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Abstract

In this paper, a Luenberger-type boundary observer is presented for a class of distributed-parameter systems described by time-varying linear hyperbolic partial integro-differential equations. First, known limitations due to the minimum observation time for simple transport equations are restated for the considered class of systems. Then, the backstepping method is applied to determine the unknown observer gain term. By avoiding the framework of Gevrey-functions, which is typically used for the time-varying case, it is shown that the backstepping method can be employed without severe limitations on the regularity of the time-varying terms. A modification of the underlying Volterra transformation ensures that the observer error dynamics is equivalent to the behaviour of a predefined exponentially stable target system. The magnitude of the observer gain term can be traded for lower decay rates of the observer error. After the theoretic results have been proven, the effectiveness of the proposed design is demonstrated by simulation examples.

Key words: Distributed-parameter System; Hyperbolic PIDE; Luenberger-type Observer; Boundary Observer; Backstepping.

1 Introduction

While the topic of boundary control for first-order hyperbolic partial differential equations (PDEs) has been thoroughly investigated over the last twenty years, see, e.g., [4,19,20,12,5,9], the problem of state observation has been addressed only recently [3,24,8,7]. Most of these contributions apply the backstepping method introduced by Smyshlyaev and Krstic [21], which maps the observer error dynamics onto a desired (exponentially stable) target system using Volterra integral transformations. The strength of this approach is its structural simplicity, the broad range of possible time-invariant and time-varying plants [22,13] and the possibility to combine it with other concepts, as for instance flatness-based feedforward control [16].

A time-invariant version of the class of linear first-order hyperbolic PIDEs considered in this paper was introduced in [12] which is closely related to the parabolic type treated in [21]. Such PIDEs usually arise from two coupled PDEs where one can be perturbed suitably. Very recently, the boundary control concept presented in [12] was extended to an adaptive output-feedback design able to deal with unknown parameters [1] and systems with Fredholm operators that do not exhibit a strict-feedback structure [2].

While a filter-based state observer with non-adjustable error dynamics is used in [1] for time-invariant plants in the course of designing an output-feedback law using a backstepping pre-transformation, this paper is concerned with a Luenberger-type observer for time-varying hyperbolic PIDEs. Time-varying backstepping designs are usually treated by employing the framework of Gevrey-functions [25,16,15,11] which imposes strict conditions on the regularity of the time-varying terms. As this paper shows, this can be avoided for linear hyperbolic equations. By using a modified backstepping transformation, a general PIDE can serve as target system and thus a desired observer error dynamics can be chosen. Therefore, it enables to trade slower error dynamics for reduced sensitivity to noise.

The paper is structured as follows: First, the problem

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under consideration is introduced in Section 2. Minimum observation times known from simple transport systems give a lower bound of what can be achieved theoretically. Thus, it is shown in Section 3 that hyperbolic PIDEs are subject to the same limitation. Section 4 follows the backstepping approach to calculate the desired observer gain term. Finally, Section 5 applies the design to specific examples and analyzes the influence of the design parameters, introduced by the general target system in Section 4, on the error dynamics.

2 Problem statement

In the following, the observer design is considered for systems of the form

$$\begin{aligned} x_t(z,t) = & \, x_z(z,t) + a(z,t)\,x(z,t) + g(z,t)\,x(0,t) \\ & + \int_0^z f(z,\xi,t)\,x(\xi,t)\,\mathrm{d}\xi, \end{aligned} \tag{1a}$$

with boundary and initial conditions

$$x(z,0) = x_0(z) \tag{1b}$$
$$x(1,t) = u(t) \tag{1c}$$

and the system output

$$y(t) = x(0,t) \tag{1d}$$

defined on the domain $(z,t) \in \Omega = (0,1) \times \mathbb{R}^+$. Here, $u(t)$ represents an external input. The functions $a(z,t)$, $g(z,t)$ and $f(z,\xi,t)$ with $z,\xi \in [0,1]$ and $t \in \mathbb{R}^+$ are assumed to be continuous in z, t and ξ, respectively and bounded in time. A distributed-parameter Luenberger-type observer with the observer state $\hat{x}(z,t)$ is formulated in the form

$$\begin{aligned} \hat{x}_t(z,t) = & \, \hat{x}_z(z,t) + a(z,t)\,\hat{x}(z,t) + g(z,t)\,\hat{x}(0,t) \\ & + p(z,t)\,(y(t) - \hat{y}(t)) + \int_0^z f(z,\xi,t)\,\hat{x}(\xi,t)\,\mathrm{d}\xi, \end{aligned} \tag{2a}$$

with the observer's boundary and initial conditions

$$\hat{x}(z,0) = \hat{x}_0(z) \tag{2b}$$
$$\hat{x}(1,t) = u(t) \tag{2c}$$

and the corresponding observer output

$$\hat{y}(t) = \hat{x}(0,t). \tag{2d}$$

In view of (1) and (2) the dynamics of the observer error $e(z,t) = x(z,t) - \hat{x}(z,t)$ follows as

$$\begin{aligned} e_t(z,t) = & \, e_z(z,t) + a(z,t)\,e(z,t) + p_1(z,t)\,e(0,t) \\ & + \int_0^z f(z,\xi,t)\,e(\xi,t)\,\mathrm{d}\xi, \end{aligned} \tag{3a}$$

with the associated boundary and initial conditions

$$e(z,0) = e_0(z) \tag{3b}$$
$$e(1,t) = 0 \tag{3c}$$

using

$$p_1(z,t) = g(z,t) - p(z,t). \tag{4}$$

The unknown observer gain $p(z,t)$ has to be determined such that the observer state $\hat{x}(z,t)$ converges to the system state $x(z,t)$ in the sense of the L^2-norm, i.e., that the error dynamics (3) is exponentially stable in the L^2-norm.

Remark 1 *If instability is introduced to* (1) *through the output feedback* $g(z,t)x(0,t)$ *only, the observer error dynamics can be stabilized by choosing* $p(z,t) = g(z,t)$.

3 Minimum observation time

It is well known that the observability of simple transport systems (i.e. with $f \equiv g \equiv 0$) requires a minimum observation time $T_m = 1$. Analyzing observability of distributed-parameter systems is usually done by using operator semigroup theory, see, e.g., [18,23]. Since these methods require a closed-form or series solution, an alternative approach is chosen to show that the same minimum observation time also serves as a necessary condition for the considered class of PIDEs (1).

Lemma 2 *The system* (1) *can only be observable for*

$$t \geq T_m = 1. \tag{5}$$

PROOF. Without loss of generality, (1) is restricted to $g \equiv 0$ since the term $g(z,t)x(0,t)$ is perfectly known. Applying the method of characteristics yields the implicit integral equation

$$\begin{aligned} x(z,t) = & \, u\,(t+z-1) + \int_z^1 a(\sigma,t+z-\sigma)x(\sigma,t+z-\sigma)\mathrm{d}\sigma \\ & + \int_z^1 \int_0^\sigma f\,(\sigma,\xi,t+z-\sigma)\,x\,(\xi,t+z-\sigma)\;\mathrm{d}\xi\,\mathrm{d}\sigma. \end{aligned} \tag{6}$$

This equation shows that the solution at a single point (z^*,t^*) depends on the solution of a whole subset of Ω, i.e. the domain of dependence

$$\Omega_{D^*} = \{(z,t) \in \Omega \mid z^* \leq \sigma \leq 1 \text{ and } 0 \leq z \leq \sigma\} \subset \Omega \tag{7}$$

with $\sigma = z^* + t^* - t$. Hence, Ω_{D^*} is determined by the inequalities

$$z \geq 0, \tag{8a}$$
$$z - z^* \leq -(t - t^*), \tag{8b}$$
$$t \leq t^*, \tag{8c}$$
$$t - t^* \geq z^* - 1. \tag{8d}$$

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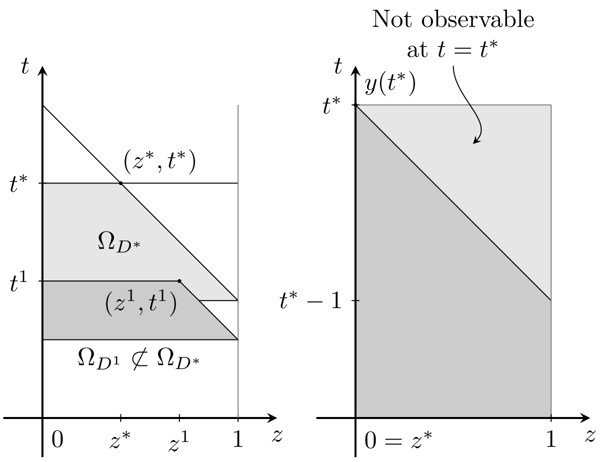

Fig. 1. The region of dependence Ω_{D^*} for an arbitrary point $(z^*, t^*) \in \Omega$ is depicted on the left. For an additional point $(z^1, t^1) \in \Omega_{D^*}$ it follows that $\Omega_{D^1} \not\subset \Omega_{D^*}$. Considering the output $y(t^*)$ by setting $z^* = 0$ as shown on the right, points that comply with $t \geq t^* - z$ might be observable at t^*, while the remaining region is certainly not.

However, the solution for points $(z, t) \in \Omega_{D^*}$ will depend on points outside of Ω_{D^*}. For example, a point $(z^1, t^1) \in \Omega_{D^*}$ has its own domain of dependence $\Omega_{D^1} \not\subset \Omega_{D^*}$ as shown in Fig. 1. Therefore, the question is if any inequality (8) will hold for iterative application of this dependence relation. It is easy to see that (8a), (8b) and (8c) indeed hold for this iterative relation. When considering the solution at the left boundary, e.g., $(z, t) = (0, t^*)$, points that comply with

$$t \geq t^* - z \tag{9}$$

do not influence the solution and thus the output $y(t^*)$. As a consequence, this part of the domain Ω can not be observable at time t^* (see Fig. 1 on the right). $\square$

Condition (5) is - without further investigation - just a necessary one for hyperbolic PIDEs of type (1).

4 Backstepping design

To determine $p_1(z, t)$ in (3a) and thus the observer gain $p(z, t)$, the backstepping method [21,12,13] is employed in a slightly different way due to the general target system that will be taken into consideration.

4.1 Selection of the target system

The choice of the target system is a crucial step when applying the backstepping method. In the following, let us presume a target system dynamics of the form

$$\begin{aligned} w_t(z,t) &= w_z(z,t) - \mu(z)\, w(z,t) \\ &\quad - \int_0^z h(z,\xi)\, w(\xi,t)\, \mathrm{d}\xi & \text{(10a)} \\ w(1,t) &= 0, & \text{(10b)} \end{aligned}$$

with the design parameters $\mu(z)$ and $h(z, \xi)$. As shown in [17], this system can be written as an abstract Cauchy problem with solutions in $L^2(0, 1)$. For the choice $\mu \equiv h \equiv 0$, the target system (10) constitutes a simple transport equation, as for instance used in [12] and [1] for control and observation problems, respectively. Introducing these additional design parameters allows to influence the decay of the observer error. However, the existence of a minimum observation time, as shown in Section 3, limits the rate of decay which is why the choice of a simple transport equation as target system cannot be surpassed in terms of convergence speed.

The system (10) is not L^2-stable for an arbitrary choice of design parameters. A sufficient condition is given by the following lemma:

Lemma 3 *The target system* (10) *is exponentially stable if*

$$\mu_{\text{inf}} - h_{\text{sup}} > 0, \tag{11}$$

where

$$\mu_{\text{inf}} = \inf_{z \in [0,1]} \mu(z), \tag{12}$$

$$h_{\text{sup}} = \sup_{(z,\xi) \in \mathcal{T}} |h(z,\xi)| \tag{13}$$

with $\mathcal{T} = \{(z,\xi) \in \mathbb{R}^2 \,|\, 0 \leq \xi \leq z \leq 1\}$. *The norm* $\|w(z,t)\|_{L^2}$ *can be bounded by*

$$\|w(z,t)\|_{L^2} \leq \mathrm{e}^{-(\mu_{\text{inf}} - h_{\text{sup}})\, t} \|w(z,0)\|_{L^2}. \tag{14}$$

PROOF. Using Cauchy-Schwarz's inequality to show that

$$\left| \int_0^1 w(\xi,t) \int_0^\xi h(\xi,y) w(y,t)\, \mathrm{d}y\, \mathrm{d}\xi \right|^2 \leq h_{\text{sup}}^2 \|w(z,t)\|_{L^2}^4, \tag{15}$$

the time derivative of

$$V(t) = \frac{1}{2}\|w(z,t)\|_{L^2}^2 = \frac{1}{2}\int_0^1 w^2(\xi,t)\, \mathrm{d}\xi \tag{16}$$

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along a solution trajectory of (10) yields

$$\begin{aligned}\dot{V}(t) = &\int_0^1 \Bigg[w(\xi,t)w_z(\xi,t) - \mu(\xi)\, w^2(\xi,t) \\ &- w(\xi,t)\int_0^\xi h(\xi,y)w(y,t)\,\mathrm{d}y\Bigg]\,\mathrm{d}\xi \\ \leq &-\frac{1}{2}w^2(0,t) - \mu_{\inf}\|w(t)\|^2_{L^2} - \int_0^1 w(\xi,t)\times \\ &\int_0^\xi h(\xi,y)w(y,t)\,\mathrm{d}y\,\mathrm{d}\xi \\ \leq &-\frac{1}{2}w^2(0,t) - \mu_{\inf}\|w(z,t)\|^2_{L^2} + h_{\sup}\|w(z,t)\|^2_{L^2} \\ \leq &-(\mu_{\inf} - h_{\sup})\,\|w(z,t)\|^2_{L^2} \end{aligned} \tag{17}$$

and thus

$$\|w(z,t)\|_{L^2} \leq \mathrm{e}^{-(\mu_{\inf}-h_{\sup})\,t}\|w(z,0)\|_{L^2}. \qquad \square$$

Remark 4 *For $h_{\sup} \leq 1$, the stronger criterion*

$$\mu_{\inf} - \ln(h_{\sup}) - 1 > 0 \tag{18}$$

can be given by using the Lyapunov function $V(t) = \int_0^1 w^2(z,t)\,\mathrm{e}^{cz}\,\mathrm{d}z$ with an arbitrary positive constant c as proposed in [5] (see Appendix for more details). However, since in either case the integral term can only be bounded with respect to absolute values (as in (15)), both are quite conservative for dominating integral action.

4.2 Determination of the kernel-PIDE

A Volterra transformation

$$e(z,t) = \alpha(z,t)\,w(z,t) - \int_0^z k(z,y,t)\,w(y,t)\,\mathrm{d}y, \tag{19}$$

with the unknown, time-varying kernel function $k(z,y,t)$ and the unknown auxiliary function $\alpha(z,t)$, is used to map (3) onto (10). As will be seen later on, the additional auxiliary function $\alpha(z,t)$ in (19) is closely linked to exponential "pre-transformations". It shall be determined in such a way that $\mu(z)$ in (10a) can be chosen arbitrarily. This is of particular interest since, according to Lemma 3, the target system's stability depends on both $h(z,\xi)$ and $\mu(z)$. A similar approach with a given auxiliary function is used in [17] for the time-invariant case.

Following the backstepping approach by differentiating (19) with respect to z and t and inserting the results into (3) yields the kernel equation

$$\begin{aligned} k_z(z,y,t) + k_y(z,y,t) - k_t(z,y,t) = &-\beta(z,y,t)k(z,y,t) \\ &+ \alpha(z,t)h(z,y) + \alpha(y,t)f(z,y,t) \\ &- \int_y^z k(z,\xi,t)h(\xi,y) + f(z,\xi,t)k(\xi,y,t)\,\mathrm{d}\xi \end{aligned} \tag{20a}$$

using the abbreviation $\beta(z,y,t) = a(z,t) + \mu(y)$. The auxiliary function $\alpha(z,t)$ is determined by

$$\alpha_t(z,t) = \alpha_z(z,t) + \beta(z,z,t)\,\alpha(z,t) \tag{20b}$$

with the arbitrary boundary condition $\alpha(1,t) = 1$. The observer gain finally is linked to the kernel via

$$p_1(z,t) = \frac{1}{\alpha(0,t)}\,k(z,0,t). \tag{20c}$$

The boundary condition

$$k(1,y,t) = 0 \tag{20d}$$

for (20a) can be obtained by evaluating (19) for $z = 1$ and using (3c) and (10b).

4.3 Well-posedness of the kernel equations

The boundary condition (20d) imposed by the backstepping design is not sufficient to determine a solution on the domain $\mathcal{T}\times\mathbb{R}^+$, additional conditions regarding times $t \leq 0$ are required. This could be accomplished by imposing initial conditions on the kernel function and the auxiliary function. Evaluating (19) for $t = 0$ gives

$$e_0(z) = \alpha(z,0)\,w_0(z) - \int_0^z k(z,y,0)\,w_0(y)\,\mathrm{d}y, \tag{21}$$

which relates the three undetermined inital values $w_0(z)$, $\alpha(z,0)$ and $k(z,y,0)$.

The approach used in this paper is to expand the domain to negative times, meaning $\mathcal{K} = \mathcal{T}\times\mathbb{R}$, and apply (20d) to all times $t \in \mathbb{R}$. Solving (20b) by the method of characteristics yields

$$\alpha(z,t) = \exp\left[\int_z^1 \beta(s,s,t+z-s)\,\mathrm{d}s\right] \tag{22}$$

which is why $\alpha(z,t)$ is uniformly bounded from above and below for all $(z,y,t) \in \mathcal{K}$ with the upper and lower bounds

$$\alpha_{\sup} = \sup_{(z,y,t)\in\mathcal{K}} \alpha(z,t) \quad \text{and} \quad \alpha_{\inf} = \inf_{(z,y,t)\in\mathcal{K}} \alpha(z,t) \tag{23}$$

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due to the boundedness of $\beta(z,y,t)$. Using this result, the kernel equation (20a) in fact forms a well-posed boundary value problem on the domain $\mathcal{K}$ according to the following theorem:

Theorem 5 (Well-posedness of (20a)) *For continuous functions $f(z,y,t)$, $h(z,y)$ and $\beta(z,y,t)$ that are bounded in time, the time-variant kernel equation* (20a) *has a unique solution $k(z,y,t) \in C^0(\mathcal{K})$ which is uniformly bounded with the upper bound*

$$|k(z,y,t)| \leq \alpha_{\sup}(h_{\sup} + f_{\sup}) \times \exp\left[(\beta_{\sup} + h_{\sup} + f_{\sup})(1-y)\right], \quad (24)$$

where

$$\beta_{\sup} = \sup_{(z,y,t)\in\mathcal{K}} |\beta(z,y,t)|,$$
$$h_{\sup} = \sup_{(z,\xi)\in\mathcal{T}} |h(z,\xi)|,$$
$$f_{\sup} = \sup_{(z,\xi,t)\in\mathcal{K}} |f(z,\xi,t)|. \quad (25)$$

PROOF. Applying the method of characteristics to (20a) using (20d) yields the implicit integral equation

$$k(z,y,t) = F_0(z,y,t) + F[k](z,y,t) \quad (26a)$$

with

$$F_0(z,y,t) = -\int_0^{1-z} \alpha(\sigma+z, t-\sigma)\, h(\sigma+z,\sigma+y) + \alpha(\sigma+y,t-\sigma)\, f(\sigma+z,\sigma+y,t-\sigma)\, \mathrm{d}\sigma \quad (26b)$$

and

$$F[k](z,y,t) = \int_0^{1-z} \beta(\sigma+z,\sigma+y,t-\sigma)\, k(\sigma+z,\sigma+y,t-\sigma)\, \mathrm{d}\sigma + \int_0^{1-z}\int_y^z \Big[k(\sigma+z,\xi+\sigma,t-\sigma)\, h(\xi+\sigma,\sigma+y) + f(\sigma+z,\xi+\sigma,t-\sigma)\, k(\xi+\sigma,\sigma+y,t-\sigma)\Big]\, \mathrm{d}\xi\, \mathrm{d}\sigma. \quad (26c)$$

Considering the iteration

$$k_{n+1}(z,y,t) = F[k_n](z,y,t) \quad (27)$$

starting with $k_0(z,y,t) = F_0(z,y,t)$, it is easy to show that the assumption

$$|k_n(z,y,t)| \leq L\, \frac{(\beta_{\sup} + h_{\sup} + f_{\sup})^n}{n!}(1-y)^n \quad (28)$$

with $L = \alpha_{\sup}(h_{\sup} + f_{\sup})$ holds for $n \geq 0$. For the next element it follows that

$$\begin{aligned}
&|k_{n+1}(z,y,t)| = \\
&\Bigg| \int_0^{1-z} \beta(\sigma+z,\sigma+y,t-\sigma)\, k_n(\sigma+z,\sigma+y,t-\sigma)\, \mathrm{d}\sigma \\
&+ \int_0^{1-z}\int_y^z \Big[k_n(\sigma+z,\xi+\sigma,t-\sigma) h(\xi+\sigma,\sigma+y) \\
&+ f(\sigma+z,\xi+\sigma,t-\sigma)\, k_n(\xi+\sigma,\sigma+y,t-\sigma)\Big]\, \mathrm{d}\xi\, \mathrm{d}\sigma \Bigg| \\
&\leq L\, \frac{(\beta_{\sup} + h_{\sup} + f_{\sup})^n}{n!} \Bigg\{ \beta_{\sup} \int_0^{1-z} (1-\sigma-y)^n\, \mathrm{d}\sigma \\
&+ \int_0^{1-z}\int_y^z h_{\sup} \underbrace{(1-\sigma-\xi)^n}_{\leq (1-\sigma-y)^n} + f_{\sup}(1-\sigma-y)^n\, \mathrm{d}\xi\, \mathrm{d}\sigma \Bigg\} \\
&\leq L\, \frac{(\beta_{\sup} + h_{\sup} + f_{\sup})^{n+1}}{n!} \int_0^{1-y} (1-\sigma-y)^n\, \mathrm{d}\sigma \\
&\leq L\, \frac{(\beta_{\sup} + h_{\sup} + f_{\sup})^{n+1}}{(n+1)!} (1-y)^{n+1}
\end{aligned} \quad (29)$$

and

$$|k_0(z,y,t)| \leq L, \quad (30)$$

thus proving the statement (28) by induction. Thus, by successive approximation there exists a continuous kernel

$$|k(z,y,t)| \leq \sum_{n=0}^{\infty} |k_n(z,y,t)| \quad (31)$$
$$\leq L \exp\left[(\beta_{\sup} + h_{\sup} + f_{\sup})(1-y)\right]. \quad (32)$$

Analogous to the proof in [12], this already implies uniqueness of the solution

$$k(z,y,t) = \sum_{n=0}^{\infty} k_n(z,y,t). \qquad \square \quad (33)$$

Remark 6 *The proof of Theorem 5 exploits the fact that* (20a) *can be converted into a single implicit integral equation without time-derivatives of the kernel function on the domain $\mathcal{K}$ by applying the method of characteristics. Thus, one avoids a convergence analysis in terms of Gevrey-functions [16,25] and the accompanied limitations on the regularity of the parameter functions. While imposing initial conditions would in principle allow the same treatment, the integration along a characteristic curve to obtain* (26) *would have to distinguish between points that have to be integrated towards the boundary condition or the initial condition, respectively. The resulting pair of coupled implicit integral equations is significantly harder to analyze. However, the extension of the domain requires additional values of the parameter*

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functions for times $t<0$. If no such values can be given due to the nature of the problem, a simple choice is to expand the values of $t=0$ to earlier times. Such an approach is utilized in Section 5. This procedure is imposing specific but unknown initial conditions defined by the parameter values chosen. By expanding the values of $t=0$, one is in fact imposing stationary solutions of (20a) *and* (20b) *as initial conditions.*

4.4 Relation to "pre-transformation"

It shall be pointed out that the modified Volterra transformation is closely related to exponential scalings called pre-transformations as used in [6,12]. Applying the time-varying exponential scaling

$$e(z,t) = \exp\left[\int_z^1 a(s,t+z-s)\,\mathrm{d}s\right]\bar{e}(z,t) \tag{34}$$

to the observer error (3) and the time-invariant scaling $w(z,t)=\exp[\int_z^1 -\mu(s)\,\mathrm{d}s]\bar{w}(z,t)$ to the target system dynamics (10) yields the "traditional" backstepping transformation $\bar{e}(z,t)=\bar{w}(z,t)-\int_0^z \bar{k}(z,y,t)\bar{w}(y,t)\,\mathrm{d}y$ with the kernel function $\bar{k}(z,y,t)=\exp[-\int_z^1 a(s,t+z-s)\,\mathrm{d}s-\int_y^1 \mu(s)\,\mathrm{d}s]k(z,y,t)$ and the corresponding kernel equation

$$\bar{k}_z(z,y,t)+\bar{k}_y(z,y,t)-\bar{k}_t(z,y,t)=\bar{h}(z,y)+\bar{f}(z,y,t) - \int_y^z \bar{k}(z,\xi,t)\bar{h}(\xi,y)+\bar{f}(z,\xi,t)\bar{k}(\xi,y,t)\,\mathrm{d}\xi. \tag{35}$$

with

$$\bar{f}(z,y,t)=\exp\left[-\int_z^1 a(s,t+z-s)\,\mathrm{d}s + \int_y^1 a(s,t+y-s)\,\mathrm{d}s\right] f(z,y,t) \tag{36}$$

$$\bar{h}(z,y)=\exp\left[-\int_y^1 \mu(s)\,\mathrm{d}s + \int_z^1 \mu(s)\,\mathrm{d}s\right] h(z,y). \tag{37}$$

Equation (34) is in fact the time-varying generalisation of the pre-transformations used in [6,12]. Thus, the auxiliary function $\alpha(z,t)$ is determined by (20b) in such a way (cf. its solution (22)) that the Volterra transformation (19) in fact comprises pre-transformations for both, the error system and the target system.

Remark 7 *Since the reaction term $a(z,t)$ could be eliminated by applying* (34)*, one could set $\mu(z)=0$ in* (10) *and remain with a single design parameter $h(z,\xi)$. While this is true in principle, the stability criterion* (11) *is especially conservative in handling the integral term. Thus, by including the reaction term into the integral term through exponential scaling, the estimates of* (11) *are becoming quite vague.*

4.5 Stability of the closed loop

Before being able to prove exponential stability of the observer error dynamics, boundedness of the inverse transformation of (19) has to be shown. This can be done by explicitly calculating its kernel function in the same way as above or by using operator theory as presented in [14] for the case $\alpha(z,t)\equiv 1$.

Lemma 8 (Bounded inverse) *The linear bounded operator $K: L^2(0,1)\to L^2(0,1)$ given by*

$$e(z,t)=(K\,w)(z,t) = \alpha(z,t)\,w(z,t)-\int_0^z k(z,y,t)\,w(y,t)\,\mathrm{d}y \tag{38}$$

has a linear inverse K^{-1} uniformly bounded for $t\geq 0$ with the upper bound

$$N=\|K^{-1}\|_{L^2}=\frac{1}{\alpha_{\inf}}\left(1+M\,\mathrm{e}^{M}\right) \tag{39}$$

and

$$M=\sup_{(z,y,t)\in\mathcal{T}\times\mathbb{R}^+}\left|\frac{k(z,y,t)}{\alpha(z,t)}\right|. \tag{40}$$

PROOF. See Appendix A.

This result enables us to prove stability of the observer error dynamics.

Theorem 9 (Stability of the error dynamics) *The observer error $e(z,t)$ for the time-varying system* (1) *described by* (3) *is exponentially stable in the L^2-norm if* (11) *is satisfied. For the case $h\equiv 0$, the observer error vanishes after the minimum observation time $T_m=1$.*

PROOF. The backstepping transformation (19) has a uniformly bounded inverse and from Lemma 8 it follows that

$$\|w(z,t)\|_{L^2}\leq N\,\|e(z,t)\|_{L^2} \tag{41}$$

with N defined in (39). Thus, combining Lemma 3 and (41), we see that

$$\begin{aligned}\|e(z,t)\|_{L^2} &= \left\|\alpha(z,t)w(z,t)-\int_0^z k(z,y,t)\,w(y,t)\,\mathrm{d}y\right\|_{L^2}\\ &\leq C\|w(z,t)\|_{L^2}\leq C\mathrm{e}^{-(\mu_{inf}-h_{sup})t}\|w(z,0)\|_{L^2}\\ &\leq C\,N\,\mathrm{e}^{-(\mu_{inf}-h_{sup})\,t}\|e(z,0)\|_{L^2}\end{aligned} \tag{42}$$

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with

$$C = \sup_{(z,t)\in\Omega} |\alpha(z,t)| + \sup_{(z,y,t)\in\mathcal{T}\times\mathbb{R}^+} |k(z,y,t)|. \quad (43)$$

This implies exponential stability of the observer error dynamics if (11) is met. If $h \equiv 0$ the target system (10) represents a transport system. Since its solution vanishes for $t > T_m = 1$ (see [12]), the observer error will also vanish due to

$$\|e(z,t)\|_{L^2} \leq C\|w(z,t)\|_{L^2}. \qquad \square \quad (44)$$

5 Simulation examples

The considerations above guarantee the existence and uniqueness of a suitable kernel function to exponentially stablise the observer error dynamics. In this section, the theoretical concepts will be applied to concrete examples. The first example presents the observer design for a simple, time-invariant case where an analytic solution for the kernel function can be obtained. Then, the effects of different choices for the design parameters $\mu(z)$ and $h(z,\xi)$ are studied. Finally, a general time-variant system is treated in the second example comparing the fully-fledged time-variant observer design with a simpler "naive" approach.

Contrary to the dual problem of boundary control, the output-feedback term $g(z,t)$ in (1a) can be compensated directly through the observer gain $p(z,t)$, see (4), and thus does not appear in the kernel equation (20a). Therefore, g is considered identical to zero for the following examples.

5.1 Time-invariant case

Let us consider the system

$$x_t(z,t) = x_z(z,t) + \int_0^z f_c e^{z-\xi}\, x(\xi,t)\, d\xi \quad (45)$$

presented in [12] for the control problem. Fig. 2 shows the behaviour of the plant for $f_c = 4$. A Luenberger-type observer

$$\hat{x}_t(z,t) = \hat{x}_z(z,t) + p(z)(y(t) - \hat{y}(t)) + \int_0^z f_c e^{z-\xi}\, \hat{x}(\xi,t)\, d\xi \quad (46)$$

is constructed by considering the integral kernel $k(z,y)$ and the auxiliary function $\alpha(z)$, which both do not depend on the time t. Therefore, (20a) reduces to

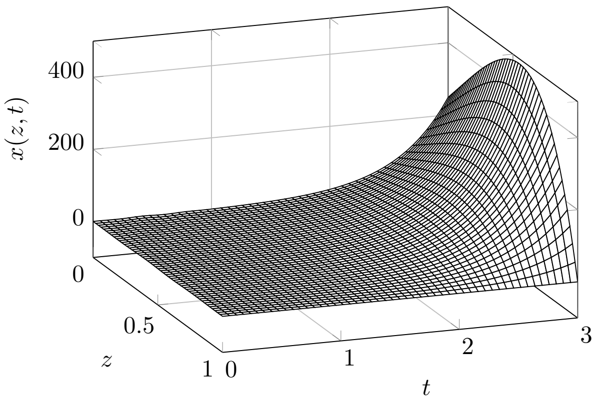

Fig. 2. Behaviour of the plant (45) for $f_c = 4$.

$$\begin{aligned} k_z(z,y) + k_y(z,y) &= -\mu(y)\, k(z,y) \\ &\quad + \alpha(z)h(z,y) + \alpha(y) f_c e^{z-y} \\ &\quad - \int_y^z k(z,\xi)h(\xi,y) + f_c e^{z-\xi} k(\xi,y)\, d\xi \end{aligned} \quad (47)$$

with

$$\alpha(z) = \exp\left[\int_z^1 \mu(s)\, ds\right]. \quad (48)$$

No closed-form solution can be given for this equation with general design parameters. For the special case of a transport-like target system with $h \equiv \mu \equiv 0$ however, it follows that $\alpha(z) = 1$ and thus

$$k_z(z,y) + k_y(z,y) = f_c e^{z-y} - \int_y^z f_c e^{z-\xi} k(\xi,y)\, d\xi \quad (49)$$

with $k(1,y) = 0$. This equation closely resembles the kernel equation presented for the control problem. Following the procedure provided in [12] and [21], the solution to (49) is given by

$$k(z,y) = -f_c\,(1-z)\, e^{z-y}\, \frac{I_1\left(2\sqrt{f_c\,(1-y)(z-y)}\right)}{\sqrt{f_c\,(1-y)(z-y)}} \quad (50)$$

using the modified Bessel functions of the first kind I_n. Thus, using (20c) and (4) the observer gain follows as

$$p(z) = f_c\,(1-z)\, e^z\, \frac{I_1\left(2\sqrt{f_c}\, z\right)}{\sqrt{f_c}\, z}. \quad (51)$$

With this solution, the observation error indeed vanishes for $t > 1$ as shown in Fig. 3.

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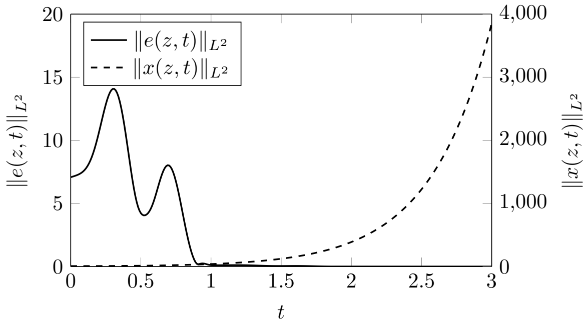

Fig. 3. Time evolution of the L^2-norms of the observer error $e(z,t)$ and the system state $x(z,t)$. While the system is unstable, the observer error vanishes for $t > 1$.

5.2 *Influence of the design parameters*

To gain insight on how to choose the design parameters, they are assumed to be constant for simplicity, i.e. $\mu(z) = \mu_c$ and $h(z,y) = h_c$. Thus, the kernel equation (47) reduces to

$$k_z(z,y) + k_y(z,y) = -\mu_c k(z,y) + h_c\alpha(z) + f_c\alpha(y)\mathrm{e}^{z-y} - \int_y^z h_c k(z,\xi) + f_c \mathrm{e}^{z-\xi} k(\xi,y)\,\mathrm{d}\xi \qquad (52)$$

with $\alpha(z) = \mathrm{e}^{\mu_c(1-z)}$. Equation (52) can be solved with sufficient precision by using a finite approximation of (33). From (42) we know that

$$\|e(z,t)\|_{L^2} \leq C\,N\,\mathrm{e}^{-(\mu_c - |h_c|)\,t}\|e(z,0)\|_{L^2}. \qquad (53)$$

Thus, an approximately similar decay of the observer error is expected for parameter sets fulfilling the condition $\mu_c - |h_c| = c$. Fig. 4 shows the L^2-norms of the observer errors and the corresponding observer gains for four parameter sets with $c = 1$. While the observer errors show some differences prior to $t = T_m = 1$, they decline at approximately the same rate as predicted except for the transport-like case $h_c = 0$, where the simple stability criterion (11) is very restrictive as indicated in Appendix A.2. However, the necessary observer gain $p(z)$ varies considerably. As one would expect, values of h_c closer to the plant's integral action with $f_c = 4$ (notice the negative sign in (10a)) exhibit significantly lower observer gain functions $p(z)$. Since higher observer gains increase the sensitivity to measurement noise, using negative values for h_c can be used to trade dynamics for robustness against noise by lowering c as shown in Fig. 5 and Fig. 6.

Remark 10 *Conversely, a target system like* (10) *could be used for boundary control applications to reduce the necessary control effort.*

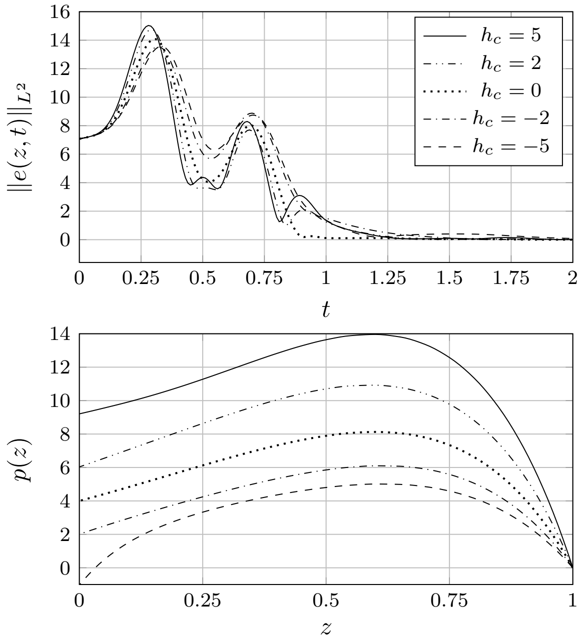

Fig. 4. Observer error $\|e(z,t)\|_{L^2}$ and observer gain $p(z)$ for the plant (45) and the observer (46) with $f_c = 4$ shown for various parameter sets h_c and $\mu_c = |h_c| + 1$.

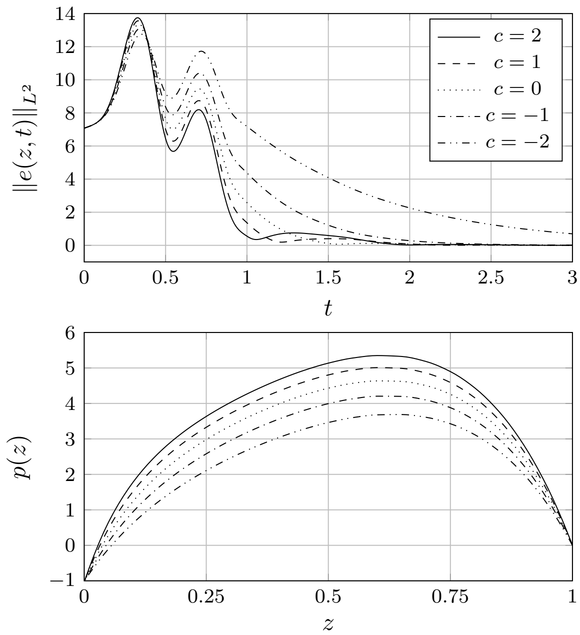

Fig. 5. Observer error $\|e(z,t)\|_{L^2}$ and observer gain $p(z)$ for the plant (45) and the observer (46) with $f_c = 4$ shown for $h_c = -5$ and various values of μ_c.

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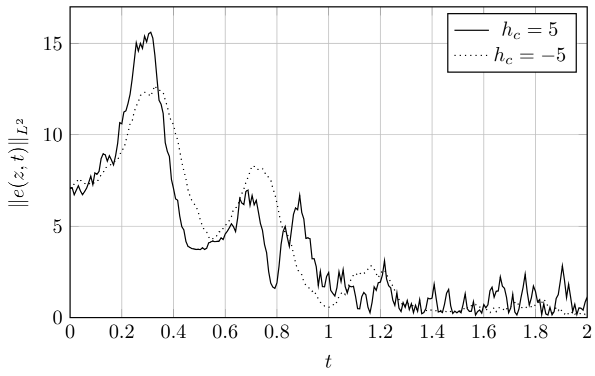

Fig. 6. Presence of noise in the observer error's L^2-norm $\|e(z,t)\|_{L^2}$ for two different observer gain functions (cf. Fig. 4).

5.3 *Time-variant case: Comparison to a naive approach*

A common approach often used in practical applications with time-varying parameters, henceforth also referred to as naive approach, is to consider all parameters time invariant for the controller or observer design and replacing them with the original time-varying versions afterwards. This approach does adjust to the set of parameters currently governing the system but lacks to incorporate their rates of change influencing the dynamic behaviour. Thus, these techniques usually work quite well for small rates of change, while for more rapid changes they fail to achieve their goal, as will be demonstrated by the following example.

Consider the scenario of Section 5.1 but now with a time-varying function

$$f(z,y,t) = f_c(t)\,\mathrm{e}^{z-y}, \tag{54}$$

where

$$f_c(t) = 8\left[1+\sin^2\left(\frac{\pi t}{3}\right)\right]. \tag{55}$$

Regarding the observer design presented in this paper, the naive approach described above essentially boils down to using the time-invariant kernel equation (49) with time-varying parameters. This neglects parts of the structure introduced by the full time-varying observer, see (20a), whose kernel equation

$$k_z(z,y,t) + k_y(z,y,t) - k_t(z,y,t) = f_c(t)\mathrm{e}^{z-y} - f_c(t)\int_y^z k(z,\xi,t)\mathrm{e}^{z-\xi}\,\mathrm{d}\xi \tag{56}$$

contains an additional time derivative $k_t(z,y,t)$ compared to (49). By simply replacing f_c with $f_c(t)$ in (51),

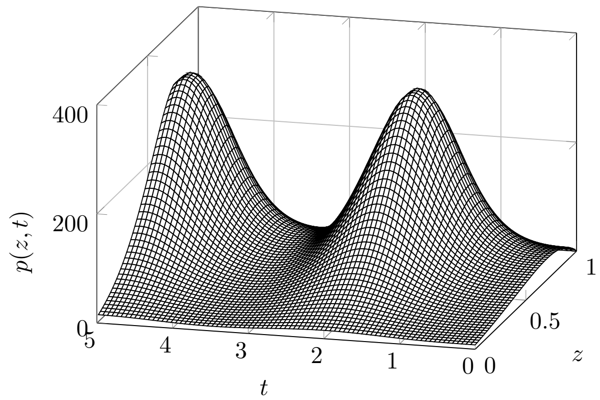

Fig. 7. The observer gain $p(z,t) = -k(z,0,t)$ by solving (56).

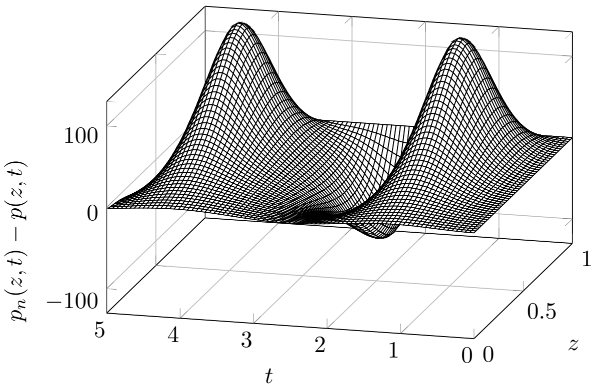

Fig. 8. Difference of observer gains between the naive design (57) and the full time-varying observer design.

the observer gain takes the form

$$p_n(z,t) = f_c(t)\,(1-z)\,\mathrm{e}^z\,\frac{I_1\left(2\sqrt{f_c(t)\,z}\right)}{\sqrt{f_c(t)\,z}} \tag{57}$$

for the naive approach. The solution of (56) (see Fig. 7) can again be obtained by using a finite approximation of (33). According to Remark 6, for times $0 \le t \le 1$ the values at time $t = 0$ were used for $f(z,y,t)$ which is equivalent to using a stationary solution of (56) as initial condition. However, precalculating the observer gain for all times clearly is not a practical solution. Therefore, a numerical method to determine the observer gain similarly to [10] has been employed.

Comparing the fully time-varying solution $p(z,t)$ to the solution (57) using the naive approach $p_n(z,t)$ in Fig. 8 shows that there are significant deviations. The effect of this difference can be seen in Fig. 9. While the full time-varying observer is able to stabilize the error and force it to zero, the naive approach fails to do so.

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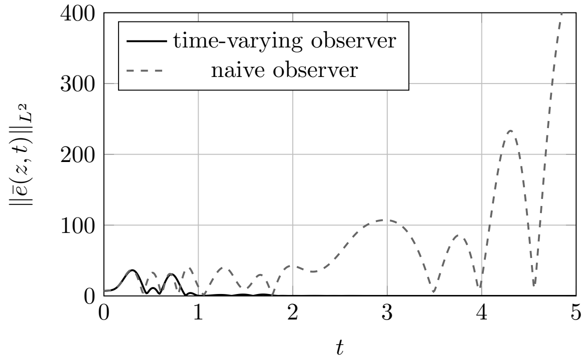

Fig. 9. Comparison of the L^2 norms over time between the naive approach and the full time-varying observer. The naive approach is not capable of bounding the observer error (dashed) while the full time-varying observer (solid) does.

6 Conclusion

In this paper, a Luenberger-type observer for a class of time-varying hyperbolic PIDEs is presented. As shown in Section 3, the class of hyperbolic PIDEs exhibits a minimum observation time analogous to simple transport systems without non-local effects. The unknown observer gain is determined by using the backstepping method. The choice of the target system and its conditions for exponential stability requires to specifically influence the reaction term. This is achieved by introducing an auxiliary function into the backstepping transformation which is determined as part of the design. As shown, this auxiliary function is closely related to exponential pre-transformations and generalizes known time-invariant results. Thus, we are able to reduce the necessary observer gain by allowing a slower decay of the observer error. While this reduces the sensitivity to measurement noise for observers it can be used to reduce the necessary control effort in boundary control applications. By avoiding the Gevrey framework, it is sufficient that the parameter functions are continuous and bounded in time.

The generality of the target system increases the complexity of determining the observer gain while the time-varying nature limits the usability of precalculated kernel functions. Therefore, an efficient algorithm is deemed necessary to determine the observer gain in real time.

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A Appendix

A.1 Proof of Lemma 8

This proof follows the one presented in [14]. By defining $v(z,t) = e(z,t) - \alpha(z,t)\, w(z,t)$, we get

$$v(z,t) = -\int_0^z \frac{k(z,y,t)}{\alpha(y,t)} \left[e(y,t) - v(y,t)\right] \mathrm{d}y. \tag{A.1}$$

To solve this implicit integral equation, again successive approximation is used. Theorem 5 ensures that the kernel function $k(z,y,t)$ is uniformly bounded in time. Due to (22) and (23), the same holds true for $1/\alpha(z,t)$. Setting

$$v_0(z,t) = -\int_0^z \frac{k(z,y,t)}{\alpha(y,t)} e(y,t)\,\mathrm{d}y \quad \text{and}$$
$$v_n(z,t) = \int_0^z \frac{k(z,y,t)}{\alpha(y,t)} v_{n-1}(y,t)\,\mathrm{d}y, \tag{A.2}$$

the absolute value of $v_0(z,t)$ can be bounded by

$$|v_0(z,t)| \le \int_0^1 \left|\frac{k(z,y,t)}{\alpha(y,t)}\right| |e(y,t)|\,\mathrm{d}y \le M\,\|e\|_{L^2} \tag{A.3}$$

with $M = \sup_{(z,y,t)\in\mathcal{T}\times\mathbb{R}^+} |\frac{k(z,y,t)}{\alpha(y,t)}|$ using Hölder's inequality. This induces a sequence of inequalities

$$\begin{aligned} |v_0(z,t)| &\le \int_0^1 \left|\frac{k(z,y,t)}{\alpha(y,t)}\right| |e(y,t)|\,\mathrm{d}y \le M\,\|e\|_{L^2},\\ |v_1(z,t)| &\le M^2 z\,\|e\|_{L^2},\\ |v_2(z,t)| &\le \frac{M^3}{2!} z^2\,\|e\|_{L^2},\\ &\cdots\\ |v_n(z,t)| &\le \frac{M^{n+1}}{n!} z^n\,\|e\|_{L^2}. \end{aligned} \tag{A.4}$$

The series

$$v(z,t) = \sum_{n=0}^{\infty} v_n(z,t) \tag{A.5}$$

therefore converges absolutely and uniformly to a continuous solution of (A.1) which is bounded by

$$|v(z,t)| \le M \sum_{n=0}^{\infty} \frac{M^n}{n!} z^n\,\|e\|_{L^2} \le M\,\mathrm{e}^{Mz}\,\|e\|_{L^2}. \tag{A.6}$$

Thus, there exists a constant $B = M\mathrm{e}^M$ that

$$\|v(z,t)\|_{L^2} \le B\|e(z,t)\|_{L^2}. \tag{A.7}$$

This implies the existence of a bounded linear operator $\Phi : L^2(0,1) \to L^2(0,1)$ mapping $e(z,t)$ onto $v(z,t) = (\Phi\, e)(z,t)$. By using the definition of $v(z,t)$, we obtain the relation

$$\begin{aligned} w(z,t) &= \frac{1}{\alpha(z,t)}\left[e(z,t) - v(z,t)\right]\\ &= \left[\frac{1}{\alpha(z,t)}(I-\Phi)e\right](z,t) = (K^{-1}e)(z,t). \end{aligned} \tag{A.8}$$

Thus, the linear operator K^{-1} is uniformly bounded by

$$\|w(z,t)\|_{L^2} \le N\,\|e(z,t)\|_{L^2} \tag{A.9}$$

with

$$N = \frac{1}{\alpha_{\inf}}(1+B) > 0. \tag{A.10}$$

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A.2 Addendum to Remark 4

Using the Lyapunov function $V(t) = \int_0^1 w^2(z,t)\mathrm{e}^{cz}\,\mathrm{d}z$ with an arbitrary positive constant c (see [5]) and proceeding analogous to Lemma 3 using

$$\left|\int_0^1 w(z,t)\,\mathrm{e}^{cz}\int_0^z h(z,\xi)w(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}z\right|^2 \le \mathrm{e}^c h_{\mathrm{sup}}^2 V^2(t) \tag{A.11}$$

one obtains

$$\dot{V}(t) \le -2\left(\frac{c}{2} + \mu_{\mathrm{inf}} - \mathrm{e}^{c/2}h_{\mathrm{sup}}\right)V(t) = -2\kappa(c)V(t). \tag{A.12}$$

Maximizing $\kappa(c)$ for $c \ge 0$ yields

$$c^* = \begin{cases} -2\ln(h_{\mathrm{sup}}) & \text{for} \quad h_{\mathrm{sup}} < 1 \\ 0 & \text{else} \end{cases} \tag{A.13}$$

and therefore $\dot{V}(t) \le -2\left(\mu_{\mathrm{inf}} - \ln(h_{\mathrm{sup}}) - 1\right)V(t)$ for $h_{\mathrm{sup}} < 1$. Fig. A.1 illustrates the stronger stability result.

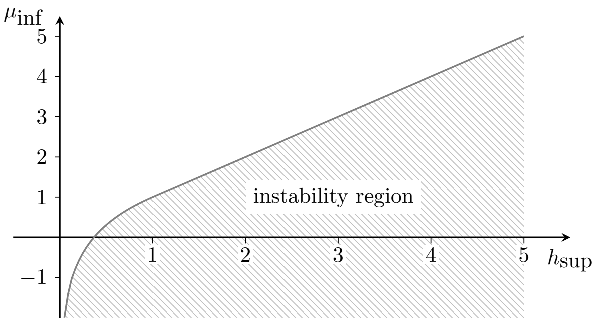

Fig. A.1. Stability of the target system.

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