theorems for organizational development of quality management, knowledge management, learning organization, change management, business reengineering and process management, as well as on the requirements of standard based holistic management systems and on interviews and practical experiences. Therefore it must be simple and intuitive to handle.

The organizational learning and knowledge system should meet possible the stated requirements as far as but it is thereby only a tool, which supports the optimization of the organization so far as this is admitted by the culture. Therefore we need an open, confident based, fault-tolerant corporate and learning culture with criticism and change readiness. The coworkers should be interested in new knowledge, able for self-driven learning, have personal employment, team ability and change willingness apart from necessary IT-competences. All managers must use constantly and actively the system and motivate their coworkers in following these principles, promoting in this way learning and new knowledge generation. By introducing this system the system manager and/or knowledge manager extend their own job, needing also the acquisition of the necessary media-pedagogical and didactical knowledge for preparing the content and the necessary skills for supporting e-learning and knowledge management.

The novelty of this project is the integration of a holistic management system, knowledge management and workplace integrated e-learning into an open confident based corporate culture.

Design and High Precision Monitoring of Detector Structures at CERN.

Dipl.-Ing.Dr.techn. Friedrich Lackner

Begutachter: O.Univ.Prof. Dr.techn. Dr.mult.h.c. P. Herbert Osanna

Situated on the outskirts of Geneva, CERN is the world wide leading center for particle physics. The Large Hadron Collider (LHC) with its 27 km ring shaped accelerator, which is currently under construction and will be operational in 2008, will begin a new era in high energy physics by revealing the basic constituents of the universe. In order to deliver the maximum center-of-mass energy within the 27 km of circumference the LHC is constructed as high-luminosity proton-proton collider based on 1232 superconducting dipole magnets. The magnets are providing a magnetic field of 8.4 T. The superconducting coils are electrically operated with a current of 11.2 kA. The magnets will be operated at temperature below 2 K using super fluid helium for the cooling. In the last few decades there has been an enormous improvement and a successive appreciation of the basic composition of matter, particles and their interactions. This definition and all the basis of knowledge are established by the standard model theory. The LHC will accelerate and collide proton beams and heavy ions configured in bunches of up to 10^{11} for the p-p runs, separated by 25 ns, giving a center of mass energy of 14 TeV and luminosities of 10^{34} $cm^{-2}s^{-1}$. The collisions will take place at four experimental sites on the accelerator ring.

One of the experiments is ALICE (A Large Ion - Colliding - Experiment), a detector consisting of multiple layers of sub detectors around the collision point to detect different types and properties of particles created in the collisions. Those particles are identified via their energy, momentum, track and decay products, and it is therefore important to align the various sub detectors very precisely to each other and monitor their position. The monitoring systems have to operate for an extended period of time under extreme conditions (e.g. high radiation) and must not absorb too many of the particles created in the collisions. The thesis describes monitoring systems developed for the ALICE and CMS (Compact Muon Solenoid) experiment. The crucial aspect within the integration of the ALICE experiment is precise alignment of the inner detectors with respect to the central beryllium beam pipe. Based on the optical position monitoring system BCAM (Brandeis CCD Angle Monitor), tests were carried out in order to approve the idea of mounting one BCAM on the external reference point and a reflecting mirror on the sub detector. Using a corner cube prism instead of a plane mirror eliminates the sensitivity to rotations of the mirror. Results obtained from the various lab tests and final setups will show that the novel BCAM application which is now used in three out of the four LHC experiments, has several advantages over the standard two BCAM based angle monitoring. One of these advantages is given in the position monitoring of sliding parts as required for the alignment of the five large barrels of the CMS super-conducting solenoid.

One further application is the monitoring of the fragile ALICE central beryllium beam pipe. This beam pipe with a diameter of 59.6 mm and 0.8 mm wall thickness is supported at three points. In order to minimize the deflections and hence stresses in the beam pipe, one of the three support structures was designed with the aid of finite element analysis. The pipe will operate in an environment of 0.5 T magnetic field and is expected to absorb a dose of 10 kGy in ten years. These special constraints and the lack of access preclude most standard force monitoring systems. Previous work has shown that strain gage based systems work well under these conditions.

The thesis presents an optimized strain gage based system for the ALICE beam pipe that is sensitive to changes in force of 1N. Both the BCAM - retroreflector system and the strain gage based force monitoring system provide critical information regarding the status of the beam pipe, ITS (Inner Tracking System) and forward detector systems. Furthermore the thesis describes first BCAM - reflector measurement results from the first CMS barrel closure.

Modellbildung und Strategien zur Luftpfadregelung eines aufgeladenen Dieselmotors mit Abgasrückführung und variabler Turboladergeometrie

Dipl.-Ing. Hannes Seyrkammer

Begutachter: o.Univ.-Prof. Dipl.-Ing. Dr. Kurt Schlacher

Bei dem in einem Dieselmotor stattfindenden Verbrennungsvorgang werden aufgrund der hohen Spitzentemperaturen vermehrt die unerwünschten Reaktionsprodukte NO und NO_2 (kurz Stickoxide oder NO_x) erzeugt. Eine effektive Methode, die Bildung von NO_x zu reduzieren ist, einen Teil des Abgases mit Hilfe des so genannten Abgasrückführventils (AGR-Ventil) über einen Abgaskühler (AG-Kühler) zurück zum Luftsammler zu leiten. Durch den Anteil des Abgases in der dem Motor zugeführten Luft wird die Spitzentemperatur bei der Verbrennung reduziert und so eine Bildung der unerwünschten Stickoxide unterdrückt. Der Anteil des rückgeführten Abgases hängt dabei von dem aktuellen Betriebszustand des Motors, welcher im Wesentlichen durch seine Drehzahl und der eingespritzten Brennstoffmenge charakterisiert wird, ab. Weiters wird, um die Leistung des Motors zu erhöhen, die Maschine oft mit einem Turbolader ausgestattet. Dieser besteht im Wesentlichen aus einer Turbine und einem Verdichter, welche durch eine Welle miteinander verbunden sind. Im Betrieb wird dadurch ein Teil der im Abgas vorhandenen Energie durch die Turbine zum Verdichter transferiert, um den Ladedruck im Luftsammler zu erhöhen. Durch den damit verbundenen Temperaturanstieg der verdichteten Frischluft ist es notwendig, diese im Anschluss zu kühlen, um den gewünschten Effekt des Turboladers nicht wieder zu verlieren. Bei Turboladern mit festem Leitapperat war es notwendig, einen Kompromiss zwischen schnellen Ansprechzeiten bei niedrigen Motordrehzahlen und einer hoher Motorleistung bei hohen Motordrehzahlen zu finden. Dies führte zur Entwicklung der variablen Turbinengeometrie (VTG), um die Menge des Abgasmassenstroms durch die Turbine, und somit die von der Turbine transferierte Leistung, gezielt und über den gesamten Betriebsbereich des Dieselmotors zu beeinflussen.

In dieser Arbeit wird die Modellbildung für das Luftsystem eines solchen Dieselmotors, ausgestattet mit einer Abgasrückführung und einem Turbolader mit variabler Turbinengeometrie, sowie Strategien zur Regelung vorgestellt. Ziel der Regelung ist es, von der Motorsteuerung vorgegebene Werte für den Verdichtermassenstrom und für den Ladeluftdruck simultan zu regeln. In dieser Arbeit werden dabei Methoden zur Regelung von nichtlinearen Mehrgrößensystemen auf das gegebene System angewandt.

Der erste Teil der Arbeit beschäftigt sich mit einer physikalischen Modellbildung der relevanten Komponenten des Luftsystems, wobei im ersten Schritt auf eine detaillierte, modulare Beschreibung Wert gelegt wird. Dieses mathematische Modell wird anschließend mittels Messungen, welche am Motorenprüfstand der BMW-Motoren GmbH Steyr durchgeführt wurden, abgeglichen. Basierend auf diesem Modell wird im Anschluss eine Reduktion vorgenommen, um eine wesentliche Steigerung der Simulationsgeschwindigkeit zu erreichen. Dieses vereinfachte Modell dient weiters als Referenzmodell für verschiedene Regelungsstrategien.

Der zweite Teil dieser Arbeit beschäftigt sich mit der Regelung des vorliegenden nichtlinearen Mehrgrößensystems. Die gewählten Zugänge bedienen sich dabei der Methoden aus dem Bereich der neueren nichtlinearen Regelungstheorie, um die gestellten Forderungen, simultane Regelung des Verdichtermassenstromes und des Ladeluftdrucks, zu erfüllen. Für die Entwicklung der Regelungsstrategien werden weitere Vereinfachungen im Modell vorgenommen und in Hinblick auf die Regelung analysiert. Es werden vier Konzepte vorgestellt, wobei zwei Strategien die Methode der Eingangs/Ausgangs-Linearisierung verwenden. Das dritte Konzept stellt einen flachheitsbasierten Zugang dar, in Kombination mit einer ingenieurmäßigen Vereinfachung betreffend die Vernachlässigung der Dynamik des Abgasgegendrucks. Für das vierte Konzept wird der ingenieurmäßige Zugang näher untersucht, wobei festgestellt wurde, dass bei dem vorliegenden System nicht nur der Abgasgegendruck, sondern auch die Differentialgleichung für den Verdichtermassenstrom einem schnellen Teilsystem zuzuordnen ist. Aufgrund dieser Erkenntnis wurde mittels der Methode der singulären Störung eine Strategie entwickelt, welches dieses natürliche Verhalten des Systems ausnutzt. Das heißt, durch diesen Zugang werden die Dynamiken des vorhandenen langsamen Teilsystems und die des schnellen Systems nicht angeglichen, sondern beibehalten. Die entwickelten Strategien wurden in der Simulation zunächst unter den für den Entwurf idealisierten Annahmen verifiziert. Dabei erzielte die flachheitsbasierte Regelung die besten Ergebnisse. Weiters ist es für eine spätere Umsetzung an einem realen System wichtig, das Reglerverhalten im Beisein von vernachlässigten Komponenten und weiteren nichtidealen Einflüssen zu untersuchen. In der Simulation zeigt sich dabei, dass der Regler, welcher auf dem Zugang der singulären Störung basiert, das robusteste Verhalten von den vorgeschlagenen Strategien aufweist.

Bei der Arbeit wurde neben der Anwendung der Regelungsverfahren auch Wert auf eine wissenschaftliche Erstellung und Analyse der Systemmodelle sowie auf die Stabilitätsnachweise der geschlossenen Regelkreise Wert gelegt. Sowohl der flachheitsbasierte Zugang als auch der Zugang basierend auf der singulären Störung zur Regelung des Luftsystems sind neue Ansätze auf diesem Gebiet. Durch die Analyse des singulär gestörten Systems wird ein guter Einblick in das Systemverhalten gegeben und ermöglicht einen einfachen Reglerentwurf.

Neuronales Data Mining unter chaostheoretischen Aspekten in Technisch-Kybernetischen Systemen

Dipl.-Ing. Werner Toplak

Begutachter: O.Univ.Prof. Dipl.-Ing. Dr.techn. Dr.mult.h.c. Peter Kopacek

Diese Arbeit befasst sich mit dem Erkennen von Systemeigenschaften. Die Forschungslandschaft gestaltet sich immer interdisziplinärer und es treffen viele unterschiedliche Persönlichkeiten aufeinander, die an der gemeinsamen Lösung von Problemen arbeiten. Ein jeder sieht ein Problem aus seinem individuellen Blickwinkel. In Summe gesehen ergeben sich hybride Anwendungen, in denen sich verschiedene Lösungsansätze ergänzen, vereinen und absichern.

Die visuelle Aufbereitung von Beobachtungsdaten, auch als Visuelles Data Mining bekannt, ist ein Grundstein dieser Arbeit. Die Kreation von informationsdichten Visualisierungen als Diskussionsgrundlage in interdisziplinären Teams hilft disziplinfremde Ansätze leichter transportabel und verständlich zu machen. In den letzten Jahrzehnten hat die Künstliche Intelligenz (KI) Einzug in viele Bereiche der Wissenschaft gehalten. Auf der Forschungsseite beschäftigt man sich in der Verkehrstelematik seit einigen Jahren mit Künstlichen Neuronalen Netzen (KNN) zur Mustererkennung, Klassifikation und Prognose. Im Vergleich zu früher ist unsere Welt mittlerweile in Echtzeit messbar. Die Datenbestände wachsen täglich und oft haben die Betreiber der Datenaufzeichnungen noch keine Ahnung, wonach sie eigentlich suchen sollen. Ökologisch, technisch und ökonomisch gibt es sehr viel Optimierungspotenzial, um unsere Systeme nachhaltig zu betreiben. Rohstoffeinsatz, Energieverbrauch und Schadstoffemissionen sind Kernpunkte des öffentlichen Interesses.

In einem mikroskopisch motivierten Ansatz wurden KNN zur Klassifikation bzw. Prognose von Verkehrskenngrößen eingesetzt. Die Kurzfristprognose (15 Minuten bis 2 Stunden) von Geschwindigkeiten rückt aufgrund von LKW-Mautdaten in das Zentrum der Betrachtungen. KNN eignen sich aufgrund ihrer sehr kurzen Recall-Phase für die Schonung von Rechnerressourcen im laufenden Betrieb. Dies ist ein maßgeblicher Unterschied im Vergleich zu Expertensystemen (XPS), die Wissen nicht assoziativ während eines Lernprozesses bilden, sondern durch Abfragen einen Datenbestand (Wissensbasis) auf Ähnlichkeit durchsuchen.

Einerseits drehen sich die Fragen um die Datenselektion hinsichtlich chaostheoretischer Konzepte zur Optimierung von Lernprozessen. Der *β-Converter* ist ein auf MatLab basierender Prototyp, mit dessen Hilfe ein erster Zugang in der Darstellung von chronologischen Messwerten gegeben wird. Weiters wird ein Ansatz vorgestellt, der sich der Indikatoren für Stabilität (Entropie und Ljapunow-Exponent) bedient: Das so erfundene ALEV-Verfahren (Aspekte von Ljapunow, Entropie und Varianz) wurde zur Patentierung angemeldet und eignet sich für die Reduktion großer Datenbestände von Zeitreihen. Auf dieser Basis wurden *Multi-Layer-Perceptrons* (MLPs) und weitere fortgeschrittene Feed-Forward Netze mit den so chaos-theoretisch gefilterten Datensätzen beschickt und hinsichtlich ihrer Prognosegüte untersucht. Dabei erfolgten auch Untersuchungen hinsichtlich der Einbettungsdimension von Geschwindigkeitssequenzen. Mit dem Einsatz von genetischen Algorithmen wurden KNN evolutionär optimiert. Auf Basis dieser Ergebnisse entstand ein Verkehrsinformationssystem, in dessen Prognosemodul mehr als 900 MLPs operativ arbeiten und das die österreichischen Autobahnen abdeckt. Sensitivitätsanalysen ermöglichen tiefere Einblicke in das abgebildete nichtlineare Systemverhalten. Perturbationsdiagramme ermöglichen das Ergründen von Kausalauswirkungen und deren zeitliche Relevanz in der Vergangenheit. Eine entsprechende Reduktion der Eingangsdimensionen wurde untersucht. Zuletzt wird ein Ausblick gegeben, der sich mit zukünftigen Anwendungen und Forschungsthemen beschäftigt.

Der übergeordnete Begriff *Neuronales Data Mining* (NDM) rückt dabei den Menschen und die Maschine erkenntnistheoretisch näher zusammen und den Menschen dabei im speziellen in den Vordergrund: Er ist es, der beschreibt und modelliert. Jede Theorie ist ein weiterer Aspekt auf der Suche nach einem Gesamtbild. Und das System, das es zu beschreiben gilt, ist von technisch-kybernetischer Natur: sozial und biologisch dominiert, technologisch getragen. Hochkomplex und nicht linear.

Basic Real-Time Reconfiguration Services for Zero Down-Time Automation Systems

Dipl.-Ing. Alois Zoitl

Begutachter: Univ.Prof. Dipl.-Ing. Dr. techn. Bernard Favre

The ever accelerating globalisation and heavy competition from low-wage countries force companies with own production to adopt new technologies. Future production facilities need to be flexible, adaptable, and capable of fast changes with little costs. Several approaches have been conducted in order to provide more flexibility and adaptability to production facilities. Most of these approaches were focused on only the higher levels of industrial control systems and did not consider measures to increase flexibility and adaptability on the lower level real-time control layers. A promising approach for providing flexibility and adaptability is dynamic reconfiguration. Dynamic reconfiguration allows changing applications during operation of a system. State of the art real-time control systems are generally not suited to adopt the feature of dynamic reconfigurability, mainly because of their central architecture. The architecture defined in the new standard from the International Electrotechnical Commission, IEC 61499, is a suitable basis for a dynamic reconfiguration approach. It provides a distributed architecture and basic reconfiguration support. However IEC 61499 is still missing definitions and mechanisms for dynamic reconfiguration and real-time execution of control applications. Both are main requirements for adaptive industrial control systems which are targeted in this work.

The goal of this thesis is to provide an execution environment for the lower level real-time control layer of automation systems, supporting dynamic reconfiguration of control applications. This includes the execution of real-time constrained control applications and the support for changing the control applications while they are executing and controlling the plant. The approach for achieving this goal is twofold: First to develop mechanisms that allow to conduct reconfiguration processes by user programmable reconfiguration applications, and second, to develop a real-time execution mechanism for control applications and reconfiguration applications.

A reconfiguration application conducts the reconfiguration process, which transforms the existing application into the target application. As the reconfiguration application is a user programmable application it can be adapted to different reconfiguration scenarios. Such an application monitors and changes the control application through a set of reconfiguration services. We identify the smallest possible set of reconfiguration services still allowing to conduct any reconfiguration process possible in an IEC 61499 control system. For conducting a dynamic reconfiguration process during the operation of a plant a timely execution of the reconfiguration applications is necessary.

The basis for the new execution approach is the existing real-time execution theory from the domain of real-time computer systems. We investigate what elements in IEC 61499 are best suited to be mapped to a real-time task. Existing approaches were limited on structural elements. We extend this investigation towards logical elements. The logical element that suits most is the so called Event Chain. An Event Chain is the execution that takes place in an IEC 61499 application triggered from outside the control device (e.g. interrupts, network messages). The Event Chain and the external triggers define the structure of the new execution environment. While applying different scheduling algorithms we are able to provide real-time constrained execution of IEC 61499 applications.

In order to test and verify the developed concepts we conduct several experiments. These are on the one hand real-time execution experiments and on the other hand real-time reconfiguration experiments. The final experiment is a real-world control example: the balancing of an inverted pendulum. Different control scenarios are applied. Furthermore the control scenarios are switched during the balancing of the pendulum.

It is the intention of the author to provide a significant technological progress towards adaptive production systems with zero down-times. The results provide means of real-time reconfiguration on the lower levels of industrial control systems. As a technological vision, similar reconfiguration mechanisms could be levered up to higher control levels, allowing self adaptive production systems with high efficiency also for very small lot sizes.

Modeling and control of the hot rolling process of heavy plates in view of the elimination of ski–ends

Dipl.-Ing. Thomas Kiefer

Begutachter: Univ.Prof. Dipl.-Ing. Dr. techn. Andreas Kugi

In heavy plate mills, the customer demands on product thickness and flatness quality are steadily increasing. Amongst others the quality is primarily influenced by the processing at the finishing mill stand where the thickness of plates of different widths and lengths is decreased in several passes. As a consequence, the improvement of the process control is a permanent subject of research. In the last years it has turned out that especially the development of accurate physics–based models, which serve as a basis for the controller design and for the system optimization, leads to very good results in view of a further improvement of the product quality. The goal of this thesis is the development of such models and control strategies for the hot rolling process at the plant of the AG der Dillinger Hüttenwerke. Thereby, special emphasis is placed on the derivation of online executable models for the plastic material behavior in the roll gap.

In the first part of the thesis a forward slip model is developed, which is used for the product tracking of the rolled plates. This product tracking constitutes an essential part for the design of a feedforward control concept for controlling the thickness profile of the rolled plates. The analytical model is based on an exact solution of the governing system of partial differential equations for the material behavior in the roll gap. A viscoplastic material behavior is assumed and a solution can be found by means of an appropriate change from Cartesian to bipolar coordinates which describe the roll gap geometry in a natural way. The results are compared with classical models as well as measurement data.

The second part is devoted to the modeling of the so–called ski–effect, i.e. the phenomenon of bended plate–ends, as well as to the development of a control concept for the avoidance of the ski–ends. This unwanted effect, which results from asymmetries in the roll gap due to different circumferential speeds, different work roll radii, different friction parameters or vertical temperature inhomogeneities, brings along a degradation of the plate quality with respect to the flatness properties and may lead to problems in the further processing steps. The mathematical model is based on the Upper Bound Method which in general is used to derive semi–analytical models representing approximate solutions of the governing system of partial differential equations. The advantage of this method is the good trade–off between high accuracy and short execution time. Thus, this model can also be used within the process control unit. In a first step, a detailed model is derived and compared with Finite Element simulations. Furthermore, the model can be simplified by taking into account the analogy between rolling and flat compression such that it is possible to further reduce the execution time. Finally, the model is validated by means of plant measurements.

Based on this semi–analytical model it is possible to design a controller for the minimization of ski–ends. The control concept consists of two parts: The first part concerns an improvement of the classical control concept for the drive train which constitutes the only effective actuator in this case. Thereby, this control concept is designed in such a way that it is directly possible to control the circumferential speed difference between the upper and the lower work roll. This speed controller is used to reject unwanted speed differences and the resulting ski–ends which may occur due to different gripping and friction conditions at the upper and the lower work rolls, especially at the beginning of a pass. In the second part, an outer controller is derived to reject ski–ends which are provoked by asymmetries in the temperature profile over the plate height. This controller is based on the semi–analytical ski–end model and provides a desired circumferential speed difference at the very beginning of a pass for the improved speed controller. Simulation results emphasize the feasibility of the combination of both controllers.

Fertiggestellte Diplomarbeiten

Nichtlineare Regelung zur Unterdrückung von Schwingungen im Kaltwalzsimulator

Bernhard Ramsebner; Institut für Regelungstechnik und Prozessautomatisierung, Universität Linz

Modellierung eines Kaltwalzsimulators für die aktive Schwingungsunterdrückung

Christian Mayrhofer, Institut für Regelungstechnik und Prozessautomatisierung, Universität Linz

Nichtlineare Beobachter für Drehfeldmaschinen

Harald Daxberger, Institut für Regelungstechnik und Prozessautomatisierung, Universität Linz

Mechanische und regelungstechnische Charakterisierung von Knickarmkinematiken am Beispiel des Bearbeitungsroboters robmill

Alois Wiesinger; Institut für Regelungstechnik und Prozessautomatisierung, Universität Linz

End-of-Life Management von Handys

Peter Lichtenwöhrer, IHRT, TU Wien

Entwicklung eines Videosystems für das Furushiki2 Projekt

Lukas Wallentin, IHRT, TU Wien

Anwendungs- und Adaptionspotentiale sechsbeiniger Roboter

Nikolaus Christoph, IHRT, TU Wien

EUROBY 2008

15. – 17. Juni 2008 in Zürich(Schweiz)
19. – 22. Juni 2008 in Linz (Österreich)

EUROBY 2008 findet, parallel zur EURO 2008 vom 15 – 17 Juni 2008 in Zürich und vom 19 – 22 Juni 2008 in Linz statt. Koordiniert vom IHRT der TU Wien, erfolgt die Organisation in Linz durch ein Konsortium unter Leitung des „ARS Electronica Center" und in Zürich durch das Intelligence Laboratory (AILab) der Universität Zürich als Spezialevent im Schweizer Jahr der Informatik 2008.

Zum Unterschied von den menschlichen Stars sind Roboterfußballer würfelartige Gebilde, welche von PC's ferngesteuert werden. Allerdings haben die 5 oder 11 Roboter mit einer eigenen Kantenlänge von 7.5 cm ein Spielfeld zwischen 180 x 210cm bis 280 x 400cm und einen Golfball zur Verfügung. Nach dem Anpfiff übernimmt der Teamrechner das Spiel. Er bekommt über eine Kamera die derzeitigen Positionen der Roboter 120mal in einer Sekunde mitgeteilt und entscheidet blitzartig über die nächsten Spielzüge seiner Roboter. Die Befehle werden diesen über Funk übermittelt.

Roboterfußball ist eine neue „High tech" Anwendung von Mechatronik, Robotik, Softwaretechnologie und künstlicher Intelligenz. Roboterfußball bildet die Basis für zukünftige industrielle Anwendungen, wie beispielsweise intelligente, mobile Roboter ,welche sich selbstständig in unbekannter Umgebung (Fabrikshallen, Museen, Krankenhäuser....) bewegen. Laut jüngsten Studien stellt diese neue Robotergeneration zukünftig eine enorme Marktchance für die Industrie dar.

Gespielt wird in den Kategorien:
Mirosot (Micro Robot Soccer Tournament) – Roboter 7.5 x 7.5 x 7.5cm.

Middle League	(5 gegen 5, Spielfeld: 2,2 m x 1,8 m);
Extended Middle League	(5 gegen 5, Spielfeld: 2,2 m x 2,8 m);
Large League	(11 gegen 11, Spielfeld: 4,0 m x 2,8 m);

sowie in jener der „ Winzlinge" mit Robotern der Größe 4x4x5cm

Narosot	(5 gegen 5 Spielfeld: 1,8 m x 2,2 m).

Eine besondere Attraktion sind zweibeinige, menschenähnliche Roboter welche zwar noch nicht richtig fußballerisch tätig sind aber in der Kategorie „Hurosot" ihr Können unter Beweis stellen.
Die Teams der EUROBY 2008 kommen aus Deutschland, England, Holland, Kosovo, Kroatien, Österreich, Russland, Slowenien und der Slowakei.
Dieses sportliche „High Tech" Großereignis wird von einem kulturell-technischen Rahmenprogramm begleitet.

Aktuelle Informationen: www.euroby2008.at

Datum	***Veranstaltung***	***Ort***	***Weitere Informationen erhältlich bei:***
21.-23.5.2008	2nd Korean-Austrian Automation Day	*Changwon Korea*	e-mail: kopacek@ihrt.tuwien.ac.at
12.-14.2008	AACC IFAC Conference American Control Conference ACC 2008 – in cooperation with IFAC	*Seattle USA*	e-mail: aanna@mit.edu http://www.a2c2.org/conferences/acc2008/
15.–17.6.2008	EUROBY 2008	*Zürich Schweiz*	e-mail: kopacek@ihrt.tuwien.ac.at http://www.euroby2008.at
19.-21.2008	CIRAS 2008 Int. Conference on Computational Intelligence, Robotics and Autonomous Systems	*Linz Austria*	e-mail: norman.weiss@ciras2008.org http://www.ciras2008.org
19.-22.6.2008	EUROBY 2008	*Linz Austria*	e-mail: kopacek@ihrt.tuwien.ac.at http://www.euroby2008.at
6.-11.7.2008	IFAC Congress World Congress	*Seoul Korea*	e-mail: secretariat@ifac2008.org http://www.ifac2008.org
8.-10.9.2008	CLAWAR 2008-02-14 11th Int.Conference on Climbing & Walking Robots & the Support Technologies for Mobile Machiens	*Coimbra Portugal*	e-mail: clawar2008@isr.uc.pt http://www.isr.uc.pt/clawar2008
15.-17.9.2008	RAAD 2008 17th Int.Workshop on Robotics in Alpe-Adria-Danube Region	*Ancona Italy*	e-mail: raad@univpm.it http://www.dipmec.univpm.it/RAAD08
9.-10.10.2008	IFAC IFIP Workshop Intelligent Manufactoring Systemy (9th) – IMS	*Szczecin Poland*	e-mail: ims08@ps.pl http://www.ims08.ps.pl
June 2009	RAAD 2009 18th Int.Workshop on Robotics in Alpe-Adria-Danube Region	*Vienna Austria*	e-mail: kopacek@ihrt.tuwien.ac.at

Datum	***Veranstaltung***	***Ort***	***Weitere Informationen erhältlich bei:***
30.6.-3.7.2009	IFAC Symposium Fault Detection, Supervision and Safety for Technical Process – SAFEPROCESS	*Barcelona Spain*	e-mail: joseba.quevedo@upc.edu http://safeprocess09.upc.es
12.-15.7.2009	IFAC Symposium Advanced Control of Chemical Prosesses - ADCHEM	*Istanbul Turkey*	e-mail: dburak@ku.edu.tr http://www.ku.edu.tr/ku/index.php?option=com_content&task=view&id=1954&Itemid=2892
9.-12.9.2009	IFAC Symposium Robot Control, SYROCO 2009-9th	*Gifu Japan*	e-mail: syroco2009_office@syroco2009.org http://www.syroco2009.org

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